Chapter 1
When the search for solutions can be terminated

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Dedicated to Peter Paule on the occasion
of his 60th birthday

Abstract As a rule, search algorithms for those solutions of differential equations and systems that belong to a fixed class of functions are designed so that nonexistence of solutions of the desired type is detected only in the last stages of the algorithm. In some cases, performing additional tests on the intermediate results makes it possible to stop the algorithm as soon as these tests imply that no solutions of the desired type exist. We will consider these questions in connection with the search for rational solutions of linear homogeneous differential systems with polynomial coefficients. (Some approaches are already known for the case of scalar equations.)

1.1 Introduction

One of actual computer algebra problems is the development of algorithms for finding solutions to differential equations and systems of such equations. Usually solutions belonging to some fixed class are discussed. Often the proposed algorithms are such that the absence of solutions of the desired form is detected only in the final stages, when many of the quantities required to construct such a (potential) solution are already computed.

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However, it is possible that in the algorithm one can choose some checkpoints and, accordingly, associate with them some tests which make it possible to ascertain already at an early stage that there are no solutions of the desired type. This will save time and other computing resources. Thus there is the problem of choosing checkpoints and tests. On the one hand, one can think about this choice already in the development of the algorithm and seek the appearance in the algorithm of such points equipped with easily performable tests; on the other hand, one can take a known algorithm and insert checkpoints in it. In this case, it may be necessary to modify the algorithm in order for suitable checkpoints to be discovered and for these points to precede some resource-consuming fragments of the algorithm.

In the present paper, we consider this problem as applied to the search for rational solutions.

Let K be a field of characteristic 0. The ring of polynomials and the field of rational functions of x are conventionally denoted as $K[x]$ and $K(x)$, respectively. The ring of formal Laurent series is denoted as $K((x))$. If R is a ring (in particular, a field), then $\mathrm{Mat}_m(R)$ denotes the ring of $m \times m$-matrices with entries from R. We consider systems of the form

$$A_r(x)D^r y(x) + \cdots + A_1(x)Dy(x) + A_0(x)y(x) = 0 \tag{1.1}$$

where $D = \frac{d}{dx}$, and $A_i(x)$, for $i = 0, 1, \ldots, r$, are matrices of size $m \times m$ with entries from $K[x]$. Here $A_r(x)$ is the *leading* matrix (we suppose that $A_r(x)$ is non-zero), and $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T$ is a column of unknown functions (T denotes transposition). The number r is called the *order* of the system. The system under study is assumed to be of full rank; i.e., the equations of the system are linearly independent over the ring of operators $K(x)[D]$. In some cases, the *trailing* matrix of a system is also considered. (If $k = \min\{l \mid A_l \neq 0\}$ then A_k is the trailing matrix of (1.1).)

The system (1.1) can be written in the form

$$L(y) = 0 \tag{1.2}$$

where

$$L = A_r(x)D^r + \cdots + A_1(x)D + A_0(x). \tag{1.3}$$

A solution $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T \in K(x)^m$ of (1.1) is called a *rational* solution. If $y(x) \in K[x]^m$, it is called a *polynomial* solution (a particular case of a rational solution). Algorithms for finding all rational solutions to a first-order system of the form

$$Dy(x) = A(x)y(x), \tag{1.4}$$

where $A(x) \in \mathrm{Mat}_m(K(x))$, are well known (see, e.g., [3], [10]). The problem of finding rational solutions for full-rank systems (1.1) in the case where the matrix $A_r(x)$ may be singular, were considered much less frequently. Nevertheless, an appropriate algorithm was suggested in [9]. This algorithm is based on finding a *universal denominator* of rational solutions to the original system (for brevity, we call it the universal denominator for the original system), i.e., a polynomial

$U(x) \in K[x]$ such that, if the system has a rational solution $y(x) \in K(x)^m$, then it can be represented as $\frac{1}{U(x)} z(x)$, where $z(x) \in K[x]^m$. If a universal denominator is known, we can make the substitution

$$y(x) = \frac{1}{U(x)} z(x) \tag{1.5}$$

where $z(x) = (z_1, \ldots, z_m)^T$ is a vector of new unknowns, and then apply one of the algorithms for finding polynomial solutions (see, e.g., [3], [11], [17]). A *denominator bound* for the original system is a rational function $S(x)$ such that any rational solution of the original system can be represented in the form $S(x) f(x)$ with $f(x) \in K[x]^m$. So a denominator bound can also be used for finding rational solutions by using the substitution

$$y(x) = S(x) z(x) \tag{1.6}$$

instead of (1.5). (If $U(x)$ is a universal denominator for (1.2) then $\frac{1}{U(x)}$ is obviously a denominator bound for the same system.)

Other approaches are also possible. For example, the approach presented in [2] is based on expanding a general solution of the original system (1.2) into a series whose coefficients linearly depend on arbitrary constants. After multiplication by a universal denominator $U(x)$ (or by $S^{-1}(x)$, where $S(x)$ is a denominator bound) the series corresponding to rational solutions turn into polynomials.

In the sequel, it will be useful to consider formal Laurent series, i.e., for example, elements of the field $K((x))$ (or the field $\bar{K}((x))$, where $\bar{K}$ is the algebraic closure of K). Recall that the *valuation* $\operatorname{val} y(x)$ of $y(x) \in K((x))$ is the minimal integer i such that the coefficient of x^i in $y(x)$ is non-zero. If $y(x)$ is the zero series then we set $\operatorname{val} y(x) = +\infty$. We can also consider the field $K((x - \alpha))$ of formal Laurent series in $x - \alpha$ and, correspondingly, $\operatorname{val}_{x-\alpha} t(x)$ for $t \in K((x - \alpha))$.

We consider also the formal series in terms of decreasing powers (this can also be viewed as expansion at ∞); the field of such series is denoted by $K((x^{-1}))$. Each series of this kind contains only a finite number of powers of x with nonnegative exponents and, possibly, an infinite number of powers with negative ones. The greatest exponent of x with a nonzero coefficient occurring in a series $y(x)$ is the valuation $\operatorname{val}_\infty y(x)$. If $y(x) \in K((x^{-1}))$ is the zero series, then we set $\operatorname{val}_\infty y(x) = -\infty$.

For a vector $f(x) = (f_1(x), \ldots, f_m(x))^T \in K((x))^m$ we set $\operatorname{val} f(x) = \min_{i=1}^m \operatorname{val} f_i$ (similarly for $\operatorname{val}_{x-\alpha} f(x)$). For $g(x) = (g_1(x), \ldots, g_m(x))^T \in K((x^{-1}))^m$ we set $\operatorname{val}_\infty g(x) = \max_{i=1}^m \operatorname{val}_\infty g_i$. It is easy to see that $\operatorname{val}_\infty p(x) = \deg p(x)$ for a polynomial $p(x)$ and $v(\frac{f(x)}{g(x)}) = v(f(x)) - v(g(x))$ for $f(x), g(x) \in K[x]$, $v \in \{\operatorname{val}, \operatorname{val}_{x-\alpha}, \operatorname{val}_\infty\}$. It is also significant that the valuation of any type under consideration of a product is the sum of the valuations of the factors.

The checkpoints and tests mentioned at the beginning of the Introduction may help detect situations where substitutions of the series in question lead to a system that obviously has no polynomial solutions. In this case, we would like to obtain tests that do not require a complete calculation of the universal denominators or denominator bounds, but involve just some preliminary estimates. For scalar difference equations, such points and tests were found by A. Gheffar in [13, 14]. In the present paper, we generalize those ideas for linear systems of differential equations with polynomial coefficients.

1.2 Preliminaries: indicial polynomials

A rational solution of a system of the form (1.1) can be represented by formal Laurent series both at an arbitrary finite point α and at ∞.

It is well known (see, e.g., [7, Sect. 7.2]) that it is possible to construct for (1.1) a finite set of irreducible polynomials over K

$$p_1(x), \ldots, p_k(x) \tag{1.7}$$

such that if for some $\alpha \in \overline{K}$ there exists a solution $F \in \overline{K}((x-\alpha))^m$ such that $\mathrm{val}_{x-\alpha} F < 0$ then $p_i(\alpha) = 0$ for some $1 \leqslant i \leqslant k$, and for each $p_i(x)$ a polynomial $I_{L,p_i}(\lambda) \in K[\lambda]$ can be constructed such that for a solution $F \in K((x-\alpha))^m$, $p_i(\alpha) = 0$, one has $I_{L,p_i}(\mathrm{val}_{x-\alpha} F) = 0$ [7]. It is also possible to construct such a polynomial $I_{L,\infty}(\lambda) \in K[\lambda]$ that if a system $L(y) = 0$ has a solution $y \in K((x^{-1}))$ then $I_{L,\infty}(\mathrm{val}_\infty y(x)) = 0$. In particular, the degree of a polynomial solution is a root of $I_{L,\infty}(\lambda)$. The polynomials $I_{L,\infty}(\lambda), I_{L,p_1}(\lambda), \ldots, I_{L,p_k}(\lambda)$ are the *indicial* polynomials connected with L.

Remark In the context of this paper, by the indicial polynomial for a given operator L we mean a certain polynomial, a root of which may give useful information on solutions of the initial differential system. Absence of roots of a certain type also gives information on solutions of the initial differential system. Note that it is not necessary that every root of such a polynomial corresponds to some specific solution of the initial system, as in classical theory. To construct the needed polynomials we can use the so-called induced recurrence system and bring its leading or trailing matrix to non-singular form. Based on the determinants of those matrices, some polynomials can be obtained that play the role of the indicial polynomials ([1, 7]). (The mentioned induced recurrence system is satisfied by the sequence of coefficients of any Laurent series solution of the original differential system; the elements of such a sequence belong to K^m or $\bar{K}^m$.)

1.3 Scheme equipped with control tests

The following proposition is the main statement of the paper.

Proposition Let L, $p_1(x), \ldots, p_k(x)$ be as in (1.3), (1.7). Let $I_{L,\infty}(\lambda)$, $I_{L,p_1}(\lambda), \ldots, I_{L,p_k}(\lambda)$ be the corresponding indicial polynomials. In this case

(i) if $I_{L,\infty}(\lambda)$ has no integer root then (1.2) has no rational solution;

(ii) if at least one of the polynomials $I_{L,p_1}(\lambda), \dots, I_{L,p_k}(\lambda)$ has no integer root then (1.2) has no rational solution;

(iii) if $b_1, \dots, b_k \in \mathbb{Z}$ are lower bounds for integer roots of polynomials $I_{L,p_1}(\lambda), \dots, I_{L,p_k}(\lambda)$ (e.g., $b_1, \dots, b_k$ can be equal to the minimal integer roots of those polynomials), N is an upper bound for integer roots of the polynomial $I_{L,\infty}(\lambda)$ (e.g., N can be equal to the maximal integer root of that polynomial), and $N - \sum_{i=1}^k b_i \deg p_i < 0$, then (1.2) has no rational solution;

(iv) if $N - \sum_{i=1}^k b_i \deg p_i \geqslant 0$ (see (iii)) and the system (1.2) has a rational solution then that solution is of the form $p_1^{b_1}(x) \dots p_k^{b_k}(x) f(x)$, where $f(x) = (f_1(x), \dots, f_m(x))^T \in K[x]^m$ with $\deg f_j(x) \leqslant N - \sum_{i=1}^k b_i \deg p_i$, $j = 1, \dots, m$.

Proof (i), (ii): If (1.2) has a rational solution $F \in K(x)^m$ then (1.2) has a solution in $K((x^{-1}))^m$ as well, since $F(x)$ can be represented by a series from $K((x^{-1}))$. Let $s(x)$ be a formal Laurent series over K^m for $F(x^{-1})$, then $t(x) = s(x^{-1}) \in K((x^{-1}))$ is the series for $F(x)$. So $I_{L,\infty}(\mathrm{val}_\infty t(x)) = 0$, proving (i). Let α be such that $p_i(\alpha) = 0$, $1 \leqslant i \leqslant k$, and let $s \in K((x-\alpha))^m$ be the Laurent series expansion of $F(x)$. Then $I_{L,p_i(x)}(\mathrm{val}_{x-\alpha} s) = 0$, proving (ii).

(iii): $S(x) = p_1^{b_1}(x) \dots p_k^{b_k}(x)$ is a denominator bound for (1.2) (among $b_1, \dots, b_k$ there may be numbers of different signs). If $F(x) \in K(x)^m$ is a rational solution of (1.2) then $F(x) = S(x) f(x)$ for some $f(x) \in K[x]^m$. We have $0 \leqslant \mathrm{val}_\infty f(x) = \mathrm{val}_\infty F(x) - \mathrm{val}_\infty S(x) \leqslant N - \mathrm{val}_\infty S(x) = N - \sum_{i=1}^k b_i \deg p_i$. Thus, if there exists a rational solution then $N - \sum_{i=1}^k b_i \deg p_i \geqslant 0$.

(iv): The upper bound $N - \sum_{i=1}^k b_i \deg p_i \geqslant 0$ for $\mathrm{val}_\infty f(x) = \max_{j=1}^m \deg f_j$ was obtained in the proof of (iii). □

A scheme equipped with control tests may be, for example, as follows.

1. Find $I_{L,\infty}(\lambda)$. If this polynomial does not have integer roots, then STOP. Otherwise, let N be the largest integer root of $I_{L,\infty}(\lambda)$.
2. Find $p_1(x), \dots, p_k(x)$ and polynomials $I_{L,p_1}(\lambda), \dots, I_{L,p_k}(\lambda)$. If at least one of $I_{L,p_1}, \dots, I_{L,p_k}$ does not have integer roots, then STOP. Otherwise, let $e_1, \dots, e_k$ be the smallest integer roots of these indicial polynomials and $d = e_1 \deg p_1 + \dots + e_k \deg p_k$.
3. If $N+d < 0$ then STOP. Otherwise, perform in (1.1) the substitution $y = Sz$, where $S(x) = p_1(x)^{e_1} \dots p_k(x)^{e_k}$, and z is a new unknown vector. Find all polynomial solutions of the new system $\tilde{L}(z) = 0$, using the fact that the degree of each such solution does not exceed $N + d$. If there are no such solutions, then STOP. Otherwise, rational solutions of the system $L(y) = 0$ are obtained from polynomial solutions of the system $\tilde{L}(z) = 0$ by multiplying each component of z by $S(x)$.

In this scheme, the STOP command means stopping all calculations with the message to the user: "The system has no rational solutions".

Having computed the upper bound $N - d$ for the degrees of polynomial solutions allows us to use the method of undetermined coefficients for finding polynomial solutions of the system $\tilde{L}(z) = 0$ (the problem of finding polynomial solutions is reduced to solving a system of linear algebraic equations). There exist methods which are more effective than the method of undetermined coefficients (see, for example, [17]). However, to apply the algorithm from [17], it is necessary to construct an induced recurrent system and bring its trailing matrices to non-singular form (we have mentioned induced recurrent systems in Remark 1). This preparatory work is equivalent in cost to obtaining the indicial polynomial $I_{\tilde{L},\infty}$. One can also use the approach from [2], for which one does not need the substitution $y = Sz$ into $L(y) = 0$ (we have mentioned it in Section 1.1).

1.4 Examples

Example For a system $L(y) = 0$ of the form

$$\begin{pmatrix} x & 1 \\ 1 & 1 \end{pmatrix} y' + \begin{pmatrix} x^2 & x \\ 1 & x \end{pmatrix} y = 0$$

we get $I_{L,\infty}(\lambda)$ as a non-zero constant. The polynomial has no integer roots and the system has no rational solutions (there is no need to look for a universal denominator and so on). If we apply the usual approach, then we would have to find the universal denominator $U(x) = x$, make the substitution (1.5) into the original system, then a search should be made for polynomial solutions. Finally, it would show that there are no such solutions. □

Example If a system $L(y) = 0$ is of the form

$$\begin{pmatrix} 2 & 0 \\ 0 & x(x+1) \end{pmatrix} y' + \begin{pmatrix} -1 & 1 \\ x & 2(x+1) \end{pmatrix} y = 0$$

then $I_{L,\infty}(\lambda) = -\lambda - 3$. The only integer root is -3. We can find $U(x) = x^2$ as a universal denominator (or $S(x) = x^{-2}$ as a denominator bound). We see that $-3 + 2 = -1 < 0$. This implies that the system has no rational solutions (there is no need to produce the substitution $y = S(x)z$ and try to find polynomial solutions). □

1.5 Conclusion

The present paper shows that an approach similar to the proposed in [13, 14] can be applied not only to scalar equations, but to systems of equations as well. Small changes in the scheme of the algorithm allow one to mark the points that we call the checkpoints, and write down the corresponding control tests so that without increasing the cost of the algorithm as a whole, in some cases, performing the tests on the intermediate results makes it possible to stop the algorithm as soon as these tests imply that no solutions of the desired type exist.

Apparently, this approach may be useful in the development of algorithms for finding solutions that are more complicated than rational solutions (we would emphasize that the search for many types of solutions ultimately boils down to finding rational solutions for some auxiliary systems).

This type of problem can also be posed for the case of systems of linear difference equations. In the book [16] of P. Paule and M. Kauers, in particular, the basic tools for working with scalar difference equations are described. Regarding systems, it is possible, for example, to mention publications [3] – [7], [8], [11], [15]. The question of checkpoints and control tests for systems of linear difference equations remains a topic for future research.

Acknowledgements The author would like to thank M.Petkovšek, A.Ryabenko and D.Khmelnov for their valuable remarks to improve the paper. Supported by the Russian Foundation for Basic Research (RFBR), project no. 19-01-00032.

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