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Denominators of Rational Solutions of Linear Difference Systems of an Arbitrary Order

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Abstract—An algorithm for finding a universal denominator of rational solutions of a system of linear difference equations with polynomial coefficients is proposed. The equations may have arbitrary orders.

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1. INTRODUCTION

Finding rational solutions (i.e., solutions that have the form of rational functions) of linear difference equations and systems is of interest by itself and, in addition, is a part of many computer algebra algorithms.

Let k be a field of characteristic 0. In what follows, we use standard notation $k[x]$ and $k(x)$ for the ring of polynomials and the field of rational functions of x with coefficients from the field k. We consider systems of the form

$$A_r(x)y(x+r) + \ldots + A_1(x)y(x+1) + A_0(x)y(x) = b(x), \quad (1)$$

where

• $A_0(x), A_1(x), \ldots, A_r(x)$ are square matrices of order m with entries from $k[x]$ (which is denoted as $A_0(x), A_1(x), \ldots, A_r(x) \in \mathrm{Mat}_m(k[x])$), with $A_0(x)$ and $A_r(x)$ being nonzero matrices;

• $b(x) = (b_1(x), b_2(x), \ldots, b_m(x))^T \in k[x]^m$ is the right-hand side of the system;

• $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T$ is the column of unknown functions.

The number r is called the order of the system.

The homogeneous system corresponding to (1) is as follows:

$$A_r(x)y(x+r) + \ldots + A_1(x)y(x+1) + A_0(x)y(x) = 0. \quad (2)$$

We assume that equations of the latter system are independent over $k[x, \phi]$, where ϕ is the shift operator

$$\phi(y(x)) = y(x+1).$$

A solution $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T \in k(x)^m$ of system (1) is called rational. If $y(x) \in k[x]^m$, then this solution is also polynomial (a particular case of the rational solution).

Algorithms for finding all rational solutions of normal systems of the first-order equations

$$y(x+1) = A(x)y(x) \quad (3)$$

are well known (see [7, 10–13]). Here, $A(x)$ is an invertible matrix in $\mathrm{Mat}_m(k(x))$. Algorithms from [7, 10–12] are based on finding a universal denominator of rational solutions of the original system (for brevity, we will refer to it as a universal denominator for the system), i.e., a polynomial $U(x) \in k[x]$ such that a rational solution of system (if exists) $y(x) \in k(x)^m$ can be represented as $\frac{1}{U(x)}z(x)$, where $z(x) \in k[x]^m$. Having known a universal denominator, one can make a substitution and transform the original system to a system for $z(x)$ and, then, apply one of the known algorithms for finding polynomial solutions (see, e.g., [5, 7, 11]).

In [12], algorithm $\mathbf{A}_U$ for constructing universal denominators for systems of form (3) was proposed. Its improved version $\mathbf{A}'_U$ proposed in [10] was shown to have lesser complexity than $\mathbf{A}_U$ and the algorithms from [7, 11]. Algorithms $\mathbf{A}_U$ and $\mathbf{A}'_U$, as well as the algorithms from [7, 11], find one and the same universal denominator $U(x)$.

In Section 2, necessary notions are introduced, and the basic idea of algorithm $\mathbf{A}'_U$ is described. In Section 3, we show how this idea can be generalized to the case of systems of form (1). Here, the construction of the so-called embracing systems, which are discussed in Section 2.3, plays an important role. The new algorithm for constructing a universal denominator, together with the algorithm for constructing polynomial solutions [5], yields an efficient way for constructing all rational solutions. To the best of authors' knowledge, this is the first algorithm of this kind that is applicable to arbitrary systems of form (1).

Construction of a universal denominator and rational solutions themselves can be performed by means of a transformation of the original system to a first-order system. In Section 5, we demonstrate advantages of the proposed algorithm compared to the algorithm based on the latter transformation.

2. PRELIMINARIES

2.1 Valuations at Irreducible Polynomials

Let us introduce some notation. The notation $f(x) \perp g(x)$ means that polynomials $f(x), g(x) \in k[x]$ are coprime; if $F(x) \in k(x)$, then $\operatorname{den} F(x)$ is a monic polynomial (with the leading coefficient equal to 1) such that $F(x) = \frac{f(x)}{\operatorname{den} F(x)}$ for some $f(x) \in k[x]$, $f(x) \perp \operatorname{den} F(x)$. The set of monic irreducible polynomials from $k[x]$ is denoted as $\operatorname{Irr}(k[x])$. If $p(x) \in \operatorname{Irr}(k[x])$, $f(x) \in k[x]$, then $\operatorname{val}_{p(x)} f(x)$ is defined to be the greatest $n \in \mathbb{N}$ such that $p^n(x) | f(x)$ ($\operatorname{val}_{p(x)} 0 = \infty$) and $\operatorname{val}_{p(x)} F(x) = \operatorname{val}_{p(x)} f(x) - \operatorname{val}_{p(x)} g(x)$ for $F(x) = \frac{f(x)}{g(x)}$, $f(x), g(x) \in k[x]$. For two arbitrary nonzero rational functions $r(x)$ and $s(x)$ and $p(x) \in \operatorname{Irr}(k[x])$, the following relations hold:

$$\operatorname{val}_{p(x)}(r(x)s(x)) = \operatorname{val}_{p(x)} r(x) + \operatorname{val}_{p(x)} s(x),$$

$$\operatorname{val}_{p(x)}(r(x) + s(x)) \geq \min\{\operatorname{val}_{p(x)} r(x), \operatorname{val}_{p(x)} s(x)\}.$$

If $F(x) = (F_1(x), F_2(x), \ldots, F_m(x))^T \in k(x)^m$, then $\operatorname{den} F(x) = \operatorname{lcm}_{i=1}^{m} \operatorname{den} F_i(x)$ and $\operatorname{val}_{p(x)} F(x) = \min_{i=1}^{m} \operatorname{val}_{p(x)} F_i(x)$, where lcm is the least common multiple of the polynomials.

For an arbitrary matrix $A(x) = (a_{ij}(x)) \in \operatorname{Mat}_m(k(x))$, we define $\operatorname{den} A(x) = \operatorname{lcm}_{i=1}^{m} \operatorname{lcm}_{j=1}^{m} \operatorname{den} a_{ij}(x)$.

2.2 Normal Systems of First-Order Equations

For system (3), we set

$$V(x) = \operatorname{den} A(x-1), \quad W(x) = \operatorname{den} A^{-1}(x).$$

Then, for any rational solution $y(x)$ of this system and any $p(x) \in \operatorname{Irr}(k[x])$, the following inequality holds [12]:

$$\operatorname{val}_{p(x)} y(x) \geq -\min\left\{ \sum_{j \in \mathbb{N}} \operatorname{val}_{p(x+j)} V(x), \sum_{j \in \mathbb{N}} \operatorname{val}_{p(x-j)} W(x) \right\}. \tag{4}$$

By means of this inequality, one can, first, efficiently find a finite set M of irreducible polynomials such that, if the denominator of some rational solution of this system is divided by $p(x) \in \operatorname{Irr}(k[x])$, then $p(x) \in M$ and, second, determine a universal denominator

$$U(x) = \prod_{p(x) \in M} p^{\gamma_{p(x)}}(x), \tag{5}$$

where $\gamma_{p(x)}$ denotes the absolute value of the right-hand side of inequality (4).

Algorithm $\mathbf{A}'_U$ differs from $\mathbf{A}_U$ in that it takes into account the fact that exponents $\gamma_{p(x)}$ may be equal to one another for different (sometimes, many) $p(x)$ differing from one another by a shift by an integer. The input data for $\mathbf{A}'_U$ are polynomials $V(x)$ and $W(x)$, and it does not matter for the algorithm whether they are related to any system of equations.

In Section 3, we show how to find $V(x)$ and $W(x)$ for system (2) in order that the universal denominator for it could be found by algorithm $\mathbf{A}'_U$. Here, construction of the so-called embracing systems for the given system plays an important role.

2.3 Embracing Systems

For any system S of form (1), one can construct an *l-embracing* system $\bar{S}$ of the form

$$\bar{A}_r(x)y(x+r) + \ldots + \bar{A}_1(x)y(x+1) + \bar{A}_0(x)y(x) = \bar{b}(x), \tag{6}$$

with the leading matrix being invertible in $\operatorname{Mat}_m(k(x))$, the set of solutions of which contains all solutions of the system S. Similarly, one can construct a *t-embracing* system $\bar{\bar{S}}$ of the form

$$\bar{\bar{A}}_r(x)y(x+r) + \ldots + \bar{\bar{A}}_1(x)y(x+1) + \bar{\bar{A}}_0(x)y(x) = \bar{\bar{b}}(x), \tag{7}$$

whose trailing matrix is invertible in $\operatorname{Mat}_m(k(x))$ and the set of solutions of which contains all solutions of the system S. Note that elements of the matrices and the right-hand sides in (6) and (7) belong to $k[x]$ and the matrices $\bar{A}_0(x)$ and $\bar{\bar{A}}_r(x)$ may be zero – either one or both of them.

The embracing systems can be constructed by algorithms EG [6] and EG' [8]; algorithm EG' is an improved version of EG.

Remark 1. *If* $\bar{S}$ *and* $\bar{\bar{S}}$ *are l- and t-embracing systems constructed by algorithm EG' for* (1), *then l- and t-embracing systems constructed by algorithm EG' for* (2) *coincide with the homogeneous systems corresponding to* $\bar{S}$ *and* $\bar{\bar{S}}$.

The operation of algorithm EG' consists in successive repetition of two—*reduction* and *shift*—steps. The repetition continues until rows of the leading (trailing) matrix remain linear dependent over k. At the reduction step, dependence coefficients are found; then, the equation corresponding to one of the dependent rows is replaced by a linear combination of other equations, and the row of the leading (trailing) matrix becomes equal to zero. At the shift step, the operator ϕ (accordingly, ϕ^{-1}) is applied to the new equation. Selection of the dependent rows to be replaced in accordance with some simple rules guarantees that the algorithm stops.

3. UNIVERSAL DENOMINATOR OF RATIONAL SOLUTIONS OF AN ARBITRARY ORDER SYSTEM

First, let us consider the homogeneous case (2). If the leading matrix $A_r(x)$ of system (2) is invertible in $\mathrm{Mat}_m(k(x))$, this system can be rewritten as a system of form (3):

$$Y(x+1) = A(x)Y(x),$$

where

$$A(x) = \begin{pmatrix} 0 & I_m & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I_m \\ \hat{A}_0(x) & \hat{A}_1(x) & \dots & \hat{A}_{r-1}(x) \end{pmatrix} \quad (8)$$

($\hat{A}_k(x) = -A_r^{-1}(x)A_k(x)$ and I_m is the identity matrix of order m),

$$Y(x) = (y(x)^T, y(x+1)^T, \dots, y(x+r-1)^T)^T, \quad (9)$$

the matrix $A(x)$ belongs to $\mathrm{Mat}_{rm}(k(x))$, and the vector $Y(x)$ has rm components. If, additionally, the matrix $A_0(x)$ is invertible in $\mathrm{Mat}_m(k(x))$, then $A(x)$ is invertible in $\mathrm{Mat}_{rm}(k(x))$:

$$A^{-1}(x) = \begin{pmatrix} \check{A}_1(x) & \dots & \check{A}_{r-1}(x) & \check{A}_r(x) \\ I_m & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & I_m & 0 \end{pmatrix} \quad (10)$$

($\check{A}_k(x) = -A_0^{-1}(x)A_k(x)$). Hence, if the matrices $A_0(x)$ and $A_1(x)$ are invertible, then the universal denominator for system (2) can be found by applying algorithm $\mathbf{A}'_U$ to

$$V(x) = \mathrm{den}A(x-r), \quad W(x) = \mathrm{den}A^{-1}(x).$$

The use of $A(x-r)$ instead of $A(x-1)$ for $V(x)$ is correct in view of the fact that we need a universal denominator only for components $y_1(x), y_2(x), \dots, y_m(x)$ of the vector $Y(x)$.

Now, let us consider system (2) without the assumption of invertibility of the matrices $A_0(x)$ and $A_r(x)$ in $\mathrm{Mat}_m(k(x))$. For the original system, one can find l- and t-embracing systems of form (6) and (7) with zero right-hand sides. Any solution of the original system will also be a solution to the system

$$\begin{aligned}&\bar{A}_r(x+1)y(x+r+1) + (\bar{A}_{r-1}(x+1)\\ &+ \bar{\bar{A}}_r(x))y(x+r) + \dots + (\bar{A}_0(x+1)\\ &+ \bar{\bar{A}}_1(x))y(x+1) + \bar{\bar{A}}_0(x)y(x) = 0,\end{aligned} \quad (11)$$

whose leading and trailing matrices are invertible in $\mathrm{Mat}_m(k(x))$, which makes it possible to apply the above-described approach. Thus, we arrive at the following lemma.

Lemma 1. *Let $\bar{A}_r(x)$ be a leading matrix of the l-embracing system and $\bar{\bar{A}}_0(x)$ be a trailing matrix of the t-embracing system for the homogeneous system* (2)*. Let $U(x)$ be a result of the application of algorithm $\mathbf{A}'_U$ to $V(x) = \mathrm{den}\bar{A}_r^{-1}(x-r)$, $W(x) = \mathrm{den}\bar{\bar{A}}_0^{-1}(x)$. Then, the polynomial $U(x)$ is a universal denominator for system* (2).

It turns out that, in the construction of universal denominators for nonhomogeneous systems, the right-hand sides can be ignored.

Theorem 1. *Let $U(x)$ be a universal denominator for system* (2) *obtained as described in Lemma* 1*. Let the l- and t-embracing systems for* (2) *be found by algorithm EG'. Then, $U(x)$ is a universal denominator for system* (1) *for any right-hand side $b(x) \in k[x]^m$.*

Proof. Let us write system (11) in the form

$$B_{r+1}(x)y(x+r+1) + \dots + B_1(x)y(x+1) + B_0(x)y(x) = 0,$$

where

$$B_{r+1}(x) = \bar{A}_r(x), \quad B_0(x) = \bar{\bar{A}}_0(x). \quad (12)$$

According to Remark 1, there is a right-hand side $c(x) \in k[x]^m$ such that any solution of system (1) is also a solution to the system

$$B_{r+1}(x)y(x+1) + \dots + B_1(x)y(x+1) + B_0(x)y(x) = c(x). \quad (13)$$

Adding the component $y_{m+1}(x)$ to $y(x)$, we can transform (13) to the homogeneous system

$$\tilde{B}_{r+1}(x)y(x+r+1) + \dots + \tilde{B}_1(x)y(x+1) + \tilde{B}_0(x)y(x) = 0 \quad (14)$$

such that, if $(y_1(x), y(2), \dots, y_m(x))^T$ satisfies system (13), then $(y_1(x), y_2(x), \dots, y_m(x), 1)^T$ satisfies system (14):

$$\tilde{B}_0(x) = \begin{pmatrix} & & -c_1(x) \\ & B_0(x) & \vdots \\ & & -c_m(x) \\ 0 & \dots \ 0 & -1 \end{pmatrix},$$

$$\tilde{B}_1(x) = \begin{pmatrix} & & 0 \\ & B_1(x) & \vdots \\ & & 0 \\ 0 & \dots \ 0 & 1 \end{pmatrix},$$

and

$$\tilde{B}_i(x) = \begin{pmatrix} & & 0 \\ & B_i(x) & \vdots \\ & & 0 \\ 0 & \dots \ 0 & 0 \end{pmatrix},$$

$$i = 2, 3, \dots, r+1.$$

The multiplication of the last equation of system (14) by −1 and the eliminations in the last column of the trailing matrix yield system S_1:

$$\begin{pmatrix} & & 0 \\ & B_{r+1}(x) & \vdots \\ & & 0 \\ 0 & \dots \ 0 & 0 \end{pmatrix} y(x+r+1) + \dots$$

$$+ \dots \begin{pmatrix} & & 0 \\ & B_0(x) & \vdots \\ & & 0 \\ 0 & \dots \ 0 & 1 \end{pmatrix} y(x) = 0.$$

Applying the operator ϕ to all but the last equations of system (14) and operator ϕ^r to the last equation, we obtain system S_2:

$$\begin{pmatrix} & & 0 \\ & B_{r+1}(x+1) & \vdots \\ & & 0 \\ 0 & \dots \ 0 & 1 \end{pmatrix} y(x+r+2) + \dots$$

$$\dots + \begin{pmatrix} & & 0 \\ & B_0(x+1) & \vdots \\ & & 0 \\ 0 & \dots \ 0 & 0 \end{pmatrix} y(x+1) = 0.$$

Adding systems S_1 and S_2 together, we obtain

$$\begin{pmatrix} & & 0 \\ & B_{r+1}(x+1) & \vdots \\ & & 0 \\ 0 & \dots \ 0 & 1 \end{pmatrix} y(x+r+2) + \dots$$

$$\dots + \begin{pmatrix} & & 0 \\ & B_0(x) & \vdots \\ & & 0 \\ 0 & \dots \ 0 & 1 \end{pmatrix} y(x) = 0. \qquad (15)$$

Any solution of system (14) is a solution of system (15); therefore, any universal denominator for (15) is a universal denominator for (14). The denominators of the leading and trailing matrices of system (15) are equal to $\mathrm{den} B_r(x+1)$ and $\mathrm{den} B_0(x)$, respectively. Using (12), we obtain

$$\begin{pmatrix} & & 0 \\ & B_{r+1}(x+1) & \vdots \\ & & 0 \\ 0 & \dots \ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} & & 0 \\ & \bar{A}_r^{-1}(x+1) & \vdots \\ & & 0 \\ 0 & \dots \ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} & & 0 \\ & B_0(x) & \vdots \\ & & 0 \\ 0 & \dots \ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} & & 0 \\ & \bar{\bar{A}}_0^{-1}(x) & \vdots \\ & & 0 \\ 0 & \dots \ 0 & 1 \end{pmatrix}.$$

With regard to this, the application of Lemma 1 to system (15) yields the desired result. □

From Theorem 1, we obtain the following algorithm for constructing a universal denominator for an arbitrary system of form (1):

By means of algorithm EG' (*Section* 2.3), *find l- and t-embracing systems* (6) *and* (7) *for system* (1) *and set* $V(x) = \mathrm{den}\bar{A}_r^{-1}$, $W(x) = \mathrm{den}\bar{\bar{A}}_0^{-1}(x)$. *Then, by means of algorithm* $\mathbf{A}'_U$, *find a universal denominator* (5) *for the original system.*

If, for the given $V(x)$ and $W(x)$, algorithm $\mathbf{A}'_U$ finds a polynomial $U(x)$ and, simultaneously, $V(x)|V'(x)$ and $W(x)|W'(x)$, then, for $V'(x)$ and $W'(x)$, algorithm $\mathbf{A}'_U$ will find a polynomial $U'(x)$ such that $U(x)|U'(x)$. Therefore, instead of $V(x) = \mathrm{den}\bar{A}_r^{-1}(x - r)$ and $W(x) = \mathrm{den}\bar{\bar{A}}_0^{-1}(x)$, one can take $V(x) = \det\bar{A}_r(x - r)$ and $W(x) = \det\bar{\bar{A}}_0(x)$. Such $V(x)$ and $W(x)$ are easier to calculate; however, they can result in a universal denominator of a greater degree. Note also that Theorem 1 reveals an analogy of the proposed algorithm with an algorithm for the scalar case

$$a_r(x)y(x+r) + \ldots + a_1(x)y(x+1) + a_0(x)y(x) = \psi(x),$$

$a_1(x), \ldots, a_{r-1}(x), y(x) \in k[x]$, $a_0(x), a_r(x) \in k[x]\backslash\{0\}$ ([2, 12, 10]): up to constant factors, $\mathrm{den}(a_r(x-r))^{-1} = a_r(x-r)$ and $\mathrm{den}(a_0(x))^{-1} = a_0(x)$. The "scalar" algorithm from [2] can be used for systems as well if we replace $a_r(x-r)$ and $a_0(x)$ with $V(x)$ and $W(x)$. Actually, the latter fact underlies the algorithm for normal systems from [7]. However, as has already been noted, algorithm $\mathbf{A}'_U$ has lesser complexity.

In the proposed algorithm for finding a universal denominator, algorithm EG' is applied to the original system of difference equations rather than to the so-called induced recurrence system, which is satisfied by the coefficients of the expansion of the desired solutions in an appropriate basis. This seems to be the first example of such use of the algorithm in the computer algebra. Note that the differential version EG_δ of algorithm EG' suggested in [4] was applied exactly to the original differential system for finding its rational solutions. This fact justifies once again likeness of the algorithms EG_δ and EG'. To emphasize this, in [4], algorithm EG' was denoted as EG_σ (the notation δ and σ is used, for example, in the theory of the Ore polynomials for mappings possessing differential and shift properties correspondingly).

4. IMPLEMENTATION

The computer algebra system Maple [15] includes package `LinearFunctionalSystems`, which provides procedures for finding solutions of systems of ordinary equations (including difference ones). The package implements algorithms based on the construction of induced recurrence systems and on finding embracing systems for them by means of EG'. Procedure `RationalSolutions` in this package finds rational solutions only for normal systems of form (3). It is based on the algorithm from [7] and uses procedure `UniversalDenominator` from the same package, which is designed for constructing universal denominators for only such systems. In our new implementation, procedure `UniversalDenominator` uses the algorithm described in Section 3 and relies on the implementation of algorithm $\mathbf{A}'_U$ [10] and the implementation of algorithm EG' [9] in the package `LinearFunctionalSystems`. As a result, procedures `UniversalDenominator` and `RationalSolutions` are applicable to any systems of form (1).

Let us demonstrate the operation of these procedures on the example of the system of difference equations presented on the help page of the `RationalSolutions` procedure in the Maple system.

```
> with(LinearFunctionalSystems):
> sys:=
  [(x+3)*(x+6)*(x+1)*(x+5)*x*y1(x+1) -
  (x-1)*(x+2)*(x+3)*(x+6)*(x+1)*
  y1(x)-x*(x^6+11*x^5+41*x^4+65*x^3
  +50*x^2-36)*y2(x)
  +6*(x+2)*(x+3)*(x+6)*(x+1)*x*t4(x),
  (x+6)*(x+2)*y2(x+1)-x^2*y2(x),
  (x+6)*(x+1)*(x+5)*x*y3(x+1)
  +(x+6)*(x+1)*(x-1)*y1(x)-
  x*(x^5+7*x^4+11*x^3+4*x^2-5*x+6)*
  y2(x)-y3(x)*(x+6)*(x+1)*(x+5)*x
  +(x+6)*(x+1)*x*3*(x+3)*y4(x),
  (x+6)*y4(x+1)+x^2*y2(x)
  -(x+6)*y4(x)];
vars:=[y1(x), y2(x), y3(x), y4(x)]:
```

$$\begin{aligned}sys := [&(x+3)(x+6)(x+1)(x+5)xy1(x+1)\\ &-(x-1)(x+2)(x+3)(x+6)(x+1)y1(x)\\ &-x(x^6+11x^5+41x^4+65x^3+50x^2-36)y2(x)\\ &+6(x+2)(x+3)(x+6)(x+1)xy4(x),\\ &(x+6)(x+2)y2(x+1)-x^2y2(x),\\ &(x+6)(x+1)(x+5)xy3(x+1)\\ &+(x+6)(x+1)(x-1)y1(x)\\ &-x(x^5+7x^4+11x^3+4x^2-5x+6)y2(x)\\ &-y3(x)(x+6)(x+1)(x+5)x\\ &+3(x+6)(x+1)x(x+3)y4(x),\\ &(x+6)y4(x+1)+x^2y2(x)-(x+6)y4(x)].\end{aligned}$$

Let us find a universal denominator and rational solutions for this system (they can be found by the old version of the program as well, since the system has form (3) written in an equivalent form free of the denominators).

```
> UniversalDenominator(sys, vars)
```

$$\frac{1}{(x+5)(x+4)^2(x+3)^2(x+2)^3(x+1)^4x^3(x-1)}$$

```
> RationalSolutions(sys, vars)
```

$$\left[\frac{(4(-7108272\_c_2 + c_1)}{(x-1)(x+2)(x+3)(x+4)}, 0,\right.$$

$$(5x^5\_c_2 + 50x^4\_c_2 + 175x^3\_c_2 + 250x^2\_c_2$$

$$- 35\,541\,240x\_c_2 + 5x\_c_1 - 28433088\_c_2$$

$$\left. + 4\_c_1)/(5x(x+1)(x+2)(x+3)(x+4)), 0\right].$$

Now, let us transform the system by shifting three of the four its equations.

```
> sys2:=[eval(sys[1],x=x+1),
         eval(sys[2],x=x+3),
         sys[3],eval(sys[4],x=x+2)];
```

$$sys := [(x+4)(x+7)(x+2)(x+6)(x+1)y1(x+2)$$

$$- x(x+3)(x+4)(x+7)(x+2)y1(x+1)$$

$$- (x+1)((x+1)^6 + 11(x+1)^5 + 41(x+1)^4$$

$$+ 65(x+1)^3 + 50(x+1)^2 - 36)y2(x+1)$$

$$+ 6(x+3)(x+4)(x+7)(x+2)(x+1)y4(x+1),$$

$$(x+9)(x+5)y2(x+4) - (x+3)^2 y2(x+3),$$

$$(x+6)(x+1)(x+5)xy3(x+1)$$

$$+ (x+6)(x+1)(x-1)y1(x)$$

$$- x(x^5 + 7x^4 + 11x^3 + 4x^2 - 5x + 6)y2(x)$$

$$- y3(x)(x+6)(x+1)(x+5)x$$

$$+ 3(x+6)(x+1)x(x+3)y4(x),$$

$$(x+8)y4(x+3) + (x+2)^2 y2(x+2)$$

$$- (x+8)y4(x+2)].$$

The system obtained has the fourth order, and its leading and trailing matrices are not invertible over the field of rational functions. Such a system cannot be solved by the old version of the program. Let us apply the new version:

```
>UniversalDenominator(sys2, vars)
```

$$\frac{1}{(x+5)(x+4)^2(x+3)^2(x+2)^3(x+1)^4x^3(x-1)}$$

```
>RationalSolutions(sys2, vars)
```

$$\left[\frac{(4(-7108272\_c_2 + \_c_1)}{(x-1)(x+2)(x+3)(x+4)}, 0,\right.$$

$$(5x^5\_c_2 + 50x^4\_c_2 + 175x^3\_c_2 + 250x^2\_c_2$$

$$- 35\,541\,240x\_c_2 + 5x\_c_1 - 28433088\_c_2$$

$$\left. + 4\_c_1)/(5x(x+1)(x+2)(x+3)(x+4)), 0\right].$$

As it could be expected, the solution found coincides with the solution of the original system.

Table 1

	$m = 3$	$m = 6$	$m = 9$	$m = 12$
$r = 5$	4.265	14.203	53.109	130.376
$r = 10$	9.812	48.828	234.969	455.719
$r = 15$	37.688	263.484	894.094	1962.578
$r = 20$	254.921	1021.390	5013.687	26160.547

Our experiments demonstrate that our new version of the program is capable of solving systems of rather large size and order. For example, experiments were carried out for which 16 sets consisting of 10 difference systems were generated with $m = 3, 6, 9, 12$ and $r = 5, 10, 15, 20$, respectively. All systems were constructed in such a way that they had randomly generated rational solutions. Rational solutions were sought for all systems in each set by means of the new program. Results of the experiments are presented in Table 1. The numbers in the cells show the total times (in seconds) spent on finding rational solutions for all systems in each set.

Note that, for $m = 12$ and $r = 20$, the major part of the total time (17314.594 s) was spent on one example, which turned out to be more inconvenient for solving than on the average.

5. TRANSFORMATION OF A SYSTEM OF AN ARBITRARY ORDER TO A FIRST-ORDER SYSTEM

If $r = 1$ in (1), i.e., the system has the form

$$A_1(x)y(x+1) + A_0(x)y(x) = b(x) \tag{16}$$

and the rank of the matrix $A_1(x)$ over $k(x)$ is s, $0 < s < m$, then, after the reduction (see Section 2.3), the system can be rewritten as a pair consisting of a linear difference and linear algebraic systems:

$$B_1(x)y(x+1) + B_0(x)y(x) = g(x),$$

$$R(x)y(x) = h(x),$$

where the matrices $B_1(x)$ and $B_0(x)$ have the size $s \times m$, with the rank of $B_1(x)$ being equal to s, and the rank of matrix $R(x)$ of the size $(m - s) \times m$ is equal to $m - s$ by virtue of the fact that equations of the original system are independent over $k[x, \phi]$.

Further, an approach similar in some way to that for the differential case considered in [14] can be applied (see also [4, Section 2.3]). Shifting the system $R(x)y(x) = h(x)$, we obtain

$$R(x+1)y(x+1) = h(x+1). \tag{17}$$

Now, using (17), we eliminate in the system $B_1(x)y(x + 1) + B_0(x)y(x) = g(x)$ some $y_i(x + 1)$ for $m - s$ different values of index i. Next, by means of the equations of the algebraic system $R(x)y(x) = h(x)$, we eliminate unknowns $y_i(x)$ corresponding to the same values of the index. If the rank of the leading matrix of the difference system obtained is less than s, these actions are repeated (go from the difference system to the pair consisting of a difference and algebraic systems), and so on. As a result, we obtain a first-order difference system

$$\tilde{A}_1(x)\tilde{y}(x+1) + \tilde{A}_0(x)\tilde{y}(x) = \tilde{b}(x)$$

with the leading matrix $\tilde{A}_1(x)$ invertible in $\mathrm{Mat}_{\tilde{m}}(k(x))$. If the rank of the matrix $\tilde{A}_0(x)$ is less than $\tilde{m}$, similar actions are applied to the new difference system with the aim to make the trailing matrix $\tilde{A}_0(x)$ invertible (the order of the difference system will further decrease), and so on. Finally, we obtain the difference system

$$\tilde{\tilde{A}}_1(x)\tilde{\tilde{y}}(x+1) + \tilde{\tilde{A}}_0(x)\tilde{\tilde{y}}(x) = \tilde{\tilde{b}}(x) \qquad (18)$$

with invertible leading and trailing matrices of the order $\tilde{\tilde{m}}$ and the relation

$$y(x) = T(x)\tilde{\tilde{y}}(x), \qquad (19)$$

which allows us to express the original unknowns (16) in terms of the unknowns occurring in (18) ($T(x)$ is a matrix of size $m \times \tilde{\tilde{m}}$ with elements from $k(x)$). System (18) can be reduced to form (3). In so doing, the introduction of an additional unknown function allows us to get rid of its right-hand side. Next, any algorithm from those discussed in [7, 10–13] can be applied for finding a universal denominator of its solutions. After this, one can either find all rational solutions of the system obtained by means of the universal denominator found and obtain solutions of the original system (16) by means of (19) or transform the universal denominator found into the universal denominator of the solutions of (16) by using the denominators of coefficients $T(x)$ and, with its help, find all rational solutions of the original system.

The introduction of additional unknown functions (9) allows us to rewrite any system of form (1) as a system of form (16):

$$M_1(x)Y(x+1) + M_0(x)Y(x) = B(x), \qquad (20)$$

where

$$M_1(x) = \begin{pmatrix} I_m & 0 & \dots & 0 \\ 0 & I_m & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_r(x) \end{pmatrix},$$

$$M_0(x) = \begin{pmatrix} 0 & I_m & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I_m \\ A_0(x) & A_1(x) & \dots & A_{r-1}(x) \end{pmatrix},$$

$$B(x) = \begin{pmatrix} 0 \\ \dots \\ 0 \\ b(x) \end{pmatrix}.$$

The matrices $M_1(x)$ and $M_0(x)$ belong to $\mathrm{Mat}_{rm}(k(m))$; the vectors $Y(x)$ and $B(x)$ have rm components. It is known that, for large r, the transition from system (1) to a first-order system is not convenient, since there arise matrices of great size, the operation on which is cumbersome.

As an example, we consider again the fourth-order system from Section 4. For this system, the matrices $M_1(x)$ and $M_0(x)$ in (20) have the size 16×16 and are not invertible. Let us turn to system (18) with invertible matrices of size 4×4 and to the corresponding relation (19) with the matrix $T(x)$ of size 16×4. Our program in Maple performs this for 104.719 s. Further, by means of the algorithm from [10], we find a universal denominator $u(x)$ for the system obtained. Our program described in [10] performs this work for 42.797 s. The construction of the corresponding rational solution takes additional 0.468 s, and the construction of rational solutions of the original system with the help of (19) takes additional 0.079 s. The alternative variant requires 0.266 s to transform $u(x)$ to the universal denominator of the solutions of the original system and additional 6.734 s to construct the corresponding rational solution. In the given case, the first variant turned out faster. It may happen that, in some other cases, the situation will be different; however, from the standpoint of comparison with the algorithm from Section 3, this does not matter: the common part of these variants took much more time than the entire search for a rational solution by the program discussed in Section 4, which required only 1.422 s. We carried out additional experiments to compare the algorithm based on the use of EG' (Section 3) and the algorithm based on the transition to a first-order system, which substantiated this conclusion. In these experiments, 16 sets consisting of 10 difference systems with $m = 5$ and $r = 5, 10, 15, 20$, respectively, were generated. The coefficients of all these systems were polynomials of degree not greater than two with integer roots; the systems were generated in such a way that the number of nonzero coefficients constituted 50%. The results of the experiments are presented in Table 2. The numbers in the cells show the total times (in seconds) spent on finding rational solutions by the compared methods for all systems in each set. The numbers in the paren-

Table 2

	Use of EG'	First order
$r = 5$	6.564	14.186
	(1.609)	(10.000)
$r = 10$	26.281	181.172
	(2.046)	(157.858)
$r = 15$	31.628	402.841
	(2.831)	(328.736)
$r = 20$	21.484	1051.188
	(2.734)	(895.952)

theses show the corresponding total times required for constructing universal denominators (for the approach based on the reduction to a first-order system, it is the time of construction of system (18) and of the universal denominator of its solutions; further, a variant of the construction of a rational solution of this system and the calculation of a rational solution of the original system by means of relation (19) was used). Note that the universal denominator of the solution of each system constructed in the above-described way is not trivial. Accordingly, in spite of the fact that each such a system almost always has only a trivial rational solution, in order to find this out, it is required to run the algorithm of finding rational solutions. Therefore, such experiments are well suited for the comparison of the considered approaches.

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REFERENCES

1. Abramov, S.A., Rational Solutions of Linear Differential and Difference Equations with Polynomial Coefficients, U.S.S.R. Comput. Math. Math. Phys., 1989, vol. 29, no. 6, pp. 7–12.
2. Abramov, S.A., Rational Solutions of Linear Differential and Difference Equations with Polynomial Coefficients, Programming Comput. Software, 1995, vol. 21, no. 6, pp. 273—-278.
3. Abramov, S.A., Gheffar, A., and Khmelnov, D.E., Rational Solutions of Linear Difference Equations: Universal Denominators and Denominator Bounds, *Programming Comput. Software*, 2011, vol. 37, no. 2, pp. 78–86.
4. Abramov, S.A. and Khmelnov, D.E., Singular Points of Solutions of Linear ODE systems with Polynomial Coefficients, *Fund. Applied Math.* (in press).
5. Khmelnov, D.E., Search for Polynomial Solutions of Linear Functional Systems by Means of Induced Recurrences, *Programming Comput. Software*, 2004, vol. 30, no. 2, pp. 61–67.
6. Abramov, S.A., EG-Eliminations, *J. Difference Equations Appl.*, 1999, vol. 5, pp. 393–433.
7. Abramov, S.A. and Barkatou, M., Rational Solutions of First Order Linear Difference Systems, *Proc. of ISSAC'98,* 1995, pp. 290–295.
8. Abramov, S.A. and Bronstein, M., On Solutions of Linear Functional Systems, *Proc. of ISSAC'2001*, ACM, 2001, pp. 1–6.
9. Abramov, S.A., Bronstein, M., and Khmelnov, D.E., Regularization of Linear Recurrence Systems, *Trans. Lyapunov Inst.*, 2003, vol. 4, pp. 158–171.
10. Abramov, S.A., Gheffar, A., and Khmelnov, D.E., Factorization of Polynomials and GCD Computations for Finding Universal Denominators, *Proc. of CASC'2010*, 2010, pp. 4–18.
11. Barkatou, M., Rational Solutions of Matrix Difference Equations: Problem of Equivalence and Factorization, *Proc. of ISSAC'99*, 1999, pp. 277–282.
12. Gheffar, A. and Abramov, S., Valuations of Rational Solutions of Linear Difference Equations at Irreducible Polynomials, *Adv. Appl. Math.*, 2010, pp. 352–364.
13. Van Hoeij, M., Rational Solutions of Linear Difference Equations, *Proc. of ISSAC'98,* 1998, pp. 120–123.
14. Quere, M.P. and Villard, G., An Algorithm for the Reduction of Linear DAE, *Proc. of ISSAC'1995*,1995, pp. 223–231.
15. Maple online help: http://www.maplesoft.com/support/help/