On Linear Dependence of Rows and Columns in Matrices over Non-commutative Domains

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ABSTRACT

Some well-known correspondences between sets of linearly independent rows and columns of matrices over fields carry over to matrices over non-commutative rings without nontrivial zero divisors.

CCS CONCEPTS

• **Computing methodologies → Symbolic and algebraic manipulation**; **Linear algebra algorithms**.

KEYWORDS

non-commutative domains, matrices over domains, linear dependence of rows (columns), maximum sets of linearly independent rows (columns), row and column ranks, quotient skew fields

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1 INTRODUCTION

Operations with matrices are widely used in basic and applied research. In carrying them out, it is important to bear in mind that their properties depend on the properties of the algebraic structure to which the matrix elements belong.

A *domain* in this paper is a ring, not necessarily commutative, which contains no nontrivial zero divisors. In the sequel, R always denotes a domain.

Definition 1. *Let A be a matrix over R. A set of rows $\{u_1, \dots, u_r\}$ of A is* linearly dependent over R *if there are $f_1, \dots, f_r \in R$, not all zero, such that $f_1 u_1 + \cdots + f_r u_r = 0$; otherwise, this set is* linearly independent *(over R). Linear dependence and independence of a set of columns of A are defined similarly, but in this case $f_1, \dots, f_r \in R$ are used as right factors. A* maximum set of linearly independent rows resp. columns of A *is a set of linearly independent rows resp. columns of A of largest possible size. The number of elements of such a set is called the* row rank *or the* left-hand rank*, resp. the* column rank *or the* right-hand rank*, of A.*

In order to determine the row resp. column rank of A, we need to test linear dependence of sets of rows resp. columns of A. Note however that in a general domain R, the familiar linear-algebraic methods for testing linear dependence of a set $S \subseteq R^k$ may not work. For instance, if R is a field, and a non-trivial linear combination of elements from S vanishes, one can represent at least one element from S as a linear combination of the others, but this need not be true for a general domain R. As a matter of fact, the question of embeddability of a domain into a skew field and, as an example, the question of existence of a quotient skew field, are non-trivial (see, e.g., [10, §13], [3], [4]).

Here we generalize some classical linear algebra results (cf. [5, Ch. 4, §3]) to matrices over a domain. The main results are stated in Sect. 3 as Proposition 1 and Theorem 1.

The paper starts with two motivating examples in Sect. 2. In Example 1, a domain R and a matrix A over R are given such that the row and column ranks of A do not coincide. Proposition 1 in Sect. 3 simplifies computation of linearly dependent columns (rows) of a matrix A when some maximum set of linearly independent rows (columns) of A is given. Based on Proposition 1, Theorem 1 in Sect. 4 asserts that the row rank of a matrix over R coincides with its column rank if and only if for all $k \in \mathbf{Z}_{>0}$, the rows of any matrix from $R^{(k+1)\times k}$, as well as the columns of any matrix from $R^{k\times(k+1)}$, are linearly dependent over R.

The point of Example 2 is to justify defining linear dependence of a set of columns of a matrix A in Definition 1 by using the multipliers $f_1, \dots, f_r$ as *right* factors rather than as left factors as we did for a set of rows of A. The domain R in this example is the ring of differential (ordinary or partial) operators over a differential field, for which Proposition 2 in Sect. 5 asserts that for every matrix over this domain, its row and column ranks coincide. However, for the matrix given in this example the ranks would not coincide if linear dependence of a set of columns of A were defined by using $f_1, \dots, f_r$ as left factors.

In Sect. 6.1, we prove the equality of the row and column ranks for the matrices over the domains of linear (q-)difference operators.

Note that equality of row and column ranks of matrices of partial differential operators was earlier proved in [7]. That proof is likewise based on a generalization of a property of matrices over a field. However, that property is somewhat more complicated than the one used in our Theorem 1.

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A sufficient condition for existence of a quotient skew field was proposed by O. Ore in [9]. This condition is the existence of a non-zero common multiple (a right multiple, say) of any non-zero $a, b \in R$. It is known that for matrices over a skew field, the row and column ranks are equal (see, e.g., [1]; such equality follows also from Theorem 1. In the present paper, we are interested in the ranks connected with the linear independence over the original domain R.

2 MOTIVATING EXAMPLES

Considering a matrix $A \in R^{m \times n}$ of the form

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}, \tag{1}$$

we will denote by $A_{i,*}$ the i-th row, $1 \le i \le m$, and by $A_{*,j}$ the j-th column, $1 \le j \le n$, of A.

We present two examples which show that row and column ranks of matrices over a domain R behave differently when R is a non-commutative domain than in the familiar case when R is a field. The first example shows that row and column ranks of matrices over a non-commutative domain need not coincide.

Example 1. Let R be the ring of polynomials in non-commuting variables x and y (hence $xy \neq yx$) over some field, e.g., the field of rational numbers $\mathbf{Q}$. Consider the matrix

$$A = \begin{bmatrix} x \\ y \end{bmatrix} \in R^{2 \times 1}.$$

Its rows $[x]$ and $[y]$ are linearly independent over R. So, the row rank of A is 2, while its column rank is 1.

Example 2 features a domain R such that, according to Proposition 2 in Sect. 5, the row and column ranks of every matrix over R coincide, but this would be false if linear dependence of a set of columns were defined alternatively using the multipliers f_i in Definition 1 as *left* rather than right factors.

Example 2. Let $K = \mathbf{Q}(x)$, and let R be the non-commutative ring $K[\frac{d}{dx}]$ of linear ordinary differential operators with coefficients in K, having the commutator $\left[\frac{d}{dx}, x\right] = \frac{d}{dx}x - x\frac{d}{dx} = 1$. If the rows $A_{1,*}, A_{2,*}$ of the matrix

$$A = \begin{bmatrix} x\frac{d}{dx} + 1 & x \\ \frac{d}{dx} & 1 \end{bmatrix}$$

satisfy $L_1 A_{1,*} + L_2 A_{2,*} = 0$ for some $L_1, L_2 \in R$, then

$$L_1 x \frac{d}{dx} + L_1 + L_2 \frac{d}{dx} = 0, \tag{2}$$

$$L_1 x + L_2 = 0, \tag{3}$$

hence $L_2 = -L_1 x$ by (3), so (2) can be rewritten as $L_1 x \frac{d}{dx} + L_1 - L_1 x \frac{d}{dx} = 0$, thus $L_1 = 0$. It follows that $L_2 = 0$, implying that the rows of A are linearly independent, and the row rank of A is 2.

Similarly, if the columns $A_{*,1}, A_{*,2}$ of A satisfy $A_{*,1}M_1 + A_{*,2}M_2 = 0$ for some $M_1, M_2 \in R$, then

$$\left(x\frac{d}{dx} + 1\right) M_1 + x M_2 = 0, \tag{4}$$

$$\frac{d}{dx} M_1 + M_2 = 0. \tag{5}$$

Multiplying (5) on the left by x and subtracting it from (4) we obtain $M_1 = 0$. From (5) we get now $M_2 = 0$, implying that the columns of A are linearly independent, and the column rank of A is 2 as well. In fact, as we will see in Proposition 2, the row and column ranks of any matrix over the ring $R = K[\frac{d}{dx}]$ are equal.

However, the columns $A_{*,1}, A_{*,2}$ of A satisfy $A_{*,1} = \frac{d}{dx} A_{*,2}$, so under the above-mentioned alternative definition of linear dependence for columns (using M_1 and M_2 as *left* multipliers) they would be linearly dependent, and the column rank of A would equal 1.

3 ROWS VS COLUMNS

Proposition 1. *Let $\{r_{i_1}, \dots, r_{i_\ell}\}$ be a maximum set of linearly independent rows of a matrix $A \in R^{m \times n}$, $m, n \in \mathbf{Z}_{>0}$. Let $B \in R^{\ell \times n}$ be the submatrix of A consisting of rows $r_{i_1}, \dots, r_{i_\ell}$. If $s_1, \dots, s_n \in R$ are such that*

$$B_{*,1}s_1 + \dots + B_{*,n}s_n = 0, \tag{6}$$

then

$$A_{*,1}s_1 + \dots + A_{*,n}s_n = 0 \tag{7}$$

as well, and an analogous statement holds if the roles of rows and columns are interchanged.

Proof. Let $i \in \{1, \dots, m\} \setminus \{i_1, \dots, i_\ell\}$. Since $\{r_{i_1}, \dots, r_{i_\ell}\}$ is a maximum set of linearly independent rows of A, the set $\{r_{i_1}, \dots, r_{i_\ell}, r_i\}$ is linearly dependent, hence there are $f_1, \dots, f_\ell, g_i \in R$, not all zero, such that

$$f_1 r_{i_1} + \dots + f_\ell r_{i_\ell} + g_i r_i = 0. \tag{8}$$

Write $f = [f_1, \dots, f_\ell]$ and $s = [s_1, \dots, s_n]$. Note that $g_i \neq 0$, or else the set $\{r_{i_1}, \dots, r_{i_\ell}\}$ would be linearly dependent. Multiplying (8) from the right by s^T yields $(f_1 r_{i_1} + \dots + f_\ell r_{i_\ell}) s^T + g_i r_i s^T = 0$, which can be rewritten as $f B s^T + g_i r_i s^T = 0$. Note that by (6), $Bs^T = 0$, hence $g_i r_i s^T = 0$ as well. Since $g_i \neq 0$ and there are no zero divisors in R, it follows that $r_i s^T = 0$. From $Bs^T = 0$ it also follows that $r_{i_j} s^T = 0$ for all $j \in \{1, \dots, \ell\}$. As $i \in \{1, \dots, m\} \setminus \{i_1, \dots, i_\ell\}$ was arbitrary, we conclude that $As^T = 0$, which is equivalent to (7). □

Remark 1. Proposition 1 establishes a one-to-one correspondence between the maximum sets of linearly independent columns of matrices A and B (the columns of A are linearly dependent if and only if their "segments", which are the columns of B, are linearly dependent). This fact is used below in the proof of Theorem 1. In practice, it can also be useful to save computations if searching for a maximum set of linearly independent columns of some matrix, when a maximum set of linearly independent rows is known. – All of this holds also if "columns" and "rows" are interchanged.

Example 3. To illustrate Proposition 1, let $A \in R^{3 \times 2}$ be the following matrix over the domain R considered in Example 1, i.e. over the ring of polynomials in two non-commuting variables x and y:

$$A = \begin{bmatrix} x + xy & xy + xy^2 \\ x & xy \\ y & y^2 \end{bmatrix}.$$

The rows of A are linearly dependent over R: $A_{1,*} - A_{2,*} - xA_{3,*} = 0$. However, the 2nd and the 3rd rows are independent: if $fx + gy = 0$ for some $f, g \in R$ then $f = g = 0$.

In the matrix B that consists of the 2nd and the 3rd rows of A:

$$B = \begin{bmatrix} x & xy \\ y & y^2 \end{bmatrix},$$

the columns are linearly dependent: $B_{*,1}y - B_{*,2} = 0$. It is easy to see that $A_{*,1}y - A_{*,2} = 0$ as well.

The columns of A are linearly dependent. A maximum set of linearly independent columns consists of the 2nd column. We set now

$$B = \begin{bmatrix} x + xy \\ x \\ y \end{bmatrix}.$$

The rows of B are linearly dependent: $B_{1,*} - B_{2,*} - xB_{3,*} = 0$, and we have $A_{1,*} - A_{2,*} - xA_{3,*} = 0$.

4 A CRITERION FOR THE EQUALITY OF ROW AND COLUMN RANKS OF AN ARBITRARY MATRIX OVER R

Proposition 1 proved above holds for any domain, in particular, it holds for the domain R in Example 1. However, as shown in Example 1, it is not true that the row and column ranks of every matrix A over this R are equal.

Theorem 1. *Let R be a domain. The row and column ranks of an arbitrary matrix A over R are equal if and only if for any positive integer k:*

(a) the rows of any matrix from $R^{(k+1)\times k}$ are linearly dependent over R, and

(b) the columns of any matrix from $R^{k\times(k+1)}$ are linearly dependent over R.

Proof. Let the row and column ranks over R of an arbitrary matrix A of the form (1) be equal. Then the column rank of any matrix in $R^{(k+1)\times k}$ does not exceed k and, as a consequence, its row rank does not exceed k either. Hence, the rows of the matrix under consideration are linearly dependent over R and property (a) is satisfied. A similar reasoning can be carried out for the columns of matrices from $R^{k\times(k+1)}$, which proves property (b).

Now let R have properties (a) and (b). By (a) and Remark 1, the number of elements in a maximum set of linearly independent columns of A, and hence its column rank, does not exceed its row rank. Changing mutually in this reasoning all references to rows, row ranks and property (a) with references to columns, column ranks and property (b), we find that the row rank of A does not exceed its column rank. As a consequence, the row and column ranks of A are equal. □

Some examples of the domains for which the criterion holds will be given in Sections 5, 6.1.

Remark 2. If the domain R satisfies the hypotheses of Theorem 1, then for any $p, q \in R \setminus \{0\}$ there are non-zero left and right common multiples. To prove this, it suffices to consider matrices

$$\begin{bmatrix} p & q \end{bmatrix}, \quad \begin{bmatrix} p \\ q \end{bmatrix},$$

the row and column ranks of each are 1. (It follows from [9] that in this case for R there are skew fields of left and right quotients, i.e. skew fields of fractions of the form $q^{-1}p$ and, respectively, pq^{-1}.)

5 A SPECIAL CASE: MATRICES OVER THE RING OF LINEAR DIFFERENTIAL OPERATORS

Let $s, t \in \mathbf{Z}_{>0}$, and let $x_1, \ldots, x_s$ be variables. It is known that the number of differential monomials of the form

$$\frac{\partial^v}{\partial^{v_1} x_1 \cdots \partial^{v_s} x_s}, \tag{9}$$

where $v_1, \ldots, v_s \in \mathbf{Z}_{\geq 0}$ and $v_1 + \cdots + v_s = v \leq t$ is equal to

$$\binom{s+t}{t}. \tag{10}$$

Partition problems of this kind are considered, for example, in [6, Sect. 7.2.1.4]. We will refer to the value of v as the *order* of monomial (9).

Let R be the domain of linear differential operators having the form of linear combinations of monomials of the form (9), with coefficients in some fixed differential field K with derivations $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s}$. Let $D_{s,t} \subset R$ be the set of all those operators whose monomials are of the form (9) with $v_1 + \cdots + v_s = v \leq t$.

Proposition 2. *For any matrix A over the ring of differential operators R, its row and column ranks are equal.*

Proof. We show that hypotheses (a) and (b) of Theorem 1 are satisfied.

For (a), assume that $A \in R^{(k+1)\times k}$ with $k > 0$. Let $s \in \mathbf{Z}_{>0}$, $t \in \mathbf{Z}_{\geq 0}$ be such that $A \in D_{s,t}^{(k+1)\times k}$, and let $d \in \mathbf{Z}_{\geq 0}$ satisfy

$$k\binom{s+t+d}{t+d} < (k+1)\binom{s+d}{d}. \tag{11}$$

Note that (11) holds for all large enough $d \in \mathbf{Z}$ because

$$\frac{\binom{s+t+d}{t+d}}{\binom{s+d}{d}} = \frac{(s+t+d)!d!}{(s+d)!(t+d)!} =$$
$$= \frac{(d+s+1)(d+s+2)\ldots(d+s+t)}{(d+1)(d+2)\ldots(d+t)}$$

is a quotient of two monic polynomials in d of degree t, hence

$$\lim_{d\to\infty} \frac{\binom{s+t+d}{t+d}}{\binom{s+d}{d}} = 1 < \frac{k+1}{k}.$$

Write $A = [a_{ij}]_{1\leq i\leq k+1, 1\leq j\leq k}$ where

$$a_{ij} = \sum_{\substack{0\leq v\leq t \\ v_1+\cdots+v_s=v}} \alpha_{i,j,v_1,\ldots,v_s} \frac{\partial^v}{\partial^{v_1} x_1 \cdots \partial^{v_s} x_s}$$

for some $\alpha_{i,j,v_1,\ldots,v_s} \in K$, and let

$$L_i \;=\; \sum_{\substack{0 \le v \le d \\ v_1+\cdots+v_s=v}} c_{i,v_1,\ldots,v_s} \frac{\partial^v}{\partial^{v_1} x_1 \cdots \partial^{v_s} x_s}$$

for $i = 1,\ldots,k+1$ be operators with undetermined coefficients $c_{i,v_1,\ldots,v_s} \in K$. The vector operator equation

$$\sum_{i=1}^{k+1} L_i A_{i,*} \;=\; 0 \tag{12}$$

is equivalent to the system of operator equations

$$\sum_{i=1}^{k+1} L_i a_{i,j} \;=\; 0, \quad j = 1,\ldots,k, \tag{13}$$

which, in turn, is equivalent to the system of scalar equations, equating the coefficient of each of the differential monomials $\frac{\partial^\mu}{\partial^{\mu_1} x_1 \cdots \partial^{\mu_s} x_s}$ for $0 \le \mu \le d+t$, $\mu_1+\cdots+\mu_s = \mu$, that appear in the left-hand sides of (13) when the operators $L_i a_{i,j}$ are written in standard form, to zero. According to (10), this is a system of $k\binom{s+t+d}{t+d}$ homogeneous linear algebraic equations for the $(k+1)\binom{s+d}{d}$ undetermined coefficients $c_{i,v_1,\ldots,v_s}$ with $1 \le i \le k+1, v_1+\cdots+v_s = v \le d$. By (11), the number of unknowns exceeds the number of equations, hence this system has a non-zero solution. It follows that not all L_i in (12) are zero, so the rows of A are linearly dependent over R, proving hypothesis (a) of Theorem 1.

The fact that hypothesis (a) of Theorem 1 is satisfied for any matrix from $R^{(k+1)\times k}$ allows us to prove that for any matrix from $R^{k\times(k+1)}$, hypothesis (b) of Theorem 1 is satisfied as well. To this end, we use adjoint matrices of operators. If $A \in D_{s,t}^{k\times(k+1)}$, then $A^* \in D_{s,t}^{(k+1)\times k}$. The linear dependence of the rows of the matrix A^* indicates the existence of a vector $c \in R^{1\times(k+1)}$ such that $cA^* = 0, c \ne 0$, and, as a consequence, $(A^*)^* c^* = Ac^* = 0, c^* \in R^{(k+1)\times 1}$, and $c^* \ne 0$, implying that the columns of the matrix A are linearly dependent.

Since hypotheses (a) and (b) of Theorem 1 are satisfied for this R, it follows from Theorem 1 that Proposition 2 is valid. □

As a particular case ($s = 1$ in (9)), Proposition 2 also covers matrices of ordinary differential operators. This case has been discussed several times in the literature. For example, in [11] the equality of the row and column ranks of such matrices was proven using the fact that the ring of ordinary differential operators is Euclidean. In [4, Sect. 8.1, Theorem 1.1] a more general case is considered when R is a principal ideal ring (each Euclidean ring is a principal ideal ring).

Our proof of Proposition 2 uses the method of undetermined coefficients, as did the earlier proof from [7, pp. 12–15] which was based on an analogue of another fact of classical linear algebra: the equality of the maximum number of linearly independent rows canceling the given matrix by left multiplication, and the maximum number of linearly independent columns canceling the given matrix by right multiplication, which is somewhat more cumbersome than checking the criterion given in Theorem 1.

6 RELATED TOPICS

6.1 Linear difference and q-difference operators

The results from Sec. 5 on matrices over the ring R of differential operators carry over to the case when R is a ring of difference or q-difference operators. We will denote by E_i the shift operator with respect to variable x_i: $E_i x_i = x_i + 1$, and by Q_i the q-shift operator with respect to x_i: $Q_i x_i = q x_i$. The corresponding analogues of monomials of the form (9) are the products

$$E_1^{v_1} \cdots E_s^{v_s} \tag{14}$$

resp.

$$Q_1^{v_1} \cdots Q_s^{v_s}. \tag{15}$$

Denoting $D_i = \frac{\partial}{\partial x_i}$, we can rewrite the monomial in (9) as

$$D_1^{v_1} \cdots D_s^{v_s}, \tag{16}$$

where $v_1 \ldots, v_s \in \mathbf{Z}_{\ge 0}$. The sum $v = v_1 + \cdots + v_s$ is the *order* of each of (14), (15), and (16).

Let R be the domain of linear operators having the form of linear combinations of monomials of one of the forms (14), (15) or (16), with coefficients in some fixed differential or (q-)difference field K with either derivations $D_1,\ldots,D_s$, or automorphisms $E_1,\ldots,E_s$, or automorphisms $Q_1,\ldots,Q_s$. Note that formula (10) holds also for (q-)difference monomials, and that the second part of the proof of Proposition 2 about matrices from $R^{k\times(k+1)}$, where adjoint matrices and operators are used, can easily be converted into analogous proofs valid in the difference and q-difference cases, with[1]

$$D_i^* = -D_i, \;\; E_i^* = E_i^{-1}, \;\; Q_i^* = Q_i^{-1}$$

for $i = 1,\ldots,s$. For an operator

$$L = \Sigma_{v_1,\ldots,v_s} a_{v_1,\ldots,v_s} Z_1^{v_1} \cdots Z_s^{v_s},$$

where $Z \in \{D, E, Q\}$, we set

$$L^* = \Sigma_{v_1,\ldots,v_s} (Z_1^*)^{v_1} \cdots (Z_s^*)^{v_s} a_{v_1,\ldots,v_s}.$$

For matrices A, B over domains of any of the three types of operators, we have $A^{**} = A$, $(AB)^* = B^* A^*$, where the adjoint matrix is obtained by transposing the original matrix and then replacing its elements with adjoint ones. Thus, both parts of the proof of Proposition 2 carry over to the difference and q-difference cases, hence Proposition 2 holds in these cases as well.

6.2 Quotient skew field of R

It follows from [2] and [1] that if R can be embedded into some skew field F and if the coefficients of linear dependencies of rows (columns) have coefficients in F (we can talk about *dependencies over F*), then for any matrix over R its row and column ranks are the same. This follows also from Theorem 1, the hypothesis of which is obviously satisfied when R is a skew field.

Remark 3. (Remark 2 continued.) For the special case of Ore algebra (an algebra of skew polynomials in several indeterminates), F. Chyzak proposed a special version of the fraction free Euclidean algorithm, — this is command `annihilators` of Maple package `Ore_algebra` [8]. The extended

[1] In a univariate Ore polynomial ring equipped with automorphism σ and derivation δ, we have $\delta^* = -\delta\sigma^{-1}$, $\sigma^* = \sigma^{-1}$ (see, for example, [4]).

version of the Euclidean algorithm allows to compute some non-zero right and left common multiplies for non-zero elements.

In conclusion, we emphasize once again that in our present paper, we are interested in the ranks connected to linear independence over the original domain R.

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