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Linear Differential and Difference Systems: EG_δ- and EG_σ- Eliminations

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Abstract—Systems of linear ordinary differential and difference equations of the form $A_r(x)\xi^r y(x) + \ldots + A_1(x)\xi y(x) + A_0(x)y(x) = 0$, $\xi \in \left\{\frac{d}{dx}, E\right\}$, where E is the shift operator, $Ey(x) = y(x + 1)$, are considered. The coefficients $A_i(x)$, $i = 0, \ldots, r$, are square matrices of order m, and their entries are polynomials in x over a number field K, with $A_r(x)$ and $A_0(x)$ being nonzero matrices. The equations are assumed to be independent over $K[x, \xi]$. For any system S of this form, algorithms EG_δ (in the differential case) and EG_σ (in the difference case) construct, in particular, the *l-embracing* system $\bar{S}$ of the same form. The determinant of the leading matrix $\bar{A}_r(x)$ of this system is a nonzero polynomial, and the set of solutions of system $\bar{S}$ contains all solutions of system S. (Algorithm EG_σ provides also a number of additional possibilities.) Examples of problems that can be solved with the help of EG_δ and EG_σ are given. The package EG implementing the proposed algorithms in Maple is described.

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1. INTRODUCTION

Let K be a number field: $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$. The ring of polynomials and the field of rational functions of x are conventionally denoted as $K[x]$ and $K(x)$, respectively. The ring of formal power series of x over K is denoted as $K[[x]]$, and the field of formal Laurent series, as $K((x))$. If R is a ring (in particular, field), then $\mathrm{Mat}_m(R)$ denotes the ring of square matrices of order m with entries from R.

We consider systems of the form

$$A_r(x)\xi^r y(x) + \ldots + A_1(x)\xi y(x) + A_0(x)y(x) = 0, \quad (1)$$

where $\xi \in \left\{\frac{d}{dx}, E\right\}$ and E is the shift operator: $Ey(x) = y(x + 1)$. Coefficients $A_i(x)$, $i = 0, \ldots, r$ are square matrices of order m with entries from $K[x]$: $A_0(x)$, $A_1(x)$, ..., $A_r(x) \in \mathrm{Mat}_m(K[x])$, with $A_r(x)$ and $A_0(x)$ being nonzero *leading* and *trailing* matrices, and $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T$ is a column of unknown functions (T denotes transposition). The number r is called the *order* of the system.

Denoting the ith row of matrix $A_j(x)$ as $A_j^{(i)}(x)$, we can write equations in system (1) in the form

$$A_r^{(i)}(x)\xi^r y(x) + \ldots + A_1^{(i)}(x)\xi y(x) + A_0^{(i)}(x)y(x) = 0,$$

$i = 1, 2, \ldots, m$. Unless stated otherwise, in what follows, these equations are assumed to be independent over $K[x, \xi]$ (in other words, system (1) has *full rank*): let $U_1 = 0$, $U_2 = 0$, ..., $U_m = 0$ be equations composing system (1) and $L_1, L_2, \ldots, L_m \in K[x, \xi]$, then $L_1(U_1) + L_2(U_2) + \ldots + L_m(U_m) = 0$ is the equation $0 = 0$ if and only if $L_1 = L_2 = \ldots L_m = 0$.

In the general case, the leading matrix is not invertible (singular) in $\mathrm{Mat}_m(K(x))$, which generates certain difficulties in calculations. If $m = 1$ (scalar equation), polynomial $A_r(x)$ vanishes on a finite set of values of x, and these values are of special interest when studying and solving the equation. If $m > 1$, the similar role is played by values of x for which the determinant of matrix $A_r(x)$ vanishes (assuming that the determinant is not equal to zero identically). Algorithms EG_δ and EG_σ make it possible to avoid difficulties of this kind. For any system S of form (1), algorithms EG_δ (in the differential case) and EG_σ (in the difference case) construct an *l-embracing* system $\bar{S}$ of the same form (1) with a nonsingular leading matrix $\bar{A}_r(x)$. In this case, the set of solutions of system $\bar{S}$ contains all solutions of system S. (Notation δ and σ for the mappings possessing differentiation and shift properties, respectively, is used in the theory of the Ore polynomials [46].)

The difference case turns out even more compliant, which is explained by the fact that, for E, there exists a uniquely determined inverse transform $E^{-1}y(x) = y(x - 1)$. For a difference system S of form (1), algorithm EG_σ constructs also a *t-embracing* system $\bar{\bar{S}}$ of form (1) with a nonsingular trailing matrix $\bar{\bar{A}}_0$, with the set of solutions of system $\bar{\bar{S}}$ containing all solutions of system S. Besides, algorithm EG_σ finds a finite set of *linear constraints*, i.e., linear relations with constant coefficients for a finite set of values $y_i(\alpha + j)$, $\alpha \in \bar{K}$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, r$ (α is fixed for any separate linear constraint and $\bar{K}$ denotes the algebraic closure of field K). The application of EG_σ results in a system $(\tilde{S}, C)$ that is equivalent to S, with the leading and trailing matrices of system $\tilde{S}$ being nonsingular.

It is well known that a system of form (1) can be reduced to a first-order system $M_1(x)\xi Y(x) + M_0(x)Y(x) = 0$, where $M_0(x), M_1(x) \in \mathrm{Mat}_{rm}(K(x))$ and

$$Y(x) = (y(x)^T, (\xi y(x))^T, \ldots, (\xi^{r-1}y(x))^T)^T.$$

This system can be rewritten as two systems. The first system is

$$B_1(x)\xi\tilde{Y}(x) + B_0(x)\tilde{Y}(x) = 0,$$

where $B_0(x), B_1(x) \in \mathrm{Mat}_s(K(x))$, $s \le rm$, matrix $B_1(x)$ is nonsingular, and vector $\tilde{Y}(x)$ contains s unknown functions from $Y(x)$. The second system is a linear algebraic system that allows us to express all components of vector $Y(x)$ that are not contained in $\tilde{Y}(x)$ in terms of the components of $\tilde{Y}(x)$ (see, for example, [4, Section 2.3; 48] for the differential case and [5, Section 5] for the difference case). When $\xi = E$ (i.e., in the difference case), it is also possible to get a nonsingular matrix B_0. If m is fixed and r increases, then complexity of EG_δ and EG_δ is $O(r^2)$. It is shown in [4, 5] that, in this case, complexity of the transformation described grows faster than r^3. This transformation to the first-order system is not considered in the paper.

In this paper, we have collected and discussed from a unified point of view results published in [4, 5, 10, 12, 15, 16, 18, 23, 24]. In Sections 2 and 3, algorithms EG_δ and EG_σ for constructing embracing systems are described. The very fact of existence of the embracing systems simplifies the proof of a number of important properties of spaces of solutions of full-rank linear and differential systems. This is discussed in Section 4. Section 5 introduces the concept of an induced recurrent system. Such systems govern coefficients of expansions (in appropriate bases) of solutions to original systems. We show how to construct the indicial equation for an original system by means of the induced recurrent system and algorithms EG_δ and EG_σ. Section 6 is devoted to meromorphic solutions of difference systems. In Section 7, we give examples of algorithms for searching solutions of different kinds; these algorithms use the induced recurrent systems. In Section 8, it is shown that algorithms EG_δ and EG_σ can be applied to nonhomogeneous systems and that, if the system has no full rank, EG_δ and EG_σ allow us to find the rank of the system and to reduce it to a convenient form. Additionally, the case of q-difference systems is briefly discussed. In Section 9, randomization and certain heuristics for EG_δ and EG_σ are suggested. In Section 10, some other approaches to solving the problems considered in the paper are briefly surveyed. Finally, in Section 11, packages LinearFunctionalSystems and EG, which implement the algorithms discussed in the computer algebra system Maple [49], are described.

2. LINEAR DIFFERENTIAL SYSTEMS

In the differential case, the original system S has the form

$$A_r(x)y^{(r)}(x) + \ldots + A_1(x)y'(x) + A_0(x)y(x) = 0. \quad (2)$$

We will show how, for a system S of form (2), to construct an l-embracing system $\bar{S}$ of the form

$$\bar{A}_r(x)y^{(r)}(x) + \ldots + \bar{A}_1(x)y'(x) + \bar{A}_0(x)y(x) = 0, \quad (3)$$

with a nonsingular leading matrix, such that the set of solutions of this system contains all solutions of system S. The matrix $\bar{A}_0$ is allowed to be zero.

Let the ith row of matrix $A_s(x)$, $0 \le s \le r$, be nonzero and the ith rows of matrices $A_{s-1}(x), A_{s-2}(x), \ldots, A_0(x)$ be zero. Let the t-th entry, $1 \le t \le m$, be the last nonzero entry of the i-th row of $A_s(x)$. Then, the number $(r - s)m + t$ is called the *length* of the ith equation of the system, and the entry of matrix $A_s(x)$ with indices i, t is called the *last nonzero coefficient* of the ith equation of the system.

2.1. EG_δ: Reduction

Algorithm EG_δ is based on alternation of reductions and shifts. Let us explain how the reduction works.

It is checked whether rows of the leading matrix are linearly dependent over $K(x)$. If they are, coefficients $v_1(x), v_2(x), \ldots, v_m(x) \in K[x]$ of the dependence are found. From the equations of the system corresponding to nonzero coefficients, we select the equation of the greatest length. Let it be the ith equation. This equation is replaced with the linear combination of the equations with the coefficients $v_1(x), v_2(x), \ldots, v_m(x)$. As a result, the ith row of the leading matrix vanishes. This step is called *reduction*. It is important that the reduction does not increase lengths of the equations.

2.2. EG_δ: Differential Shift of the Equation with Zero Leading Part

Let the ith row of the leading matrix be zero, and let $a(x)$ be the last nonzero coefficient of the ith equation. Let us divide this equation by $a(x)$, differentiate it, and clear the denominators. This operation is called *differential shift* of the ith equation of the system. The word "shift" indicates that, owing to the performed division by the last nonzero coefficient, this operation reduces the length of the ith equation in the system ([4]).

2.3. EG_δ: Sequence of Steps "Reduction + Differential Shift"

The scheme of algorithm EG_δ is as follows. If rows of the leading matrix are linearly dependent over $K(x)$, then the reduction is performed. Suppose that this makes the ith row of the leading matrix zero. Then, we perform the differential shift of the ith row and continue the process of alternated reductions and differential shifts until the leading matrix becomes nonsingular. (Note that we will never obtain equation $0 = 0$, since, by the assumption, the equations of the original system are independent over $K\left[x, \frac{d}{dx}\right]$.)

Theorem 1 ([4]). *Algorithm* EG_δ *always terminates.*

Thus, for a differential system S of form (2) algorithm EG_δ constructs an l-embracing system $\bar{S}$, with the leading matrix $\bar{A}_r(x)$ being nonsingular and the set of solutions of system S being a subset of the set of solutions of system $\bar{S}$.

Example 1. Consider the system

$$\begin{pmatrix} 2x^2(x+2)(x+1) & -x(x+2)(x+1) \\ 2x^2(x+2) & -x(x+2) \end{pmatrix} y'' + \begin{pmatrix} 2x(x+1)(x-4) & -x^2 \\ 2x(x-4) & -x(x+4) \end{pmatrix} y' + \begin{pmatrix} -2(x+1)(x-4) & -2 \\ -2x+8 & 2 \end{pmatrix} y = 0. \quad (4)$$

Rows of the leading matrix are dependent with the coefficients $v_1(x) = -1$ and $v_2(x) = x + 1$. The equations are of the same length. Let us replace the second equation:

$$\begin{pmatrix} 2x^2(x+2)(x+1) & -x(x+2)(x+1) \\ 0 & 0 \end{pmatrix} y'' + \begin{pmatrix} 2x(x+1)(x-4) & -x^2 \\ 0 & -x(x+2)^2 \end{pmatrix} y' + \begin{pmatrix} -2(x+1)(x-4) & -2 \\ 0 & 2x+4 \end{pmatrix} y = 0.$$

The differential shift of the second equation yields:

$$\begin{pmatrix} 2x^2(x+2)(x+1) & -x(x+2)(x+1) \\ 0 & -x(x+2) \end{pmatrix} y'' + \begin{pmatrix} 2x(x+1)(x-4) & -x^2 \\ 0 & -2x \end{pmatrix} y' + \begin{pmatrix} -2(x+1)(x-4) & -2 \\ 0 & 0 \end{pmatrix} y = 0. \quad (5)$$

This system has the nonsingular leading matrix and is the result of application of EG_δ to system (4). It is an l-embracing system of the original system.

Remark 1. If, after a reduction, the ith rows of matrices $A_r(x)$, $A_{r-1}(x)$, ..., $A_{u+1}(x)$ are zero, the ith row of matrix $A_u(x)$ is nonzero, and $u < r - 1$, then the equation corresponding to the ith row can be differentiated $r - u - 1$ times without dividing it by the last nonzero coefficient, and only the last, $(r - u)$-th, differentiation needs preliminary division followed by the elimination of the denominators (this idea was put forward by M. Barkatou). Note also that the length of the equation replaced by a linear combination of other equations can be reduced after the reduction. In this case, is not necessary to divide the equation by the last nonzero coefficient before the differentiation.

Remark 2. Theoretically, algorithm EG_δ can be applied to differential systems with arbitrary analytical coefficients. However, in this case, it will be required to recognize whether the entries of the leading matrix are equal to zero, which is not possible in the general case. At the same time, the above reasoning proves *existence* of an l-embracing system for the general differential full-rank system with analytical coefficients.

2.4. On Linear Dependence on the Reduction Step

Search for coefficients $v_1(x)$, $v_2(x)$, ..., $v_m(x)$ of the linear dependence (Section 2.1) is equivalent to solving a homogeneous system of linear algebraic equations with polynomial coefficients. This problem is efficiently solved by a number of algorithms, in particular, by modular ones, which successfully cope with the coefficient growth in intermediate calculations (see, e.g., [15, Section 6; 16, Section 4.2; 44; 45]. If s independent solutions of this system are found, then it is possible to obtain s zero rows in the leading matrix. These s solutions are first written as rows of matrix V of dimension $s \times m$. The first row of matrix V is used for nullifying the ith row of the leading matrix and differential shift of the ith equation. Then, by means of the first row, the ith entries in the rows with numbers from 2 to s are eliminated in matrix V. As a result, all rows of matrix V, starting from the second row, contain coefficients of the linear dependences of rows of the leading matrix with numbers $1, \ldots, i-1, i+1, \ldots, m$. Continuing this process, we can perform s steps "reduction + differential shift." Heuristic strategies of selection of rows in matrix V can also be applied (see Section 9.3).

2.5. EG_δ: Complexity

Suppose that complexity of the reductions of a leading matrix of order m is $O(m^\omega)$, $2 < \omega \le 3$, in terms of the number of operations over elements of ring $K[x]$. The number of steps "reduction + differential shift" of algorithm EG_δ does not exceed rm^2, and the cost of each step is estimated as $O(rm^2 + m^\omega)$. Then, complexity of EG_δ in terms of the number of operations in $K[x]$ is $O(r^2m^4 + rm^{\omega+2})$.

It seems likely that the estimate rm^2 of the number of steps of algorithm EG_δ is too high (which is justified by results of experiments). Anyway, if every differential shift results in the appearance of an additional solution, then the number of steps cannot exceed rm (see Section 4.1 below), and, in this case, the total asymptotic estimate of complexity is $O(r^2m^3 + rm^{\omega+1})$.

3. LINEAR DIFFERENCE SYSTEMS

In the difference case, system S has the form

$$\begin{aligned} A_r(x)y(x+r) + \ldots + A_1(x)y(x+1) \\ + A_0(x)y(x) = 0. \end{aligned} \tag{6}$$

We will discuss how to construct an *l-embracing* system

$$\begin{aligned} \bar{A}_r(x)y(x+r) + \ldots + \bar{A}_1(x)y(x+1) \\ + \bar{A}_0(x)y(x) = 0 \end{aligned} \tag{7}$$

for this system such that its trailing matrix is invertible and the set of solutions contains all solutions of system S. Similarly, one can construct a t-embracing system $\bar{\bar{S}}$ given by

$$\begin{aligned} \bar{\bar{A}}_r(x)y(x+r) + \ldots + \bar{\bar{A}}_1(x)y(x+1) \\ + \bar{\bar{A}}_0(x)y(x) = 0 \end{aligned} \tag{8}$$

with a nonsingular trailing matrix and the set of solutions containing all solutions of system S. The case where matrices $\bar{A}_0(x)$ and $\bar{\bar{A}}_r(x)$ are equal to zero (either one of them or both) is not excluded.

The concept of the equation length introduced for the differential case is naturally extended to the difference case as well.

3.1. EG_σ: Reduction

The reduction for the leading matrix coincides with that for the differential case described in Section 2.1. However, in the difference case, we also construct the so-called linear constraints corresponding to the roots of polynomial $v_i(x)$ (see Section 3.4 for detail).

3.2. EG_σ: Shift of the Row with the Zero Leading Part

Suppose that the ith row of the leading matrix consists completely of zeros. Then, the shift operator E is applied to the ith equation of the system.

3.3. EG_σ: Sequence of Steps "Reduction + Shift"

The scheme of algorithm EG_σ is similar to that for algorithm EG_δ presented in Section 2.3. However, instead of the differential shifts, the shifts described in Section 3.2 are used. Perform the sequence of steps "reduction + shift" until the leading matrix becomes nonsingular (we will never obtain equation $0 = 0$, since the equations of the original system are independent over $K[x, E]$). The discussion in Section 2.4 is valid for the difference case as well.

Theorem 2 ([12]). *Algorithm* EG_σ *always terminates.*

Thus, for a difference system S of form (6) algorithm EG_σ constructs an l-embracing system $\bar{S}$, with the leading matrix $\bar{A}_r$ of system $\bar{S}$ being nonsingular and the set of solutions of system S being a subset of the set of solutions of system $\bar{S}$.

Algorithm EG_σ can also be used for constructing a t-embracing system $\bar{\bar{S}}$ with a nonsingular trailing matrix and the set of solutions that is a subset of the set of solutions of system $\bar{\bar{S}}$. In this case, the reduction is applied to the trailing matrix $A_0(x)$, and the shift is the application of E^{-1} to the ith equation of the system. The definition of the equation length is changed accordingly. Let the ith row of matrix $A_s(x)$, $0 \le s \le r$, be nonzero and the ith rows of matrices $A_r(x), A_{r-1}(x), \ldots, A_{s+1}(x)$ be zero. Let the lth entry, $1 \le l \le m$, be the first nonzero entry of the ith row of $A_s(x)$. Then, the number $m(s+1) - l + 1$ is called the *length* of the ith equation of the system.

The discussion in Remark 2 can accordingly be extended to the difference case.

3.4. Linear Constraints

If, at some reduction step, the ith equation of the system is replaced with the combination of the equations with the coefficients $v_1(x), v_2(x), \ldots, v_m(x)$, then the linear constraints arise by virtue of the fact that $v_i(x)$ vanish at some values of x. Each value α of this kind is substituted for x into the ith equation before it is replaced. Such linear constraint has the form of a linear relation for

$$y_i(\alpha + j), \quad i = 1, 2, \ldots, m, \quad j = 0, 1, \ldots, r,$$

with constant coefficients.

Let S and $\bar{S}$ be difference systems of form (1) and C and $\tilde{C}$ be finite sets of linear constraints. Systems (S, C) and $(\tilde{S}, \tilde{C})$ are said to be equivalent if the set of solutions of system S satisfying C coincides with the set of solutions of system $\tilde{S}$ satisfying $\tilde{C}$. Both l- and t-variants of algorithm EG_σ construct system (S', C) that is equivalent to $(S, \varnothing)$, such that its leading or, accordingly, trailing matrix is nonsingular.

Example 2. System S

$$\begin{pmatrix} x-1 & 0 \\ -2 & 0 \end{pmatrix} y(x+2) + \begin{pmatrix} 0 & 0 \\ 0 & x-2 \end{pmatrix} y(x+1)$$

$$+\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} y(x) = 0$$

is equivalent to the system

$$\begin{pmatrix} 0 & x(x-1) \\ -2 & 0 \end{pmatrix} y(x+2) + \begin{pmatrix} 0 & -2 \\ 0 & x-2 \end{pmatrix} y(x+1)$$

$$+\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} y(x) = 0$$

with the empty set of linear constraints.

System S is also equivalent to the system

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} y(x+2) + \begin{pmatrix} (x-4)(x-2) & 0 \\ -2 & 0 \end{pmatrix} y(x+1)$$

$$+\begin{pmatrix} -2 & 0 \\ 0 & x-3 \end{pmatrix} y(x) = 0$$

with the linear constraint $2y_1(5) - y_2(3) = 0$.

3.5. Double-Sided Embracing

Let the l- and t-embracing systems $\bar{S}$ and $\bar{\bar{S}}$ of an original system have forms (7) and (8), respectively. Any solution of the original system will also be a solution to the system

$$\begin{aligned} &\bar{A}_r(x+1)y(x+r+1) + (\bar{A}_{r-1}(x+1) \\ &+ \bar{\bar{A}}_r(x))y(x+r) + \ldots + (\bar{A}_0(x+1) \\ &+ \bar{\bar{A}}_1(x))y(x+1) + \bar{\bar{A}}_0(x)y(x) = 0 \end{aligned} \qquad (9)$$

with the nonsingular leading and trailing matrices. If $\bar{C}$ and $\bar{\bar{C}}$ are sets of linear constraints for $\bar{S}$ and $\bar{\bar{S}}$, then $\bar{C} \cup \bar{\bar{C}}$ is a set of linear constraints for (9). However, it is not excluded that, even with regard to these linear constraints, system (9) may have solutions that are not solutions to the original system (note that the order of the system has been increased).

3.6. EG_σ: Complexity

As in the case of EG_δ, we assume that complexity of the reduction of the leading matrix of order m in terms of the number of operations in $K[x]$ is $O(m^\omega)$, $2 < \omega \le 3$. The number of steps "reduction + shift" of algorithm EG_σ does not exceed rm (unlike in the differential case, each shift here is by the entire row of the matrix). Complexity of each step is $O(rm^2 + m^\omega)$. This brings us to the asymptotic estimate $O(r^2m^3 + rm^{\omega+1})$ of the EG_σ complexity in terms of the number of operations in $K[x]$.

4. POSSIBILITY TO PROVE PROPERTIES OF SYSTEM SOLUTIONS

The very fact of existence of the embracing systems makes it possible to prove certain important properties of solutions to systems of form (1). To consider examples of such properties (some of them might have already been known), we need some definitions.

For a nonzero element $a(x) = \sum a_i x^i$ from $K((x))$, its *valuation* $\mathrm{val}_x a(x)$ is defined by the equality

$$\mathrm{val}_x a(x) = \min\{i: a_i \neq 0\}. \qquad (10)$$

Additionally, $\mathrm{val}_x 0 = \infty$. In a similar way, we can define $\mathrm{val}_{x-\alpha}\psi(x)$ for a meromorphic function $\psi(x)$ and $\alpha \in \mathbb{C}$.

The notation $f(x) \perp g(x)$ denotes that polynomials $f(x), g(x) \in K[x]$ are relatively prime; if $F(x) \in K(x)$, then $\mathrm{den}F(x)$ is a monic (with the leading coefficient equal to 1) polynomial such that $F(x) = \frac{f(x)}{\mathrm{den}F(x)}$ for some $f(x) \in K[x]$, $f(x) \perp \mathrm{den}F(x)$. The set of monic irreducible polynomials from $K[x]$ is denoted as $\mathrm{Irr}(K[x])$. If $p(x) \in \mathrm{Irr}(K[x])$, $f(x) \in K[x]$, then $\mathrm{val}_{p(x)} f(x)$ is defined to be the maximum $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ such that $p^n(x) | f(x)$ ($\mathrm{val}_{p(x)} 0 = \infty$) and

$$\mathrm{val}_{p(x)} F(x) = \mathrm{val}_{p(x)} f(x) - \mathrm{val}_{p(x)} g(x) \qquad (11)$$

for $F(x) = \frac{f(x)}{g(x)}$, $f(x), g(x) \in K[x]$. The expansion of $F(x)$ into a formal Laurent series and the use of (10) yields the same value of valuation as (11) for $p(x) = x$.

For two arbitrary nonzero rational functions $R(x)$, $S(x)$ and $p(x) \in \mathrm{Irr}(K[x])$, the relations similar to those for the valuations of the Laurent series hold:

$$\mathrm{val}_{p(x)}(R(x)S(x)) = \mathrm{val}_{p(x)} R(x) + \mathrm{val}_{p(x)} S(x),$$

$$\mathrm{val}_{p(x)}(R(x) + S(x)) \ge \min\{\mathrm{val}_{p(x)} R(x), \mathrm{val}_{p(x)} S(x)\}.$$

If $F(x) = (F_1(x), F_2(x), \ldots, F_m(x))^T \in K(x)^m$, then $\mathrm{den}F(x) = \mathrm{lcm}_{i=1}^m \mathrm{den}F_i(x)$ and $\mathrm{val}_{p(x)} F(x) = \min_{i=1}^m \mathrm{val}_{p(x)} F_i(x)$, where lcm denotes the least common multiple of the polynomials.

For $A(x) = (a_{ij}(x)) \in \mathrm{Mat}_m(K(x))$, we set $\mathrm{den}A(x) = \mathrm{lcm}_{i=1}^m \mathrm{lcm}_{j=1}^m \mathrm{den}a_{ij}(x)$ and $\mathrm{val}_{p(x)} A(x) = \min_{i,j} \mathrm{val}_{p(x)} a_{ij}(x)$.

4.1. Dimension of the Space of Solutions of Differential Systems

As it is known (see, for example, [9, Chapter 3, Section 7]), if a matrix of order m is defined and analytical in a one-connected domain D of the complex plane, then, for any $\alpha \in D$ and $w \in \mathbb{C}^m$, the first-order normal system $y'(x) = A(x)y(x)$ has only one analytical solution in D for which $y(\alpha) = w$ and, thus, the space of analytical solutions defined in domain D has dimension m. Let $A(x) \in \mathrm{Mat}_m(K(x))$ and $\det A(x) \neq 0$ in a neighborhood of point α. Then, it follows that the dimension of the space of analytical solutions of the system defined in this neighborhood has dimension m.

If S is a system of form (2) with a nonsingular leading matrix, then it can be rewritten in the equivalent form $Y'(x) = A(x)Y(x)$ by taking

$$A(x) = \begin{pmatrix} 0 & I_m & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I_m \\ \hat{A}_0(x) & \hat{A}_1(x) & \dots & \hat{A}_{r-1}(x) \end{pmatrix}, \qquad (12)$$

where I_m is the identity matrix of order m, $\hat{A}_k(x) = -A_r^{-1}(x)A_k(x)$, $k = 0, 1, \dots, r-1$, and

$$Y(x) = (y'(x)^T, y''(x)^T, \dots, y^{(r-1)}(x)^T).$$

Hence, we arrive at the following theorem.

Theorem 3. *Let $\bar{S}$ be an l-embracing system for system S of form* (2), *and let $\bar{A}_r(x)$ be the leading matrix of system $\bar{S}$. Let* $\det\bar{A}_r(x)$ *do not vanish at some point* $\alpha \in \mathbb{C}$. *Then*, *analytical solutions of system S have no singularities at* α, *and the dimension of the space of such solutions does not exceed rm*; *if the leading matrix of system S is not singular* ($\bar{S}$ *coincides with S*), *then the dimension is equal to rm.*

If the leading matrix of the system is singular, then this dimension may be less than rm (see Example 3 below). In this case, the number of solutions of system $\bar{S}$ constructed by algorithm EG_σ is greater than that of S.

Example 3. System

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} y' + \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} y = 0 \qquad (13)$$

has the one-dimensional space of solutions $y = (c, 0)^T$. The application of EG_δ to this system results in the construction of the following system $\bar{S}$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y' + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} y = 0,$$

which has the two-dimensional space of solutions $y = (c_1x + c_2, c_1)^T$. Note that the equations of system (13) are independent over $K\left[x, \frac{d}{dx}\right]$.

Thus, the fact of existence of an l-embracing system in the differential case allows us to prove that the set of points where an analytical solution to the system may have singularity is finite and that the dimension of the space of solutions does not exceed rm.

Remark 3. If K is an arbitrary field of characteristic 0, then the dimension of the space of solutions of systems of form (2) belonging to $\bar{K}[[x-\alpha]]$ is equal to rm for all $\alpha \in \bar{K}$, except for, possibly, a finite set of values α where this dimension is reduced.

If the equations of the original differential system are not independent over $K\left[x, \frac{d}{dx}\right]$, the set of singular points of solutions of the system may be infinite.

Example 4. For the system $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} y' + \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} y = 0$, any point is singular for some solution with equal components y_1 and y_2 (note that the equations of the system are still independent over K).

4.2. Dimension of the Space of Solutions of Difference Systems

In the difference case, finite dimensionality can be proved, for example, for the space of solutions that are sequences. A sequence $c: \mathbb{Z} \longrightarrow K^m$, $c = \{c_n\}_{n \in \mathbb{Z}}$, is called a *sequential* solution (or a solution in the form of a double-sided sequence) to system (6) if

$$A_r(n)c_{n+d} + \dots + A_1(n)c_{n+1} + A_0(n)c_n = 0$$

for all $n \in \mathbb{Z}$.

Let $A_r(x)$ and $A_0(x)$ be invertible in $\mathrm{Mat}_m(K(x))$. A segment of the set of integers

$$I = \{v, v+1, \dots, w\}, \quad v, w \in \mathbb{Z}, \quad v \leq w,$$

is called an *essential segment* for (6) if the polynomial $A_r(x - r)$ has no integer roots that are greater than w, the polynomial $\det A_0(x)$ has no integer roots that are less than v, and $w - v + 1 \geq 1$.

Let I be an essential segment for (6). Then, any sequential solution c is uniquely determined by vectors c_n, $n \in I$. Hence, in order to describe the linear space of sequential solutions over K, it is sufficient to find its restriction to I. This restriction consists of all sets of vectors $(c_v, c_{v+1}, \dots, c_w)$ that satisfy the equations

$$\sum_{i=0}^{r} A_i(n)c_{n+i} = 0, \quad n = v, v+1, \ldots, w-r.$$

This gives us the system of linear algebraic equations that the components of vectors c_v, c_{v+1}, ..., c_w must satisfy. The dimension of the space of sequential solutions of the considered difference system coincides with the dimension of the space of solutions to this algebraic system.

If the leading, or trailing, matrix is singular, we may turn to system (9), which does not reduce the dimension of the space of sequential solutions.

Hence, we have the following theorem.

Theorem 4. *The space of sequential solutions of a system of form* (6) *is finite-dimensional.*

In the scalar case, additional properties of the space of sequential solutions are proved in [14, Section 2]. Some of them hold for systems.

As stated above, singular points of solutions of the considered differential systems form finite sets. The situation is different in the difference case, even for scalar equations: just recall that the gamma-function, which satisfies the scalar equation $y(x + 1) - xy(x) = 0$, has poles at all integer nonpositive values of x.

The solution spaces of linear differential and difference systems also differ considerably: solutions of difference systems can be multiplied not only by constants but also by functions of period 1 (i.e., functions f such that $f(x + 1) = f(x)$). Together with a meromorphic solution $y(x)$, the system will also has, for example, solutions

$$(\sin 2\pi(x+\beta))y(x) \text{ and } (\sin 2\pi(x+\beta))^{-1}y(x) \quad (14)$$

for any $\beta \in \mathbb{C}$. Assuming that $K = \mathbb{C}$, let us consider the space of meromorphic solutions of a system of form (6) over field $\mathbb{C}(e^{2\pi ix})$. All elements of this field are meromorphic functions of period 1, including, for example, functions $\sin 2\pi(x + \beta)$ and $(\sin 2\pi(x + \beta))^{-1}$. For the scalar equations of order r, it is shown in [28] that such space of meromorphic solutions has dimension r. In the same work, it is proved that, for the first-order normal systems consisting of m equations

$$y(x+1) = A(x)y(x), \quad (15)$$

$A(x) \in \mathrm{Mat}_m(K(x))$, such a solution space has dimension m, since, by means of an appropriate cyclic vector [28, 36], system (15) can be rewritten as a scalar equation of the order not greater than m.

Let us turn to system (6). If the leading matrix $A_r(x)$ is invertible in $\mathrm{Mat}_m(K(x))$, then (6) can be rewritten as the system

$$Y(x+1) = A(x)Y(x), \quad (16)$$

where $A(x)$ has form (12) and

$$Y(x) = (y(x)^T, y(x+1)^T, \ldots, y(x+r-1)^T)^T$$

(transition from (6) to (16) is possible, of course, not only in the case of $K = \mathbb{C}$, but also in the case of an arbitrary K). Hence, if matrix $A_r(x)$ is not singular, then the dimension of the space of meromorphic solutions over $\mathbb{C}(e^{2\pi ix})$ is not greater than rm. Note also that the transition from field $\mathbb{C}$ to $K \subseteq \mathbb{C}$ does not increase dimension of the space of meromorphic solutions. We may return to the original assumption about field K.

Theorem 5. *The dimension of the space of meromorphic solutions of system* (6) over the field $K(e^{2\pi ix})$ *does not exceed rm.*

Additionally, it can be shown that, if both the leading and trailing matrices for an original system S of form (6) are not singular for $K = \mathbb{C}$ (i.e., systems S, $\bar{S}$, and $\bar{\bar{S}}$ coincide), then the discussed dimension is equal to rm.

4.3. Boundedness of Valuations of Meromorphic Solutions to Difference Systems

In the difference case, existence of l- and t-embracing systems allows us, for example, to prove that, for a meromorphic solution $y(x)$ and a fixed $\alpha \in \mathbb{C}$, the values $\mathrm{val}_{x-\alpha-n}y(x)$ are bounded from below for $n \in \mathbb{Z}$. This assertion is formulated in the following theorem.

Theorem 6 ([24, Proposition 1]). *Let* $y(x)$ *be a meromorphic solution of system* (6). *Let* (7) *and* (8) *be the* l*- and* t*-embracing systems for this system. Let*

$$V(x) = \det\bar{A}_r(x-r), \quad W(x) = \det\bar{\bar{A}}_0(x). \quad (17)$$

Then, (i) *if* N_0 *is such that* $V(\alpha + n_0)W(\alpha + n_0) \neq 0$ *for all integer* $n_0 \geq N_0$, *then*

$$\exists_{\lambda \in \mathbb{Z}}(\forall_{n_0 \geq N_0} \min_{n=n_0}^{n_0+r-1} \mathrm{val}_{x-\alpha-n}y(x) = \lambda);$$

(ii) *if* N_1 *is such that* $V(\alpha + n_1)W(\alpha + n_1) \neq 0$ *for all integer* $n_1 \leq N_1$, *then*

$$\exists_{\mu \in \mathbb{Z}}(\forall_{n_1 \leq N_1} \min_{n=n_1}^{n_1+r-1} \mathrm{val}_{x-\alpha-n}y(x) = \mu).$$

5. INDUCED RECURRENCE SYSTEMS

In this and the next sections, we describe algorithms for constructing certain basic objects (recurrence systems, polynomials, and numeric quantities) that play an important role in finding various solutions of a given system.

5.1. Construction of Induced Systems

We will use double-sided sequences of rational functions

$$\{x^n\}_{n \in \mathbb{Z}} \quad (18)$$

and

$$\left\{\begin{matrix} x^n, \text{ if } n \geq 0 \\ 1/x^{|n|}, \text{ if } n < 0 \end{matrix}\right\}_{n \in \mathbb{Z}} \quad (19)$$

($x^{\underline{n}} = \prod_{k=1}^{n}(x-k+1)$, $x^{\overline{n}} = \prod_{k=1}^{n}(x+k)$) as bases for expanding certain solutions of systems. Basis (18) will be used in the differential case, and basis (19) will be used in the difference case. Let coefficients of the expansion of a solution in the corresponding basis be $z(n)$, $n \in \mathbb{Z}$, where $z(n) = (z_1(n), \ldots, z_m(n))^T$ is the column vector of number sequences. Then, the sequence of the column vectors $\{z(n)\}_{n\in\mathbb{Z}}$ satisfies the *induced recurrence* system

$$B_l(n)z(n+l) + B_{l-1}(n)z(n+l-1) + \ldots \\ + B_t(n)z(n+t) = 0, \tag{20}$$

where l and t are integers such that $l \geq t$ and $B_t(n), \ldots, B_l(n) \in \mathrm{Mat}_m(K[n])$. We use the term "induced recurrence system," rather than the "induced difference system," in order to emphasize the special role of the induced systems.

To describe the method for constructing an induced recurrence system, it is convenient to rewrite the original system (1) by using the matrix whose entries are scalar operators:

$$\begin{pmatrix} L_{11} & \ldots & L_{1m} \\ \ldots & \ldots & \ldots \\ L_{m1} & \ldots & L_{mm} \end{pmatrix} y(x) = 0, \tag{21}$$

$L_{ij} \in [x, \xi]$, $i, j = 1, 2, \ldots, m$. (The transformation to such system representation and the inverse transformation present no difficulties.) Then, the induced recurrence system takes the form

$$\begin{pmatrix} \tilde{L}_{11} & \ldots & \tilde{L}_{1m} \\ \ldots & \ldots & \ldots \\ \tilde{L}_{m1} & \ldots & \tilde{L}_{mm} \end{pmatrix} z(n) = 0, \tag{22}$$

where $\tilde{L}_{ij} \in K[x, E_n, E_n^{-1}]$, $i, j = 1, 2, \ldots, m$, and E_n is the shift operator by n: $E_n z(n) = z(n+1)$. Each operator $\tilde{L}_{ij}$ is obtained from L_{ij} through the transformation described in the following theorem.

Theorem 7 ([17; 12, Section 3; 26; 25]). *In the differential case, the induced recurrence system is constructed by system* (21) *by means of the transformation*

$$x \longrightarrow E_n^{-1}, \quad \frac{d}{dx} \longrightarrow (n+1)E_n,$$

and, in the difference case, by the transformation

$$x \longrightarrow n + E_n^{-1}, \quad E \longrightarrow 1 + (n+1)E_n. \tag{23}$$

Example 5. Rewriting the difference system

$$\begin{pmatrix} x^2+x & 0 \\ 1 & 0 \end{pmatrix} y(x+1) \\ + \begin{pmatrix} -2x^2-4x & x^2+3x+2 \\ -1 & 0 \end{pmatrix} y(x) = 0 \tag{24}$$

in form (21), we obtain

$$\begin{pmatrix} (x^2+x)E - 2x^2 - 4 & x^2+3x+2 \\ -1 & E \end{pmatrix} y(x) = 0.$$

Transformation (23) results in the system of form (22) with $m = 2$ in which

$$\tilde{L}_{11} = (n^3+2n^2+n)E_n + n^2 - 3n - (n+3)E_n^{-1} - E_n^{-2},$$
$$\tilde{L}_{12} = n^2+3n+2+(2n+2)E_n^{-1} + E_n^{-2},$$
$$\tilde{L}_{21} = -1,$$
$$\tilde{L}_{22} = (n+1)E_n + 1,$$

The induced recurrence system for system (24) is rewritten in form (20) as

$$\begin{pmatrix} n^3+2n^2+n & 0 \\ 0 & n+1 \end{pmatrix} z(n+1) \\ + \begin{pmatrix} n^2-3n & n^2+3n+2 \\ -1 & 1 \end{pmatrix} z(n) \\ + \begin{pmatrix} -n-3 & 2n+2 \\ 0 & 0 \end{pmatrix} z(n-1) \\ + \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} z(n-2) = 0. \tag{25}$$

For example, if system (24) has a polynomial solution, this solution can be written in basis (19) as

$$a_0 x^{\underline{0}} + a_1 x^{\underline{1}} + \ldots + a_k x^{\underline{k}}, \quad a_i \in K^m.$$

Then, the double-sided sequence

$$z(n) = \begin{cases} a_n, & \text{if } 0 \leq n \leq k, \\ 0, & \text{otherwise} \end{cases}$$

must satisfy system (25). We will return to the discussion of this in Example 8.

Remark 4. In fact, a stronger assertion than just fulfillment of Eq. (20) for the expansion coefficients holds. Let $y = (y_1, y_2, \ldots, y_m)^T$ be a vector whose entries admit expansions in terms of basis (18) or (19), and let

$$y^*(x) = A_r(x)\xi^r y(x) + \ldots + A_1(x)\xi y(x) + A_0(x)y(x)$$

(here, $y(x)$ is not necessarily a solution to the original system). Let $z(n)$ and $z^*(n)$ be vectors whose components are sequences of the expansion coefficients for components $y_1, y_2, \ldots, y_m$ and $y_1^*, y_2^*, \ldots, y_m^*$, respectively, in terms of basis (18) or (19). Then,

$$z^*(n) = B_l(n)z(n+l) + B_{l-1}(n)z(n+l-1) + \ldots + B_t(n)z(n+t).$$

It is not difficult to show that the equations of the induced recurrence system are independent over $K[n, E_n]$ if and only if the equations of the original system are independent over $K\left[x, \frac{d}{dx}\right]$, or, accordingly, $K[x, E]$. At the same time, the fact that leading matrix of the original differential system is nonsingular does not guarantee that the leading matrix of the induced recurrence system is also nonsingular, and vice versa.

Example 6. The differential system

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} y' + \begin{pmatrix} 0 & x \\ 1 & 1 \end{pmatrix} y = 0$$

has the singular leading matrix, whereas the leading matrix of the induced recurrence system

$$\begin{pmatrix} n & 0 \\ 1 & 1 \end{pmatrix} z(n) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z(n-1) = 0$$

is nonsingular [4, Section 3.2]. On the other hand, for the differential system

$$\begin{pmatrix} x & 0 \\ 0 & x^2 \end{pmatrix} y' + \begin{pmatrix} -1 & -x^3 \\ -2 & -x \end{pmatrix} y = 0,$$

the induced recurrence system

$$\begin{pmatrix} n-1 & 0 \\ -2 & 0 \end{pmatrix} z(n) + \begin{pmatrix} 0 & 0 \\ 0 & n-2 \end{pmatrix} z(n-1) + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} z(n-3) = 0$$

has singular leading matrix [12, Example 8].

There also exist simple examples where the leading matrices of the differential and induced recurrence systems are both (non)singular. In the difference case, similar examples can be presented for the trailing matrices.

5.2. Indicial Equations

In the classical theory of ordinary differential equations, the so-called *indicial* (algebraic) equations are used for finding valuations of analytical solutions in the scalar case (see, for example, [9, Chapter IV]). To find upper bounds for degrees of polynomial solutions of systems of form (1) in the differential and difference cases and lower bounds for valuations of such solutions at a given point in the differential case, we need analogues of the indicial equations. The induced recurrence systems, which were introduced in Section 5.1, make it possible to construct such equations.

If the leading matrix $B_l(n)$ of system (20) is singular, one can construct the l-embracing system

$$\bar{B}_l(n)z(n+l) + \bar{B}_{l-1}(n)z(n+l-1) + \ldots + \bar{B}_t(n)z(n+t) = 0$$

for (20) with the help of EG_δ. In a similar way, one can construct the t-embracing system

$$\bar{\bar{B}}_l(n)z(n+l) + \bar{\bar{B}}_{l-1}(n)z(n+l-1) + \ldots + \bar{\bar{B}}_t(n)z(n+t) = 0.$$

The fact that the value of t in (20) is not necessarily equal to zero and even can be negative (and the value of l is not necessarily positive) does not prohibit the application of algorithm EG_δ.

Theorem 8. (i) *Let $y(x) \in K((x))^m$ be a solution of the differential system for which* (20) *is the induced recurrence system. Then, the value $n = \mathrm{val}_x y(x)$ satisfies the equation*

$$\det \bar{B}_l(n-l) = 0. \qquad (26)$$

(ii) *Let $y(x) \in K[x]^m$ be a solution of the differential or difference system for which* (20) *is the induced recurrence system. Then, the value $n = \deg y(x)$ satisfies the equation*

$$\det \bar{\bar{B}}_t(n-t) = 0 \qquad (27)$$

(here, $\deg y(x) = \max_{i=1}^{m} \deg y_i(x)$ *for* $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T \in K[x]^m$).

Equation (27) may be viewed as an indicial equation for the original system at point ∞. The greatest integer nonnegative root of this equation yields the upper bound for the degree of polynomial solutions. If Eq. (27) has no integer nonnegative roots, then the original system has no polynomial solutions. Similarly, in the differential case, Eq. (26) can be used for finding lower bounds of valuations of solutions at point 0. (Substitution of $x + \alpha$ for x in the original system turns point α to 0.)

Example 7. For the differential system

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} y' + \begin{pmatrix} 1 & 1+x^2 \\ -x & 0 \end{pmatrix} y = 0, \qquad (28)$$

the induced recurrence system is given by

$$\begin{pmatrix} n+1 & n+1 \\ n & n \end{pmatrix} z(n) + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} z(n-1) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z(n-2) = 0.$$

The application of EG_σ to the leading matrix yields

$$\begin{pmatrix} n+2 & 0 \\ n & n \end{pmatrix} z(n) + \begin{pmatrix} 0 & n+1 \\ -1 & 0 \end{pmatrix} z(n-1) = 0 \quad (29)$$

and the linear constraint

$$z_1(0) + z_2(0) + z_2(-2) = 0. \quad (30)$$

The determinant of the leading matrix of system (29) has roots 0 and -2. Therefore, the differential system (28) cannot have solutions from $K((x))^m$ at point $x = 0$ with valuations different from 0 and -2. Whether it has solutions with these valuations we do not know yet. The answer to this question will be given in Example 9.

We see that, in the differential case, algorithms EG_δ and EG_σ work jointly: the first algorithm is used to find the set of possible solution singularities, and the second one is used to compute the bounds of solution valuations at these points.

In what follows, we will call (26) and (27) "indicial equations." This is a provisional name, since, for example, the equations obtained in this way depend on the constructed l- and t-embracing systems and, thus, are not unique.

6. CALCULATION OF LOWER BOUNDS OF VALUATIONS OF COMPONENTS OF MEROMORPHIC SOLUTIONS TO DIFFERENCE SYSTEMS

We continue study of meromorphic solutions of difference systems started at the end of Section 4.1. The question is the lower bound for $\mathrm{val}_{x-\alpha}y(x)$, where $y(x)$ is a meromorphic solution to system (6) and α is a point in the complex plane.

6.1. Bounds of Solution Valuations

Consider the problem of calculating the lower bound for $\mathrm{val}_{x-\alpha}y(x)$, assuming that

$$\min_{n=n_0}^{n_0+r-1} \mathrm{val}_{x-\alpha+n}y(x) \ge v \quad (31)$$

for some nonnegative integer n_0 and integer v, or, similarly,

$$\min_{n=n_1}^{n_1+r-1} \mathrm{val}_{x-\alpha-n}y(x) \ge w \quad (32)$$

for some nonnegative integer n_1 and integer w. The following theorem and remark are borrowed from [24, Section 3.2].

Theorem 9. *Let $y(x)$ be a meromorphic solution to system* (6), $\alpha \in \mathbb{C}$, *and $p(\alpha) = 0$ for $p(x) \in \mathrm{Irr}(K[x])$. Let $V(x)$ and $W(x)$ be defined by* (17), *and let a nonnegative n_0 and integer v satisfy inequality* (31). *Then*,

$$\mathrm{val}_{x-\alpha}y(x) \ge v - \sum_{n=0}^{n_0-1} \mathrm{val}_{p(x+n)}V(x).$$

Similarly, let a nonnegative n_1 and integer w satisfy inequality (32). *Then*,

$$\mathrm{val}_{x-\alpha}y(x) \ge w - \sum_{n=0}^{n_1-1} \mathrm{val}_{p(x-n)}W(x).$$

Remark 5. Let λ and μ be defined like in Theorem 6. If $v \le \lambda$ and $w \le \mu$, then the following inequality holds for any mutual location of point α and the roots of the polynomials $W(x)$ and $V(x)$:

$$\mathrm{val}_{x-\alpha}y(x) \ge \max\left\{ v - \sum_{n\in\mathbb{N}} \mathrm{val}_{p(x+n)}V(x),\; w - \sum_{n\in\mathbb{N}} \mathrm{val}_{p(x-n)}W(x) \right\} \quad (33)$$

(the sums on the right-hand side of the inequality are finite).

Theorem 9 and Remark 5 yield algorithms for solving the problem under consideration.

6.2. Bounds for Valuations of Solution Components

It is also possible to consider the problem of calculation of lower bounds for valuations $\mathrm{val}_{x-\alpha}y_i(x)$, $i = 1, 2, \ldots, m$, assuming that, for some nonnegative integer n_0, lower bounds for valuations

$$\mathrm{val}_{x-\alpha+n}y_i(x), \quad n = n_0, n_0+1, \ldots, n_0+r-1, \quad i = 1, 2, \ldots, m,$$

are separately given, or, for some nonnegative integer n_1, lower bounds for valuations

$$\mathrm{val}_{x-\alpha-n}y_i(x), \quad n = n_1, \quad n_1+1, \ldots, n_1+r-1, \quad i = 1, 2, \ldots, m,$$

are separately given. In [24], an algorithm is presented that, in the general case, finds more accurate bounds, compared to the previous algorithm, but has greater complexity. This algorithm is based on the so-called tropical calculations [6, Section 2]. Here, we will not go in detail of this algorithm.

7. SOLUTION CONSTRUCTION

In this section, we give examples of algorithms for searching solutions of various types. The algorithms use induced recurrence systems and indicial equations.

7.1. Polynomial Solutions

Polynomial and rational solutions are of interest by themselves. Besides, finding of such solutions may serve as an intermediate stage in the construction of more complex solutions.

After the upper bound of degrees of all polynomial solutions has been found with the help of (27), the solutions themselves can be found by the method of undetermined coefficients. However, there are more efficient methods for computing coefficients of polynomial solutions with the help of the induced recurrence system (see, for example, [10]).

Example 8. Let us return to consideration of system (24). The induced recurrence system (25) has the singular trailing matrix. Algorithm EG_σ constructs the t-embracing recurrence system

$$\begin{pmatrix} (n-1)^2-(n-1)^3 & 0 \\ 0 & n-1 \end{pmatrix} z(n-1)$$

$$+\begin{pmatrix} -(n-1)^2+5n-9 & -2 \\ -1 & 1 \end{pmatrix} z(n-2) = 0$$

with the empty set C of linear constraints. The roots of Eq. (27) are equal to 1 and 2. This yields the upper bound 2 for the degrees of possible polynomial solutions.

By means of the t-embracing recurrence system obtained, we successively find coefficients of the desired polynomial solution, starting from the leading coefficients to the lower ones. In doing so, we take into account the fact that the coefficients with indices greater than 2 and less than 0 are necessarily equal to zero. Here, this system is used for n varying from 4 to -1. The solution contains arbitrary constants:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_2+(2c_1+c_2)x^{\underline{1}}+c_1x^{\underline{2}} \\ c_2x^{\underline{1}}+c_1x^{\underline{2}} \end{pmatrix}$$

$$= \begin{pmatrix} c_2+(2c_1+c_2)x+c_1x(x-1) \\ c_2x+c_1x(x-1) \end{pmatrix}.$$

7.2. Rational Solutions

Let us describe how the search of rational solutions can be reduced to that of polynomial solutions. Consider first the differential case. As noted in Section 4.1, for a differential system S of the considered form, one can find a polynomial $d(x)$ such that $d(\alpha) = 0$ at the point α where system S has singularity. The fact that we can find the lower bound e_α for the valuation at point α for an arbitrary solution of system S allows us to construct a rational function $F(x) = \prod_{d(\alpha)=0}(x-\alpha)^{e_\alpha}$ such that any rational solution $y(x)$ of system S has the form

$$y(x) = F(x)z(x), \tag{34}$$

where $z(x) \in K[x]^m$. Substitution (34) transforms S to a system in $z(x)$, and it remains to find polynomial solutions of the system obtained [12, Example 10]. If the indicial equation corresponding to some α has no integer roots, then S has no rational solutions.

In addition to the above-said, we note that the bounds e_i are identical for all roots α_i of each irreducible polynomial $d(x)$. Therefore, $F(x)$ can be written as

$$\prod_{\substack{p(x)\in \mathrm{Irr}(K[x]) \\ p(x)|d'(x)}} p(x)^{e_{p(x)}},$$

where $d'(x)$ is the polynomial obtained from $d(x)$ after the elimination of the squares. When $\deg_{p(x)} > 1$ the exponents $e_{p(x)}$ are found by the calculation over the field $K(\alpha)$, $p(\alpha) = 0$.

Now, let us consider the difference case. When searching denominators of rational solutions, the a priori known lower bounds v and w of valuations are set equal to zero in (33). Let us consider an algorithm for finding a *universal denominator* of rational solutions of the original system, or, for brevity, the universal denominator for the original system, i.e., a polynomial $U(x) \in K[x]$ such that, if the system has a rational solution $y(x) \in K(x)^m$, then the solution may be represented as

$$y(x) = \frac{1}{U(x)}z(x), \tag{35}$$

where $z(x) \in K[x]^m$. If the universal denominator is known, we can make substitution (35), transform the original system to a system in $z(x)$, and apply one of the algorithms for searching polynomial solutions (see Section 7.1).

For $p(x) \in \mathrm{Irr}(K[x])$ and $f(x) \in K[x]\backslash\{0\}$, we define the finite set

$$\mathcal{N}_{p(x)}(f(x)) = \{k \in \mathbb{Z}: p(x+k)|f(x)\}.$$

If $\mathcal{N}_{p(x)}(f(x)) = \varnothing$, we set $\mathcal{N}_{p(x)}(f(x)) = -\infty$ and $\min\mathcal{N}_{p(x)}(f(x)) = +\infty$.

One of the practical algorithms for constructing the universal denominators consists of two steps. On the first step, the finite set of irreducible polynomials

$$M = \{p(x) \in \mathrm{Irr}(K[x]): \min\mathcal{N}_{p(x)}(W(x)) \le 0,$$
$$\max\mathcal{N}_{p(x)}(V(x)) \ge 0\},$$

where $V(x)$ and $W(x)$ are defined by (17), is constructed. To construct set M, complete factorization of $V(x)$ and $W(x)$ is used. On the second step, the universal denominator is calculated in the form of the product

$$\prod_{p(x)\in M} p^{\gamma_{p(x)}}(x), \tag{36}$$

where, in accordance with (33),

$$\gamma_{p(x)} = \min\left\{\sum_{n\in\mathbb{N}} \mathrm{val}_{p(x+n)}V(x),\ \sum_{n\in\mathbb{N}} \mathrm{val}_{p(x-n)}W(x)\right\}. \tag{37}$$

In [39], an algorithm for finding the universal denominator based on formulas (36) and (37) was suggested. Then, it was shown that, if we take into account that the exponents $\gamma_{p(x)}$ for different (sometimes, many) $p(x)$ that differ from one another by a shift on an integer number may coincide, the calculations can significantly be sped up [3, 20].

Remark 6. In [5, Theorem 1], it was proved that, if an algorithm for constructing a universal denominator $U(x)$ uses, like the earlier proposed algorithm, only $V(x)$ and $W(x)$, then $U(x)$ is also a universal denominator for any nonhomogeneous system with the same left-hand side and an arbitrary polynomial right-hand side $(b_1(x), b_2(x), \ldots, b_m(x))^T \in K[x]^m$.

Given $V(x)$ and $W(x)$, $U(x)$ can also be found by algorithms that calculate the so-called *dispersion* of polynomials [1; 2, Section 8]. Originally, these algorithms were designed for the scalar case. Later, in [13], an algorithm for normal systems (15) was proposed, with $V(x) = \operatorname{den}A(x-1)$ and $W(x) = \operatorname{den}A^{-1}(x)$. (These dispersion algorithms are used, in particular, in Maple.) A similar approach was used in [29] for solving a more general problem.

However, complexity of the algorithm based on formulas (36) and (37) that takes into account the possibility of identical $\gamma_{p(x)}$ for different $p(x)$ is less than that of the dispersion algorithm [20, Section 4.2].

7.3. Solutions with Series Components

Here, we discuss, in particular, how to search for solutions of differential systems whose components are Laurent series. These solutions will be referred to as *Laurent* solutions. The convergence is not considered here.

In the case of a singular leading matrix of the induced system, we apply EG_σ. The finite set C of linear constraints arising in this case will allow us to discard those roots of the denominator of the leading matrix that are certainly not valuations of the Laurent solutions (this root separation can also be used in searching for rational solutions). Series solutions are represented by initial terms. The number of the initial terms is selected in such a way that, first, the computation of the following terms does not already require to take into account the linear constraints and, second, the determinant of the leading matrix of the l-embracing recurrence system does not vanish in the course of the computation. Having found the lower bound of the solution valuation (Theorem 8 (i)), we can write down and solve a system of linear algebraic equations for the initial terms. The following terms may be obtained with the help of the l-embracing recurrence system.

Example 9. Let us return to the differential system (28) from Example 7. The determinant of the leading matrix of system (29) has roots 0 and −2. However, no Laurent solution of (28) corresponds to the larger root: constraint (30) and the first equation of system (29) show that, if $z(-1) = z(-2) = 0$, then $z(0) = 0$ as well. As for the root −2, the corresponding Laurent solutions are easily constructed. We select $z(-2)$ that satisfies the equation

$$\begin{pmatrix} 0 & 0 \\ -2 & -2 \end{pmatrix} z(-2) = 0.$$

For the basis solution of this linear system of algebraic equations, we may take, for example, $z_1(-2) = (1, -1)^T$. From (29), we obtain

$$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} z(-1) + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} z(-2) = 0,$$

which yields $z_1(-1) = (0, -1)^T$. For $n = 0$, system (29) takes the form

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} z(0) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z(-1) = 0$$

and, together with (30), yields $z_1(0) = (1/2, 1/2)^T$. Using (29), we obtain

$$z(n) = \begin{pmatrix} 0 & -\dfrac{n+1}{n+2} \\ \dfrac{1}{n} & \dfrac{n+1}{n+2} \end{pmatrix} z(n-1)$$

for $n \geq 1$.

We see that, at point $x = 0$, the differential system (28) has one-dimensional space of the Laurent solutions with the basis given by the series

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} x^{-2} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} x^{-1} + \sum_{n=0}^{\infty} \begin{pmatrix} z_1(n) \\ z_2(n) \end{pmatrix} x^n,$$

where

$$\begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

and

$$\begin{pmatrix} z_1(n) \\ z_2(n) \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{n+1}{n+2} \\ \dfrac{1}{n} & \dfrac{n+1}{n+2} \end{pmatrix} \begin{pmatrix} z_1(n-1) \\ z_2(n-1) \end{pmatrix} \tag{38}$$

for $n \geq 1$.

Remark 7. It was noted in [31, Section 6] that the transformations performed by algorithm EG_σ on the induced recurrence system are associated with certain transformations of the original differential system. The authors of the paper [31] refers to this as a differential variant of EG_σ, which works without transforming to the recurrence system. Such an approach may be helpful when it is required to find a very few terms of the Laurent series. However, when the desired number of terms is great, the induced recurrence system is more efficient (in Example 9, we obtained convenient recurrent formula (38)). Besides, the original variant of EG_σ additionally yields a finite set C of linear constraints, which, in certain cases, allows us not to consider some roots of the determinant of the corresponding matrix (in Example 9, this set consists of only one relation (30), which allows us not to consider the root 0).

In a similar way, in the difference case, one can apply the induced recurrence systems for constructing solutions in the form of Newton's series [7, Chapter II]. At point 0, such a solution is given by

$$a_0 x^{\underline{0}} + a_1 x^{\underline{1}} + a_2 x^{\underline{2}} + \dots, \quad a_i \in K^m.$$

These series are of interest; however, they are more difficult to deal with compared to the Laurent series. For example, even in the case of an entire function, its representation in terms of the Newton series is generally not unique (however, there are sufficient conditions of uniqueness [7, Chapter II, Section 2.3]).

7.4. Regular Solutions of Differential Systems

Regular solutions of systems of differential equations are solutions of the form

$$y(x) = x^{\lambda} v(x), \tag{39}$$

where $\lambda \in K$ and $v(x) \in K((x))^m[\log x]$. Any regular solution is written as

$$x^{\lambda}\sum_{s=0}^{k} g_s(x)\log^s(x),$$

where $k \in \mathbb{N}$ and $g^s(x) \in K((x))^m$, $s = 0, 1, \dots, k$. If

$$\min_{s=0}^{k} \mathrm{val}_x g_s(s) = 0, \tag{40}$$

then λ is called *exponent* of solution (39).

In the scalar case, the problem of finding regular solutions can be solved by means of algorithms of the theory of differential equations. The Frobenius algorithm is based on studying roots of the indicial equation [9, Chapter IV; 38; 47, Chapter V]. When constructing a solution, not only values of roots of the indicial equation but also their multiplicities are taken into account, as well as the existence of roots that differ by an integer. The Heffter algorithm [40, Chapters II and VIII; 47, Chapter V] constructs a basis (possibly, empty) of regular solutions with multiplier x^{λ} not using multiplicity of root λ or existence of other roots that differ from λ by an integer.

The application of the Frobenius and Heffter algorithms requires transformation of the system to a scalar equation (for example, by means of the cyclic vector method [36]), which is not convenient from the point of view of practice. There is a need in algorithms that can directly be applied to the system. In [30, 33], the Heffter algorithm was extended to the case of first-order normal systems of form (15) with the help of the approach based on super-irreducibility [41]. On the other hand, the Heffter algorithm can be generalized by means of the approach based on the construction of the induced recurrence systems and the corresponding embracing systems [19, 22]. This variant is already applicable to systems of arbitrary order. To describe it, it is convenient to write system (2) as $L(y) = 0$, where L is a differential operator with matrix coefficients of the form

$$L = A_r(x)\frac{d^r}{dx^r} + \dots + A_1(x)\frac{d}{dx} + A_0(x).$$

For an arbitrary integer $i \geq 0$, the application of L to $g(x)\log^i(x)/i!$ yields

$$L_{i,i}(g)\frac{\log^i x}{i!} + \dots + L_{i,1}(g)\frac{\log x}{1!} + L_{i,0}(g),$$

where the coefficients of the differential operators $L_{i,j}$ belong to $\mathrm{Mat}_m(K[x, x^{-1}])$ and $L_{0,0} = L$ and $L_{i+j,j} = L_{i,0}$ for all $i, j \geq 0$ [40; 43, Section 3.2.1]. Let us introduce the notation $L_i = L_{i,0}$ $(= L_{i+j,j}$ for all $j \geq 0)$.

Generalization of the Heffter algorithm to the case of systems of arbitrary order relies on the consideration of a sequence of systems $S_0, S_1, \dots$, where S_k is a system of the form

$$L_0(g_i) = -\sum_{j=1}^{i} L_j(g_{i-j}), \quad i = 0, 1, \dots, k. \tag{41}$$

Similar to the result proved by Heffter in the scalar case, the following assertion holds.

Theorem 10 ([19, 22]). *The set of nonnegative integers k for which system S_k has a Laurent solution*

$$(g_0(x)^T, g_1(x)^T, \dots, g_k(x)^T)^T, \quad g_0(x) \neq 0,$$

is finite; *if it is empty*, *then the equation $L(y) = 0$ has no nonzero solutions in $K((x))^m[\log x]$. If this set is not empty and $\tilde{k}$ is its greatest element*, *then any solution of system $L(y) = 0$ belonging to $K((x))^m[\log x]$ has the form*

$$\sum_{s=0}^{\tilde{k}} g_{\tilde{k}-s}(x)\frac{\log^s x}{s!}, \tag{42}$$

where

$$(g_0(x)^T, g_1(x)^T, \dots, g_{\tilde{k}}(x)^T)^T, \quad g_0(x) \neq 0, \tag{43}$$

is a Laurent solution of system $S_{\tilde{k}}$. At the same time, *any Laurent solution of form* (43) *of system $S_{\tilde{k}}$ generates solution* (42) *of system $L(y) = 0$.*

If the value λ is known, then the search of the regular solution (39) reduces to the search of solution $y_{\lambda}(x) \in K((x))^m[\log x]$ by means of the substitution $y(x) = x^{\lambda} y_{\lambda}(x)$.

For the possible candidates for the role of λ in (39), roots of the indicial equation (26) of the original system are used. If necessary, EG_{σ} is applied; the set C of arising linear constraints, in certain cases, allows us to eliminate some candidates for the role of the exponent λ. The linear constraints with noninteger values of n can help to discard wrong noninteger values of λ. If $\lambda_1 - \lambda_2 \in \mathbb{Z}$, then the sets of regular solutions with the multipliers x^{λ_1} and x^{λ_2} will, clearly, be identical. Therefore, among all values of λ differing by integers, it is sufficient to consider the least one. If a regular solution with such λ exists, then, by adding (if necessary) an integer to λ, it is easy to fulfill (40), having found, thus, the desired solution exponent.

Remark 8. Possible values of λ belong either to field K or to its extension. In the latter case, coefficients of the Laurent series in (42) belong to the same extension.

Example 10. For the differential system

$$\begin{pmatrix} 18x^3 & 0 \\ 0 & 50x^2 - 15x \end{pmatrix} y'(x) + \begin{pmatrix} 0 & 7 \\ 0 & 26x - 27 \end{pmatrix} y(x) = 0,$$

the induced recurrence system has the singular leading matrix. Applying EG_σ, we obtain the recurrence system with the leading matrix

$$\bar{B}_0(n) = \begin{pmatrix} 0 & 14(25n+13) \\ 162(5n+14)(5n+19)n & 28(25n+38)(25n+13) \end{pmatrix} \tag{44}$$

and, additionally, the linear constraints

$$-\frac{96}{5}z_2\left(-\frac{13}{25}\right) - 50z_2\left(-\frac{13}{25} - 1\right) = 0, \tag{45}$$

$$7z_2\left(-\frac{9}{5}\right) - \frac{342}{5}z_1\left(-\frac{9}{5} - 2\right) = 0. \tag{46}$$

The set $\left\{-\frac{19}{5}, -\frac{14}{5}, -\frac{13}{25}, 0\right\}$ of roots of the determinant of matrix $\bar{B}_0(n)$ is divided into three classes with the least elements $-\frac{19}{5}$, $-\frac{13}{25}$, and 0. The linear constraint (45) helps to eliminate the root $-\frac{13}{25}$. The linear constraint (46) eliminates nothing, so that it is required to consider two values of λ: $-\frac{19}{5}$ and 0. Calculations yield the resulting solution

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x^{-\frac{19}{5}} \begin{pmatrix} \frac{35}{342}c_1 - \frac{16}{27}c_1 x + \frac{1568}{3645}c_1 x^2 + O(x^3) \\ c_1 x^2 + O(x^3) \end{pmatrix} + \begin{pmatrix} c_2 + O(x) \\ O(x) \end{pmatrix}.$$

The expansions of all $g_s(x)$ into series are represented by initial terms (see Section 7.3).

The representation of series in the form of their initial segments is a tricky part of the algorithm. In the course of the calculations, it is required to determine the length of the initial segments on the right-hand side of (41) and, accordingly, to enlarge the initial segments of solutions of the previous systems, when needed, upon the calculation of the right-hand side of the next system in the sequence $S_0, S_1, \ldots$. Our algorithm overcomes this difficulty (see [19, 22] for detail).

It is easy to see that systems of form (41) differ by only their right-hand sides, whereas the left-hand side is always the result of application of L_0 (i.e., L) to the vector of unknown functions. Such systems are efficiently solved by successive application of EG_σ to nonhomogeneous systems, which is briefly discussed in Section 8.1.

Remark 9. The problem of construction of regular solutions of an arbitrary-order differential systems was completely solved by means of EG_σ in [19, 22]. The problem of construction of such solutions is also solved in [31] for the case where the leading matrix of the original differential system is nonsingular. Since the set of relations C is not considered in [31], the redundant solutions are eliminated by substituting them into the equations of the original system. If the induced recurrence system and the set C have been constructed, such elimination, in our opinion, is more expensive than that where the relations similar to (30) are taken into account. (In [31], all power series in a regular solution are given in a truncated form, which makes the eliminating substitution–check even more complicated.)

7.5. Rational Logarithmic Solutions of Differential Systems

Rational logarithmic solutions of differential systems are solutions belonging to $K(x)^m[\log x]$. Their components are written as

$$\sum_{s=0}^{k} g_s(x)\log^s(x), \tag{47}$$

where $k \in \mathbb{N}$ and $g_s(x) \in K(x)$, $s = 0, 1, \ldots, k$. Whereas the search of regular solutions is a local problem, i.e., the problem solved at some point, the search of rational logarithmic solutions is a similar global problem

The algorithm for searching regular solutions that was discussed in Section 7.4 constructs initial segments of the Laurent series. It can easily be adapted to searching solutions in $K[x]^m[\log x]$. To this end, instead of dynamical calculation of the required number of the initial terms of the series, the upper bound for the powers of the corresponding polynomial solutions is immediately calculated, and just this number of terms is found. Moreover, the necessity of calculation of possible values of λ is also eliminated.

Like the search of rational solutions described in Section 7.1, the search of rational logarithmic solutions relies on the construction of a function $F(x) \in K(x)$ such that any rational logarithmic solution $y(x)$ of system S has form (34); i.e., $y(x) = F(x)z(x)$, but $z(x) \in$

$K[x]^m[\log x]$. Substitution (34) reduces the search of a solution in $K(x)^m[\log x]$ to that in $K[x]^m[\log x]$. It is shown in [30] that, for searching rational and rational logarithmic solutions, one and the same function $F(x)$ can be used.

Example 11. For the differential system

$$\begin{pmatrix} x^2+x & 0 \\ 0 & x^2+x \end{pmatrix} y' + \begin{pmatrix} 2x+1 & x+1 \\ 0 & 2x+1 \end{pmatrix} y = 0,$$

we find $F(x) = \dfrac{1}{x^2(x+1)}$ and obtain the corresponding rational logarithmic solution

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \dfrac{c_1 + c_2 \log x}{x(x+1)} \\ \dfrac{c_2}{x(x+1)} \end{pmatrix},$$

with the rational solution of the same system having the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \dfrac{c_1}{x(x+1)} \\ 0 \end{pmatrix}.$$

The rational logarithmic solution found coincides with this rational solution when $c_2 = 0$.

8. ADDITIONAL POSSIBILITIES

8.1. Nonhomogeneous Systems

Given a nonhomogeneous system S whose left-hand side coincides with the left-hand side of (1) and the right-hand side has the form

$$b(x) = (b_1(x), b_2(x), \ldots, b_m(x))^T \in K[x]^m,$$

by adding component y_{m+1} equal to 1 into $y(x)$, we transform this system to the homogeneous system S_1 of order $m + 1$ in $m + 1$ unknown functions. The difference system

$$A_r(x)y(x+r) + \ldots + A_1(x)y(x+1) + A_0(x)y(x) = b(x)$$

is transformed to system S_1 of the form

$$\tilde{A}_r(x)y(x+r) + \ldots + \tilde{A}_1(x)y(x+1) + \tilde{A}_0(x)y(x) = 0,$$

where

$$\tilde{A}_0(x) = \begin{pmatrix} & & -b_1(x) \\ & A_0(x) & \vdots \\ & & -b_m(x) \\ 0 & \ldots\ 0 & -1 \end{pmatrix},$$

$$\tilde{A}_1(x) = \begin{pmatrix} & & 0 \\ & A_1(x) & \vdots \\ & & 0 \\ 0 & \ldots\ 0 & 1 \end{pmatrix},$$

and

$$\tilde{A}_i(x) = \begin{pmatrix} & & 0 \\ & A_i(x) & \vdots \\ & & 0 \\ 0 & \ldots\ 0 & 0 \end{pmatrix},$$

$i = 2, 3, \ldots, r$. In a similar way, the transition to system S_1 in the differential case can be described, with the additional equation being $y'_{m+1}(x) = 0$ instead of $y_{m+1}(x+1) - y_{m+1}(x) = 0$.

Let a homogeneous system S_2 be obtained by discarding the right-hand sides in the equations of the original system. If the equations in S_2 are independent over $K\left[x, \dfrac{d}{dx}\right]$, then the equations in S_1 are also independent. Taking this into account, we may confine ourselves to the consideration of only homogeneous systems for many problems in hand. For example, to find the polynomial whose roots contain all singular points of the solutions of the differential system S, it is sufficient to find the corresponding polynomial for S_1. Besides, it happens that the right-hand side can be ignored on some step of calculations (see, for example, Remark 6).

Algorithms EG_δ and EG_σ can be applied directly to nonhomogeneous systems (for EG_σ, this was shown in [12]) to obtain the embracing nonhomogeneous systems. The components of the column consisting of the components of the right-hand side do not leave their places (never are shifted to other columns) in the course of the application of EG_δ and EG_σ. This makes it possible to achieve certain efficiency in the algorithm for constructing regular solutions, which was mentioned in Section 7.4. Solution $g_0(x)$ of system $L_0(g_0) = 0$ contains arbitrary constants. We use $g_0(x)$ as the right-hand side of system $L_0(g_1) = -L_1(g_0)$, so that this right-hand side is a linear function of the above-mentioned arbitrary constants. Applying the same technique as in the case of the scalar equation with a parameterized right-hand side (see, for example, [17]), we find, together with $g_1(x)$, linear relations for the constants occurring in $g_0(x)$ and $g_1(x)$. Continuing this process, on each step, we obtain $g_0(x), \ldots, g_i(x)$ containing unknown constants and a linear algebraic system in these constants. To guarantee termination of this process, we impose condition $g_0(x) \neq 0$; in this case, we necessarily reach a k such that (41) has no Laurent solutions. All these systems have one and the same left-hand side, but their right-hand sides are different. In order to perform transformations of EG_σ

only once, EG_σ is applied to the system with the right-hand side of the general form. This yields an l-embracing recurrence system with a nonsingular leading matrix, a finite set of linear constraints, and a transformed right-hand side. Each component of this transformed right-hand side is a linear combination of the components (possibly, shifted) of the original right-hand side. Hence, the same l-embracing recurrence system can be used for solving any system of form (41) after the substitution of the components of the right-hand side of the particular system into the transformed right-hand side.

8.2. EG_δ and EG_σ as Rank-Revealing Transformations

Throughout this paper, we assumed that the equations of the original system S of form (1) are independent over $K[x, \xi]$, in other words, the system has rank m (i.e., full rank). If this assumption is not fulfilled, then algorithms EG_δ and EG_σ find the rank of the system, since the transformations performed preserve the number of the equations that are independent over $K[x, \xi]$. If the rank is $m_0 \le m$, then, by means of EG_δ (or, respectively, EG_σ), it is possible to construct a system S' of m_0 equations independent over $K[x, \xi]$ such that its leading matrix (of size $m_0 \times m$) has rank m_0 over $K[x]$, with each solution of system S being a solution of system S' [4, Section 2.1; 16, Section 2].

8.3. q-Difference Systems

Let us briefly discuss the q-difference case. While differential equations are constructed on the basis of the differentiation operation $\frac{d}{dx}$ and difference equations, on the basis of the shift operation E, the q-difference equations are based on the q-shift Q:

$$Q(f(x)) = f(qx),$$

where q is either a fixed number or an additional variable (indefinite quantity). We will assume that $K = K_0(q)$, where K_0 is a subfield of field K and that q is transcendent over K_0. Accordingly, $f(x)$ may be an analytical function (as a rule, of two variables x and q), or a formal series, or, for example, a sequence $f(q^n)$, $n \in \mathbb{Z}$ (in this case, x denotes q^n). The q-difference equations are met, for example, in the subfield of number theory, partitioning theory [11, Section 8.4], and combinatorics [27]; besides, the q-difference differential and integral calculus has been constructed [8].

All discussed in Sections 2, 3, and 5 is transferred (with appropriate revisions) to the q-difference case ($\xi = Q$ in (1)), see [12, 15] for detail. For the construction of the induced recurrence system, it is convenient to assume that the solution is written in the basis $\{x^n\}_{n \in \mathbb{Z}}$. Then, the induced recurrence system is obtained from the original system by the transformation $Q \longrightarrow q^n$, $x \longrightarrow E_n^{-1}$.

9. RANDOMIZATION AMD HEURISTICS

9.1. Singular Points of Differential Systems: Singsys Algorithm

In Section 4.1, from the existence of the l-embracing system, we derived that the set of points in which analytical solutions of a differential system may have singularities is finite. Given a differential system S, algorithm Singsys[1] finds polynomial $d(x) \in K[x]\backslash\{0\}$ that satisfies the equation $d(\alpha) = 0$ at the point $\alpha \in \bar{K}$ where S has an analytical solution with a singularity of any kind. The operation of the algorithm consists in the application of algorithm EG_δ to the given system and subsequent calculation of the determinant of the leading matrix of the l-embracing system obtained. The polynomial found is then made square-free.

9.2. Randomization of EG_δ and Singsys

Algorithms EG_δ and Singsys can be randomized. Let $0 < p \le 1$. Let the probability of performing the differential shift (as described in Section 2.2) be p and that of the differentiation without division by the last nonzero coefficient be $1 - p$. We will call this p-shift (if $p = 1$, then the p-shift is the differential shift).

After each step "reduction + p-shift," the total length of all equations reduces by a quantity the average of which is not less than p. Since $p > 0$, the average time of waiting for termination of the transformations based on the steps "reduction + p-shift" is finite. Thus, we arrived at the randomized variant of algorithm EG_δ.

Further, we can use the following scheme for Singsys. The first time, algorithm EG_δ is applied in the way described in Section 2.2. Then, without changing the selected value of p, the randomized version of EG_δ is applied several times. Each such application can give us a new leading matrix and a new polynomial. The greatest common divisor of the polynomials obtained will, possibly, have a lesser degree. The process is terminated when a current application did not give rise to the reduction of the degree of the result or when the result is a zero-degree polynomial. This gives us the randomized variant of the Singsys algorithm.

Example 12. Consider the system

[1] Singsys stands for singularities of solutions of linear ordinary differential systems.

$$\begin{pmatrix} x & 0 \\ x(x-2) & 0 \end{pmatrix} y'' + \begin{pmatrix} 0 & 0 \\ -x+1 & 0 \end{pmatrix} y'$$

$$+ \begin{pmatrix} 0 & x-2 \\ 0 & -5x+6 \end{pmatrix} y = 0.$$

Applying EG_δ, we obtain

$$\begin{pmatrix} x(x+2)(x-2) & -x(x+2)^3(x-2) \\ x+2 & 0 \end{pmatrix} y''$$

$$+ \begin{pmatrix} -x^2-4 & 8(x+2)^2 \\ -1 & (x+2)^2 \end{pmatrix} y'$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} y = 0.$$

The polynomial $d_1(x) = x(x+2)(x-2)$ is the result of operation of algorithm Singsys. The application of the randomized variant of algorithm EG_δ with $p = \frac{1}{2}$ additionally yields other systems, in particular,

$$\begin{pmatrix} x & 0 \\ -x & -x(x+2)(x-1) \end{pmatrix} y''$$

$$+ \begin{pmatrix} 0 & 0 \\ -1 & -2(x+2)(2x-1) \end{pmatrix} y' \qquad (48)$$

$$+ \begin{pmatrix} 0 & x-2 \\ 0 & -2x-4 \end{pmatrix} y = 0.$$

Then, the result of operation of the randomized variant of algorithm Singsys is polynomial $d(x) = x(x+2)$, which is equal to the greatest common divisor of the polynomial $d_1(x)$ and polynomial $d_2(x) = x(x+2)(x-1)$ corresponding to (48), since other transformations of the system obtained by applying the randomized variant of EG_δ do not result in the reduction of the degree of the revealing polynomial (polynomials $d_3(x) = x(x+2)(x-1)(x^2+4x-2)$ and $d_4(x) = x(x+2)$ correspond to these transformed systems, with $d_4(x)$ being in this case the final result of operation of the randomized variant of algorithm Singsys).

9.3. Heuristics on the Reduction Step

The rows of the matrix V of linear dependencies considered in Section 2.4 are selected in arbitrary order; therefore, various heuristic strategies of such selection on each reduction step are possible. In the implementation presented in Section 11, we applied the following heuristic aimed at the reduction of the number of eliminations to be performed in the system:

1. Among the rows of matrix V that have not been used yet, a row is selected that will be used for the elimination in the equation of the greatest length.

2. If the number of such rows is greater than one, the row with the least number of nonzero entries is selected.

This heuristics makes it possible to slow down the growth of degrees of system coefficients when applying EG_δ and EG_σ.

10. OTHER APPROACHES

There exist other approaches to transforming a given differential or difference system to a form that is convenient for subsequent finding solutions of one or another kind. Such approaches were proposed, for example, by Barkatou, Cluzeau, Pflügel, El Bacha, and Stan in [28–33]. Transformations of systems were considered also by Beckermann, Cheng, and Labahn in [34, 35].

Different approaches have their advantages and disadvantages depending on a particular system. A user may apply algorithms based on different approaches, which improves chances to obtain a solution in complicated cases: sometimes, with one approach, sometimes with another one.

11. PACKAGES `LINEARFUNCTIONALSYSTEMS` AND `EG`

The discussed algorithms were implemented in the computer algebra system Maple [49]. Its standard version includes package `LinearFunctionalSystems`, which contains earlier implementations of many of the above-described algorithms, such as, for example, EG_σ and algorithms for searching various kinds of solutions of differential, difference, and q-difference systems. These algorithms are based on the construction of the induced recurrence systems and the corresponding embracing systems with the help of algorithm EG_σ (at the time of the implementation, this algorithm was called EG'). In particular, the Maple user can invoke the following procedures:

• Starting from version Maple 7,

`SeriesSolution` finds a solution in a series form (initial terms),

`ExtendSeries` extends the solution in the series form the initial terms of which were calculated by means of `SeriesSolution`,

`PolynomialSolution` finds a polynomial solution,

`RationalSolution` finds a rational solution,

`UniversalDenominator` finds a universal denominator.

• Starting from version Maple 10,

Table

	30%	50%
5	3.189	3.578
10	6.468	4.361
20	13.516	11.955
40	29.642	31.875
100	187.375	255.063
250	3305.172	5358.843
500	39183.048	82052.204

`RegularSolution` finds a regular solution (initial terms),

`ExtendRegularSolution` extends the regular solution the initial terms of which were found by `RegularSolution`,

`LogarithmicSolution` finds a logarithmic solution.

Note that the finding of a solution of some kind (for example, a polynomial solution) here implies the finding of a general solution of this kind, which may contain arbitrary constants.

Further development (implementation of new algorithms and improvement of the earlier developed implementations) is carried out in the framework of the new package EG. Its code and examples of using the package are available at the address:

http://www.ccas.ru/ca/doku.php/eg.

This package includes implementations of algorithms EG_δ and Singsys, which were not included in `LinearFunctionalSystems`, as well as algorithms for finding lower bounds for valuations of components of meromorphic solutions of difference systems and rational and logarithmic solutions of differential systems of arbitrary order.

11.1. Experiments

Efficiency of algorithms EG_σ and EG_δ, as well as algorithms for searching various solutions of differential, difference, and q-difference systems based on them, was substantiated by a number of experiments [18, 19, 22, 24], which compared them with the known alternative programs. These experiments demonstrated that our algorithms are able to work with quite large systems.

As an example, we mention two series of experiments for algorithm Singsys. For each series, seven sets consisting of ten differential systems each have been generated. For each set, $m = 10$ and $r = 5, 10, 20, 40, 100, 250, 500$. Coefficients of all systems were random polynomials (the standard Maple command `randpoly(x)` was used; i.e., coefficients of the polynomials were selected from the interval $[-99, 99]$, and their degrees did not exceed 5). The systems were generated such that the number of nonzero coefficients constituted 30% in the first series and 50% in the second series. Results of the experiments are presented in the table. Each cell contains the total time (in seconds) spent on the construction of the resulting polynomials (including the time of the execution of EG_δ) for all systems in the corresponding set of the series.

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