m-Sparse Solutions of Linear Ordinary Differential Equations with Polynomial Coefficients

*Sergei A. Abramov**
Computer Center of the Russian Academy of Science
Vavilova 40, Moscow 117967, Russia
abramov@ccas.ru, sabramov@cs.msu.su

Abstract

We introduce the notion of m-sparse power series (e.g. expanding $\sin x$ and $\cos x$ at $x = 0$ gives 2-sparse power series: a coefficient a_n of the series can be nonzero only if $remainder(n, 2)$ is equal to a fixed number). Then we consider the problem of finding all m-points of a linear ordinary differential equation $Ly = 0$ with polynomial coefficients (i.e., the points at which the equation has a solution in the form of an m-sparse series). It is easy to find an upper bound for m. We prove that if m is fixed then either there exists a finite number of m-points and all of them can be found or all points are m-points and L can be factored as $L = \tilde{L} \circ C$ where C is an operator of a special kind with constant coefficients. Additionally we formulate simple necessary and sufficient conditions for the existence of m-points for an irreducible L.

Résumé

On introduit la notion de série de puissances m-creuse. (Les dévéloppements de $\sin x$ et de $\cos x$ autour de $x = 0$ sont des exemples de séries 2-creuses: on demande que le coefficient a_n de la série soit non-nul seulement si n appartient à une classe fixée de residus modulo 2). On considère le problème de déterminer tous les m-points d'une équation différentielle linéaire $Ly = 0$ à coefficients polynomiaux (i.e. les points où l'équation admet une solution sous forme m-creuse). Il est facile de trouver une borne supérieure pour m. Pour m fixé on démontre qu'ou bien il existe un nombre fini de m-points et on peut les déterminer, ou bien tous les points sont des m-points et L peut se factoriser en $L = \tilde{L} \circ C$ où C est un opérateur d'un type particulier à coefficients constants. En plus, on donne des critères nécessaires et suffisants simples pour l'existence de m-points lorsque l'opérateur L est irréductible.

Keywords: Linear differential equations, Formal solutions, Recurrences for coefficients, Sparse power series.

---

*Supported in part by the RFBR and INTAS under Grant 95-IN-RU-412.

1 Preliminaries

Let $\mathcal{C}$ be the set of infinite sequences $(c_0, c_1, \ldots) \in \mathbb{C}^\infty$, $\mathcal{S}$ the set of formal power series $c_0 + c_1 x + \cdots$ with $(c_0, c_1, \ldots) \in \mathcal{C}$ and m an integer ≥ 2. Call $c = (c_0, c_1, \ldots) \in \mathcal{C}$ an *m-sparse* sequence if there exists an integer N such that

$$(c_n \neq 0) \Rightarrow (n \equiv N \pmod{m}). \tag{1}$$

Call $c_0 + c_1 x + \cdots \in \mathcal{S}$ an m-sparse power series if $(c_0, c_1, \ldots)$ is an m-sparse sequence. For example, the series $x + x^4 + x^7 + \cdots + x^{3n+1} + \cdots$ is 3-sparse with $N = 1$. Denote by $\mathcal{C}^{(m)}$ (resp. $\mathcal{S}^{(m)}$) the set of all m-sparse elements of $\mathcal{C}$ (resp. of $\mathcal{S}$). It is obvious that

$$(m_1 | m_2) \Rightarrow (\mathcal{C}^{(m_2)} \subset \mathcal{C}^{(m_1)}, \mathcal{S}^{(m_2)} \subset \mathcal{S}^{(m_1)}).$$

Consider a linear ordinary differential equation $Ly = 0$ with

$$L = p_r(x)D^r + \cdots + p_1(x)D + p_0(x), \tag{2}$$

$p_0(x), \ldots, p_r(x) \in \mathbb{C}[x]$, $p_r(x) \neq 0$. It is well known that the coefficients $(c_0, c_1, \ldots)$ of a power series solution $c_0 + c_1 x + \cdots$ of a linear differential equation with polynomial coefficients satisfy a linear recurrence (a difference equation) $Rc = 0$:

$$q_l(n)c_{n+l} + q_{l-1}(n)c_{n+l-1} + \cdots + q_t(n)c_{n+t} = 0, \tag{3}$$

$q_t(n), q_{t+1}(n), \ldots, q_l(n) \in \mathbb{C}[n]$; $q_l(n), q_t(n) \neq 0$. The operator R which is equal to

$$q_l(n)E^l + q_{l-1}(n)E^{l-1} + \cdots + q_t(n)E^t \tag{4}$$

is the $\mathcal{R}$-image of L where $\mathcal{R}$ is the isomorphism of $\mathbb{C}[x, x^{-1}, D]$ onto $\mathbb{C}[n, E, E^{-1}]$:

$$\mathcal{R}D = (n+1)E, \ \mathcal{R}x = E^{-1}, \ \mathcal{R}x^{-1} = E; \tag{5}$$

resp.

$$\mathcal{R}^{-1}E = x^{-1}, \ \mathcal{R}^{-1}E^{-1} = x, \ \mathcal{R}^{-1}n = xD$$

(see [4]). Note that it is possible that $t < 0$ in (3), (4). For R of the form (4) we denote $\omega(R) = l - t$. If the coefficient of x^i in the polynomial $p_j(x)$ is not equal to zero in (2) then we write $x^i D^j \in L$. It is easy to check that if L is of the form (2) and $R = \mathcal{R}L$ then

$$l = \max_{x^i D^j \in L} \{j - i\}, \ t = \min_{x^i D^j \in L} \{j - i\}; \tag{6}$$

and therefore

$$\omega(R) = \max_{x^i D^j \in L} \{j - i\} - \min_{x^i D^j \in L} \{j - i\}.$$

We will call any solution of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n \tag{7}$$

of a differential equation *local* at the point a. Local solutions at a fixed point a form a linear space over $\mathbb{C}$. We will consider points $a \in \mathbb{C}$ and formal power series solutions $y_a(x)$ of the form (7) such that

$$Ly_a(x) = 0 \tag{8}$$

and $(c_0, c_1, \ldots) \in \mathcal{C}^{(m)}$ for an integer $m > 1$. Observe that $y_a(x)$ satisfies (8) iff

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

satisfies $L^a y(x) = 0$ where

$$L^a = p_r(x + a)D^r + \cdots + p_1(x + a)D + p_0(x + a). \tag{9}$$

In this paper we propose an algorithm for finding all m and a such that the equation $L^a y = 0$ has a solution in $\mathcal{S}^{(m)}$. A preliminary version of this paper has appeared as [1].

2 m-Points

We call a difference operator of the form (4) m-sparse if for some N

$$(q_j(n) \neq 0) \Rightarrow (j \equiv N \pmod m)$$

and we call a differential operator L m-sparse if for some N

$$(x^i D^j \in L) \Rightarrow (j - i \equiv N \pmod m).$$

It is easy to check that L is an m-sparse differential operator iff $\mathcal{R}L$ is an m-sparse difference operator.

Let $c = (c_0, c_1, \ldots) \in \mathcal{C}$. Denote by (c, x) the formal series $c_0 + c_1 x + \cdots$ and by $(c)_{\geq k}$ the sequence $(c_k, c_{k+1}, \ldots) \in \mathcal{C}$ with $c_k = c_{k+1} = \cdots = c_{-1} = 0$ if $k < 0$. It can be shown that if $R = \mathcal{R}L$ and R has the form (4) then

$$L(c, x) = 0 \Leftrightarrow R(c)_{\geq t} = 0 \tag{10}$$

(see [2, 4]). Let R be of the form (4) and let r_0 be the maximal nonnegative integer root of $q_l(n)$ if such roots exist, and -1 otherwise. Set

$$\iota^*(R) = l + r_0.$$

If $L \in \mathbb{C}[x, D]$ and $R = \mathcal{R}L$, then we set $\iota^*(L) = \iota^*(R)$. For any $(c_0, c_1, \ldots) \in \mathcal{C}$ such that $L(c, x) = 0$ the values $c_0, \ldots, c_{\iota^*(L)}$ allow one to compute (by means of $\mathcal{R}L$) the values $c_{\iota^*(L)+1}, c_{\iota^*(L)+2}, \ldots$ (these $c_{\iota^*(L)+1}, c_{\iota^*(L)+2}, \ldots$ are uniquely determined because the leading coefficient of the operator $\mathcal{R}L$ does not vanish when we compute c_n with $n > \iota^*(L)$).

Lemma 1 *The equation $Ly = 0$ has a nonzero local solution at 0 iff $\iota^*(L) \geq 0$.*

Proof: Thanks to (10) and to the mentioned property of $\iota^*(L)$ we have that if $\iota^*(L) < 0$ then $Ly = 0$ has only the zero local solution.

Let $\iota^*(L) \geq 0$. Then set $s = \iota^*(L)$, take the initial segment

$$0 + 0x + \cdots + 0x^{s-1} + x^s \tag{11}$$

and extend it to a local solution using the mentioned property of the value $\iota^*(L)$. □

Lemma 2 *Let L be an operator of the form (2) which can be factored as $L_1 \circ L_2$, where L_2 is an operator with polynomial coefficients such that* $\operatorname{ord} L_2 \geq 1$ *and L_2 has no local solution at 0. Then 0 is a singularity of L.*

Proof: If 0 is an ordinary point of L then L has $r = \operatorname{ord} L$ linearly independent local solutions $f_1, f_2, \ldots, f_r$ at 0. If the equation $L_2 y = 0$ has no nonzero local solution then it is injective on the space of formal power series. Then $L_2 f_1, L_2 f_2, \ldots, L_2 f_r$ are still linearly independent, and L_1 annihilates them all because $L = L_1 \circ L_2$. But this is impossible because $\operatorname{ord} L_1 < r$. □

Lemma 3 *Let L be an m-sparse differential operator with polynomial coefficients. Let the equation $Ly = 0$ have a local solution at 0. Then it has an m-sparse local solution at 0.*

Proof: If $Ly = 0$ has a local solution at 0 then by Lemma 1 there is such a local solution whose initial segment is of the form (11). The operator $\mathcal{R}L$ is an m-sparse difference operator. Using this operator the initial segments (11) can be extended to m-sparse local solutions. □

We can prove the following lemma on the possible values of m.

Lemma 4 *Let L be of the form (2). Let $R = \mathcal{R}L$ and let $Ly = 0$ have a non-polynomial solution $f(x) = c_0 + c_1 x + \cdots \in \mathcal{S}^{(m)}$. Then $m \leq \omega(R)$.*

Proof: If $m > \omega(R)$ then there is $k > \max\{\omega(R), \iota^*(R)\}$ such that $c_k = \cdots = c_{k+\omega(R)-1} = 0$. But then $c_n = 0$ for all $n \geq k$, i.e., $f(x) \in \mathbb{C}[x]$. Contradiction. □

From now on we will deal only with non-polynomial solutions. Polynomial solutions can be found by the algorithm described in [2]. Furthermore we will suppose that L is of the form (2), $R = \mathcal{R}L$ is of the form (4) and m is a fixed integer ≥ 2.

First we discuss the existence in $\mathcal{S}^{(m)}$ of solutions of $Ly = 0$ (i.e., $L^0 y = 0$). Section 3 will be devoted to the search for all a such that the equation $L^a y = 0$ has solutions in $\mathcal{S}^{(m)}$.

We will consider along with operators L and $R = \mathcal{R}L$ the set of m-sparse differential operators $L_0, \ldots, L_{m-1}$ and the set of m-sparse difference operators $R_0, \ldots, R_{m-1}$ which are called an *m-splitting* of the operators L and resp. R:

$$L_\tau = \sum_{\substack{x^i D^j \in L \\ j-i-t \equiv \tau \pmod{m}}} p_{ji} x^i D^j, \tag{12}$$

$$R_\tau = \sum_{\substack{t \le j \le l \\ j-t \equiv \tau \pmod m}} q_j(n) E^j, \tag{13}$$

$\mathcal{R}L_\tau = R_\tau$, $\tau = 0, \dots, m-1$.

Lemma 5 *Let $R_0, \dots, R_{m-1}$ be the m-splitting of R. Let $c \in \mathcal{C}^{(m)}$. Then*

$$R(c)_{\ge t} = 0 \Leftrightarrow (R_i(c)_{\ge t} = 0,\ i = 0, \dots, m-1). \tag{14}$$

Proof: a direct check. □

The lemma allows one to write down a necessary condition for the existence in $\mathcal{S}^{(m)}$ of solutions of $Ly = 0$.

Theorem 1 *Let $R_0, \dots, R_{m-1}$ be the m-splitting of R. Let $Ly = 0$ have a solution in $\mathcal{S}^{(m)}$. Then the greatest common right divisor* (GCD) *of the operators $R_0, \dots, R_{m-1}$ has positive ω:*

$$\omega(\mathrm{GCD}(R_0, \dots, R_{m-1})) \ge 1. \tag{15}$$

(We suppose as usual that R has the form (4) and that t is the lowest exponent of E in $\mathrm{GCD}(R_0, \dots, R_{m-1})$*.)*

Proof: Due to (10) and Lemma 5. □

The operator

$$V = \mathrm{GCD}(R_0, \dots, R_{m-1}) \tag{16}$$

can be found by the (right) Euclidean algorithm. We can assume V to be an operator with polynomial coefficients. If we apply the Euclidean algorithm to m-sparse difference operators then we obviously obtain again an m-sparse operator. Hence, $V \in \mathbb{C}[n, E]$ is an m-sparse operator. By $R = R_0 + \dots + R_{m-1}$ we have that R is right-divisible by V, but the coefficients of the quotient can be in $\mathbb{C}(n)$. For some $w(n) \in \mathbb{C}[n]$ we have

$$w(n)R = Q \circ V, \tag{17}$$

where $Q \in \mathbb{C}[n, E]$.

It is useful to define ι_* which will work together with ι^*. Let R be of the form (4). Let r_1 be the maximal nonnegative integer root of $q_t(n)$ if such roots exist, and -1 otherwise. Set

$$\iota_*(R) = \max\{t + r_1, -1\}.$$

Let $L \in \mathbb{C}[x, D]$ and $R = \mathcal{R}L$, then we set $\iota_*(L) = \iota_*(R)$. For any $(c_0, c_1, \dots)$ such that $L(c, x) = 0$ the values $c_k, c_{k+1}, \dots$ with $k > \iota_*(L) + 1$ let one compute (by means of $\mathcal{R}L$) the values $c_{\iota_*(L)+1}, c_{\iota_*(L)+2}, \dots, c_{k-1}$ (these $c_{\iota_*(L)+1}, c_{\iota_*(L)+2}, \dots, c_{k-1}$ are uniquely determined because the lowest coefficient of the operator $\mathcal{R}L$ does not vanish when we compute c_n with $n > \iota_*(L)$).

Going back to (16), (17) assume $\omega(V) \ge 1$ in (16). Set

$$u = \max\{\iota_*(V),\ \iota_*(w(n)R),\ \iota^*(V),\ \iota^*(w(n)R)\},$$

$$v = u + \omega(R).$$

Using an algorithm proposed in [2] we can find a basis for the space $\mathcal{B}$ of vectors $(c_0, \dots, c_v) \in \mathbb{C}^{v+1}$ which can be extended to infinite sequences $c = (c_0, c_1, \dots) \in \mathcal{C}$ which satisfy the equation $R(c)_{\geq l} = 0$. After a basis $d_0, \dots, d_w$, $w \leq v$, for $\mathcal{B}$ is found one can check (a linear problem) whether there exist $\alpha_0, \dots, \alpha_w \in \mathbb{C}$ such that $\alpha_0 d_0 + \cdots + \alpha_w d_w$ is an m-sparse vector whose last $\omega(R)$ components satisfy the recurrence $Vc = 0$. If such $\alpha_0, \dots, \alpha_w$ exist then we can extend the corresponding initial values using the recurrence $Vc = 0$. It will give us an infinite m-sparse sequence c which satisfies $R(c)_{\geq t} = 0$.

Later we will need the following theorem:

Theorem 2 *Let $L_0, \dots, L_{m-1}$ be the m-splitting of L. Let the equation $Ly = 0$ have a solution in $\mathcal{S}^{(m)}$. Then*

$$\operatorname{ord} \operatorname{GCD}(L_0, \dots, L_{m-1}) \geq 1. \tag{18}$$

Proof: Let $f(x) = c_0 + c_1 x + \cdots \in \mathcal{S}^{(m)}$, $Lf = 0$. Then $R(c)_{\geq t} = 0$ where $c = (c_0, c_1, \dots)$. Let $R_0, \dots, R_{m-1}$ be the m-splitting of R. By Lemma 5 we have $R_i(c)_{\geq t} = 0$, $i = 0, \dots, m-1$. By (10) we get $L_i f = 0$, $i = 0, \dots, m-1$. □

Now for the last remark of this section. Suppose we know that for a fixed m the equation $Ly = 0$ has a solution in $\mathcal{S}^{(m)}$. Then the next step could be, for example, the attempt to find an m-sparse series solution which is at the same time *m-hypergeometric* [8] (a power series is m-hypergeometric if its sequence of coefficients $(c_0, c_1, \dots)$ is m-hypergeometric, i.e., $c_{n+m} = r(n)c_n$, $n = 0, 1, \dots$, for a rational function $r(n)$).

Let the operator from (16) have the form

$$V = v_{t+km}(n)E^{t+km} + v_{t+(k-1)m}(n)E^{t+(k-1)m} + \cdots + v_t(n)E^t$$

and let an m-hypergeometric sequence c satisfy $Vc = 0$. Let c be m-sparse, and assume that equality (1) holds for some N, $0 \leq N \leq m-1$. It is evident that the sequence

$$c'_N = c_N,\ c'_{N+1} = c_{N+m}, \dots, c'_{N+k} = c_{N+km}, \dots$$

is hypergeometric. The sequence satisfies the recurrence $V'c' = 0$ with

$$V' = v_{t+km}(n)E^{t+k} + v_{t+(k-1)m}(n)E^{t+k-1} + \cdots + v_t(n)E^t.$$

Algorithm Hyper [7] allows one to find hypergeometric solutions of linear recurrences whose coefficients are rational functions.

If we are only interested in m-hypergeometric m-sparse series solutions then there is no need to compute $\operatorname{GCD}(R_0, \dots, R_{m-1})$. We can find solutions in the form of m-hypergeometric elements of $\mathcal{S}$ and then select the m-sparse ones among them. Using an algorithm proposed in [8] we can find all m-hypergeometric solutions of the recurrence $Vc = 0$ and then answer the question about m-hypergeometric m-sparse solutions of the original differential equation. Note that the mentioned algorithm from [8] allows one to find only primitive m-hypergeometric solutions of a recurrence (an m-hypergeometric sequence $(c_k, c_{k+1}, \dots)$ is *primitive* if it satisfies no linear homogeneous recurrence with rational coefficients of order $< m$). But it is obvious that an m-sparse m-hypergeometric solution having $c_i \neq 0$ with arbitrary large i is primitive m-hypergeometric. Thus the algorithm from [8] is sufficient for our goal.

However the usage of (18) is convenient when we solve the problem of searching for the points a at which there exist solutions of the form (7) with $(c_0, c_1, \ldots) \in \mathcal{C}^{(m)}$. We will call these points the *m-points* both of the operator L and of the equation $Ly = 0$.

3 The search for m-points

Let again $Ly = 0$ be an equation with operator of the form (2) and m be a fixed non-negative integer ≥ 2. We formulate the problem of the search for m-points as follows: to find all complex values of a such that the equation $L^a y = 0$, with L^a of the form (9), has a solution in $\mathcal{S}^{(m)}$. Consider the operator L^a regarding a as an indeterminate over $\mathbb{C}$. We can find the m-splitting of L^a, i.e., $L^a_0, \ldots, L^a_{m-1}$. We can also construct $R^a = \mathcal{R} L^a$ and $R^a_0, \ldots, R^a_{m-1}$ (the m-splitting of R^a). The coefficients of the operators

$$L^a, L^a_0, \ldots, L^a_{m-1}. \tag{19}$$

are polynomials in x over $\mathbb{C}[a]$. In turn the coefficients of the operators

$$R^a, R^a_0, \ldots, R^a_{m-1} \tag{20}$$

are polynomials in n over $\mathbb{C}[a]$. If a_0 is a value of the parameter a then we can consider, on the one hand

$$L^{a_0}, L^{a_0}_0, \ldots, L^{a_0}_{m-1} \text{ and } R^{a_0}, R^{a_0}_0, \ldots, R^{a_0}_{m-1} \tag{21}$$

and, on the other hand, the *specialization* of the operators (19), (20) for $a = a_0$:

$$L^a|_{a=a_0}, L^a_0|_{a=a_0}, \ldots, L^a_{m-1}|_{a=a_0} \text{ and } R^a|_{a=a_0}, R^a_0|_{a=a_0}, \ldots, R^a_{m-1}|_{a=a_0} \tag{22}$$

as the result of substitution of a_0 for a in (21). Operators (21) and (22) are equal. This equivalence takes place due to the fact that evaluation of coefficients at $a = a_0$ commutes both with m-splitting and with $\mathcal{R}$.

This equivalence leads to the construction of a set including all m-points. We will show how one can gather together all a_0 such that

$$\operatorname{ord} \operatorname{GCD}(L^{a_0}_0, \ldots, L^{a_0}_{m-1}) \geq 1. \tag{23}$$

Observe that (18) is a particular version of (23) with $a_0 = 0$.

We will denote below by a a parameter while $a_0, a_1, \ldots$ denote concrete values of a ($a_0, a_1, \ldots \in \mathbb{C}$).

Theorem 3 *Let $R^a = \mathcal{R} L^a$. Let R^a be*

$$g_{l'}(n, a) E^{l'} + \cdots + g_{t'}(n, a) E^{t'}. \tag{24}$$

Then

$$r = l' \geq l, \ t' = t, \ \deg_a g_{t'}(n,a) = 0, \tag{25}$$

i.e., (24) can be written as

$$g_r(n,a)E^r + \cdots + g_{t+1}(n,a)E^{t+1} + g_t(n)E^t. \tag{26}$$

Proof: We prove $r = l'$ using (6) and $x^0 D^r \in L^a$ (this follows from the fact that a is an indeterminate). If $x^0 D^r \in L$ then $r = l$ else $r > l$. The equality $t = t'$ is obvious. The equality $\deg_a g_{t'}(n,a) = 0$ (i.e., $\deg_a g_t(n,a) = 0$) is a consequence of $\deg_a \mathrm{lc}_x\, p_j(x+a) = 0$. □

The last theorem allows one to assume that

$$2 \leq m \leq \omega(R^a) = l' - t' = r - t = \operatorname{ord} L - \min_{x^i D^j \in L} \{j - i\}.$$

Lemma 6 *Let L be an m-sparse operator and 0 be an ordinary point of it (i.e., $p_r(0) \neq 0$). Then the equation $Ly = 0$ has $r = \operatorname{ord} L$ linearly independent solutions in $\mathcal{S}^{(m)}$.*

Proof: At an ordinary point, any r initial coefficients $c_0, \ldots, c_{r-1}$ determine a series which satisfies the equation $Ly = 0$. We can take $c_i = \delta_{ij}$ in the j-th element of the basis for the space of vectors $(c_0, \ldots, c_{r-1}) \in \mathbb{C}^r$, $j = 0, \ldots, r-1$. Let us extend every element of the basis by elements $c_{-1} = 0, c_{-2} = 0, \ldots, c_t = 0$. Applying m-sparse recurrence $Rc = 0$ to any of the extended vectors as to a vector of initial elements, we obtain an infinite m-sparse sequence. □

Lemma 7 *The operator* $\mathrm{GCD}(L_0^a, \ldots, L_{m-1}^a)$ *is an m-sparse differential operator.*

Proof: Indeed, if $M_1 = v_r(x)D^r + \cdots + v_0(x)$, $M_2 = w_s(x)D^s + \cdots + w_0(x)$ are two m-sparse differential operators, $r \geq s$, then the operators $w_s(x)M_1$, $v_r(x)D^{r-s} \circ M_2$ are m-sparse. The number N mentioned in the definition of an m-sparse differential operator is the same for both operators, thereby the difference of these operators is an m-sparse operator. The order of the difference is $< r$. □

Observe that the operator $\mathrm{GCD}(L_0^a, \ldots, L_{m-1}^a)$ is defined up to a factor from $\mathbb{C}(a, x)$, and a more precise statement is that one can assume this operator to be an m-sparse differential operator.

Now the question is: for which values of the parameter a do the operators $L_0^a, \ldots, L_{m-1}^a$ have a nontrivial right common divisor? If one has two differential operators with coefficients which are polynomials in x over $\mathbb{C}[a]$ then the answer can be obtained, for example, by applying the Euclidean algorithm to them. There is a finite set of values of the parameter a for which the value of the leading coefficient of some remainder vanishes. The actual implementation would instead use differential subresultants for reasons of efficiency [5, 6]. One way or the other, we can construct a pair $(\mathbf{S}_m(L), \mathbf{T}_m(L))$, where

- $\mathbf{S}_m(L)$ is an m-sparse differential operator whose coefficients are polynomials over $\mathbb{C}[a]$,

- $\mathbf{T}_m(L)$ is a finite subset of $\mathbb{C}$,

such that the substitution of any $a_0 \notin \mathbf{T}_m(L)$ into $\mathbf{S}_m(L)$ for a gives the operator $S_{a_0} = \mathrm{GCD}(L_0^{a_0}, \ldots, L_{m-1}^{a_0})$ with $\operatorname{ord} S_{a_0} = \operatorname{ord} \mathbf{S}_m(L)$. If $a_0 \in \mathbf{T}_m(L)$ then either $S_{a_0} \neq \mathrm{GCD}(L_0^{a_0}, \ldots, L_{m-1}^{a_0})$ or the leading coefficient of $\mathbf{S}_m(L)$ vanishes at $a = a_0$.

Theorem 4 *Either the operator L has only a finite set of m-points, or any ordinary point of L which does not belong to $\mathbf{T}_m(L)$ is an m-point.*

Proof: Let
$$S = \mathbf{S}_m(L),\ T = \mathbf{T}_m(L). \tag{27}$$

If $\operatorname{ord} S = 0$ then there exists only a finite set of a_0 which satisfy (23) and there is nothing to prove.

Let $\operatorname{ord} S > 0$. Let $S_{a_0} = S|_{a=a_0}$. For $a_0 \notin T$ the operator L^{a_0} is right divisible by S_{a_0}. If for some a_0 the equation $S_{a_0} y = 0$ has no local solution at 0 then by Lemma 2 a_0 is a singularity of L. If this equation has a local solution at 0 then by Lemma 3 it has a solution in $\mathcal{S}^{(m)}$. □

The last theorem provides us with an algorithm to find all m-points of the given equation since all elements of $\mathbf{T}_m(L)$ and the singular points of L can be investigated by the approach described in Section 2.

But we will show in the next section that Theorem 4 may be strengthened. This will allow improving the algorithm.

4 The case of an infinite set of m-points

The purpose of this section is proving the following theorem:

Theorem 5 *If L has an infinite set of m-points, then $L = \tilde{L} \circ C$ where C is an m-sparse differential operator with constant coefficients, and* $\operatorname{ord} C > 0$.

Let S be as in (27). The operator S belongs to the ring $\mathbb{C}(a)[x, D]$, and we write $S(a, x, D)$ for S.

Lemma 8 *The operators $S(a+1, x, D)$ and $S(a, x-1, D)$ are equal up to a factor from $\mathbb{C}(a, x)$.*

Proof: Let $a_0 \in \mathbb{C}$ be such that
$$a_0 + 1 \notin \mathbf{T}_m(L). \tag{28}$$
Set $S' = S(a_0 + 1, x, D), S'' = S(a_0, x-1, D)$. Then
$$S' f = 0 \Leftrightarrow S'' f = 0 \tag{29}$$
for any $f \in \mathcal{S}^{(m)}$ since both equalities are equivalent to $L^{a_0+1} f = 0$. If a_0 is an ordinary point of both S' and S'' then by Lemma 6 these operators become equal after making

them monic (i.e., $S'/(\mathrm{lc}\, S') = S''/(\mathrm{lc}\, S'')$). Thus those two monic operators are equal for all $a_0 \in \mathbb{C}$ except for a finite set of values. Therefore the operators $S(a+1, x, D)$ and $S(a, x-1, D)$ taken in the monic form are equal which proves the lemma. $\square$

Call a differential operator L of the form (2) *primitive* if

$$\gcd(p_0(x), \ldots, p_r(x)) = 1, \tag{30}$$

where gcd denotes the polynomial greatest common divisor. It is easy to see that L^a (as an operator whose coefficients are polynomials in x over $\mathbb{C}(a)$) and L^{a_0} are primitive iff L is primitive.

The operator $S(a, x, D)$ can be constructed in the form of a primitive m-sparse operator, whose coefficients are polynomials over $\mathbb{C}(a)$ (this is due to Lemma 7, as well as to the fact that the greatest common divisor of polynomials of the form $x^s f(x^m)$, $f(x) \in \mathbb{C}(a)[x]$, and the quotient of such polynomials are polynomials of the same form). Then operators $S(a+1, x, D)$ and $S(a, x-1, D)$ are also primitive and by Lemma 8 should be equal up to a factor from $\mathbb{C}(a)$ (due to the primitivity). But if the operator $S(a, x, D)$ has at least one coefficient which belongs to $\mathbb{C}(a, x) \setminus \mathbb{C}(a)$, then $S(a, x-1, D)$ is not m-sparse. At the same time any operator of the form $r(a)S(a+1, x, D)$, $r(a) \in \mathbb{C}(a)$, is m-sparse. Therefore all the coefficients of $S(a, x, D)$ are in $\mathbb{C}(a)$. But the operators $S(a+1, x, D)$ and $S(a, x-1, D)$ are equal after putting them in the monic form, and this implies that the coefficients of $S(a, x, D)/(\mathrm{lc}\, S(a, x, D))$ do not depend on a, i.e., these coefficients are constants. Thus Theorem 5 is proven.

Theorem 6 *If $S = \mathbf{S}_m(L)$ then $C = S/(\mathrm{lc}\, S)$ is the maximal m-sparse differential operator with constant coefficient which divides L.*

Proof: If $\mathrm{ord}\, S > 0$ then L has an infinite set of m-points and by Theorem 5, the operators L, L^a, S are right-divisible by an m-sparse operator with constant coefficients. Let C be the maximal m-sparse operator with constant coefficients dividing S. Then C divides L and L^a. Similarly to Lemma 7 it can be shown that the m-splitting of L^a/C is equal to $L_0^a/C, \ldots, L_{m-1}^a/C$. Observe that $S/C = \mathrm{GCD}(L_0^a/C, \ldots, L_{m-1}^a/C)$ and that S/C has no positive-order right divisor with constant coefficients. Hence $\mathrm{ord}(S/C) = 0$. $\square$

Note that if L is right-divisible by an m-sparse operator C, $\mathrm{ord}\, C > 0$, with constant coefficients, then all points are m-points. But the points of $\mathbf{T}_m(L)$ are of special interest because the number of m-sparse linearly independent solutions can increase at those points (each of them can be investigated by the approach described in Section 2).

Example 1 $L = (-x^6 + 9x^4 - 7x^2 - 1)D^3 + (-2x^5 + 28x^3 + 22x)D^2 + (x^6 - 9x^4 + 7x^2 + 1)D + (2x^5 - 28x^3 - 22x)$. Take $m = 2$. We have

$L_0^a = (-x^6 + (-15a^2 + 9)x^4 + (-15a^4 + 54a^2 - 7)x^2 - a^6 - 7a^2 + 9a^4)D^3 + (-2x^5 + (-20a^2 + 28)x^3 + (-10a^4 + 84a^2 + 22)x)D^2(+x^6 + (15a^2 - 9)x^4 + (15a^4 - 54a^2 + 7)x^2 + a^6 + 7a^2 - 9a^4)D + 2x^5 - (20a^2 - 28)x^3 + (10a^4 - 84a^2 - 22)x,$

$L_1^a = (-6ax^5 + (-20a^3 + 36a)x^3 + (-6a^5 + 36a^3 - 14a)x)D^3 + (-10ax^4 + (-20a^3 + 84a)x^2 - 2a^5 + 28a^3 + 22a)D^2 + (+6ax^5 + (20a^3 - 36a)x^3 + (6a^5 - 36a^3 + 14a)x)D^1 + 10ax^4 + (20a^3 - 84a)x^2 + 2a^5 - 28a^3 - 22a$.

The algorithm from [6] allows to determine that $\mathrm{GCD}(L_0^{a_0}, L_1^{a_0})$ is $(-x^6+9x^4-7x^2-1)D^3+(-2x^5+28x^3+22x)D^2+(x^6-9x^4+7x^2+1)D+2x^5-28x^3-22x$ if $a_0 = 0$, and $D^2 - 1$ otherwise. The equation $Ly = 0$ has two linearly independent 2-sparse solutions

$$\sum_{n=0}^{\infty} \frac{(x-a_0)^{2n}}{(2n)!}, \quad \sum_{n=0}^{\infty} \frac{(x-a_0)^{2n+1}}{(2n+1)!}$$

at any $a_0 \neq 0$. It has three linearly independent 2-sparse solutions

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad \sum_{n=0}^{\infty} x^{2n}$$

at $x = 0$.

5 When L is irreducible

An irreducible operator L cannot have a non-trivial right factor over $\mathbb{C}(x)$ of order $< \operatorname{ord} L$ and we deduce the following theorem.

Theorem 7 *Let L be a primitive irreducible operator. Then $a_0 \in \mathbb{C}$ is an m-point of L iff L^{a_0} is m-sparse and $\iota^*(L^{a_0}) \geq 0$.*

Proof: Since L is irreducible, it cannot have a factor S such that $0 < \operatorname{ord} S < \operatorname{ord} L$. Therefore if a_0 is an m-point then $\mathrm{GCD}(L_0^{a_0}, \ldots, L_{m-1}^{a_0}) = L^{a_0}$. In the case of a primitive L it is possible only if one of $L_0^{a_0}, \ldots, L_{m-1}^{a_0}$ is equal to L^{a_0} and all others are equal to zero. Then L^{a_0} is m-sparse. Together with Lemma 1 this proves the necessity. The sufficiency follows from Lemmas 1, 3. □

We have as a consequence that if L is an irreducible primitive operator and a_0 is an ordinary point of L then a_0 is an m-point of L iff L^{a_0} is m-sparse. By Lemma 6 we get also that if an irreducible primitive operator L is such that L^{a_0} is m-sparse for $a_0 \in \mathbb{C}$, then the equation $Ly = 0$ has $r = \operatorname{ord} L$ linearly independent local solutions at a_0.

Additionally we can prove the following simple necessary condition.

Theorem 8 *Let L be a primitive irreducible operator of the form (2). Let $p_{s_1}(x), \ldots, p_{s_k}(x)$ be all the nonzero coefficients of L and let $t_1, \ldots, t_k$ be their degrees. Let there exist an m-point of L. Then*

$$s_1 - t_1 \equiv \ldots \equiv s_k - t_k \pmod{m}. \tag{31}$$

Proof: If a_0 is an m-point of L then by the previous theorem L^{a_0} is m-sparse. But the leading coefficients of the polynomial coefficients of L^{a_0} do not depend on the value of a_0. □

Theorem 9 *An irreducible operator L has no more than one m-point.*

Proof: Without loss of generality we can assume L to be a primitive operator (otherwise we can divide L by the gcd of its coefficients). If L is an operator with constant coefficients then $\operatorname{ord} L = 1$ due to its irreducibility and there is no $a_0 \in \mathbb{C}$ such that L^{a_0} is m-sparse. If L is not an operator with constant coefficients and L^{a_0} is m-sparse then for any $a_1 \in \mathbb{C}$, $a_1 \neq a_0$, the operator L^{a_1} is not m-sparse. Due to Theorem 7 we obtain what was claimed. $\square$

Let L be again a primitive irreducible operator of the form (2). Does there exist a_0 such that L_0^a is m-sparse and, if yes, how to construct such a_0? First check condition (31) and, if it is satisfied, set $N = remainder(s_1 - t_1, m)$. Construct L^a and, using the condition

$$(x^i D^j \in L^{a_0}) \Rightarrow (j - i \equiv N \pmod{m}),$$

we get algebraic equations for a that allow getting a_0.

Example 2 $y'' + (x-1)y = 0$. We have $2 \leq m \leq 3$. The operator $L = D^2 + (x-1)$ is irreducible over $\mathbb{C}(x)$; $L^a = D^2 + (x + a - 1)$.

$m = 2$. Necessary conditions (31) is not satisfied: $2 - 0 \not\equiv 0 - 1 \pmod{2}$. The equation has no 2-points.

$m = 3$. Necessary conditions (31) is satisfied: $2 - 0 \equiv 0 - 1 \pmod{3}$. The equation $a - 1 = 0$ gives us the value $a_0 = 1$. This is an ordinary point, hence it is a 3-point. By Lemma 6 the given equation has two linearly independent 3-sparse solutions. In this case they can be taken as 3-hypergeometric:

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1) \cdot 3n},$$

$$(x-1) + \sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^{3n+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdots 3n \cdot (3n+1)}.$$

Acknowledgement

I thank Prof. Marko Petkovšek for many valuable discussions.

References

[1] S. Abramov, Solutions of Linear Differential Equations in The Class of Sparse Power Series, in: Proc. FPSAC'97, Univ. of Wien, 1997, pp. 1–12.

[2] S. Abramov, M. Bronstein, M. Petkovšek, On Polynomial Solutions of Linear Operator Equations, in: T.Levelt (Ed.), Proc. ISSAC'95, ACM Press, New York, 1995, pp. 290–296.

[3] S. Abramov, M. Petkovšek, Special Power Series Solutions of Linear Differential Equations, in: D.Stanton (Ed.), Proc. FPSAC'96, Univ. of Minnesota, 1996, pp. 1–8.

[4] S. Abramov, M. Petkovšek, A. Ryabenko, Special Formal Series Solutions of Linear Operator Equations, Discrete Math., to appear.

[5] M. Chardin, Differential Resultants and Subresultants, In: Fundamentals of Computation Theory, Lecture Notes in Computer Science 529, Springer-Verlag, 1991, pp. 180–189.

[6] P. Glotov, On the Greatest Common Right Divisor of Ore Polynomials with Polynomial Coefficients which Depend on a Parameter, in: G.Labahn, M.O.Rayes (Eds.), Poster Session Abstracts of ISSAC'98, Univ. of Rostock, 1998, p. 20.

[7] M. Petkovšek, Hypergeometric solutions of linear recurrences with polynomial coefficients, J. Symb. Comput. **14**, 1992, pp. 243–264.

[8] M. Petkovšek, B. Salvy, Finding All Hypergeometric Solutions of Linear Differential Equations, in: M.Bronstein (Ed.), Proc. ISSAC'93, ACM Press, New York, 1993, pp. 27–33.