Linear Differential Systems with Infinite Power Series Coefficients

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Abstract. Infinite power series may appear as inputs for certain mathematical problems. This paper examines two possible solutions to the problem of representation of infinite power series: the algorithmic representation (for each series, an algorithm is specified that, given an integer i, finds the coefficient of x^i, — any such algorithm defines a so called computable, or constructive, series) and a representation in an approximate form, namely, in a truncated form.

1 Introduction

Infinite power series play an important role in mathematical studies. Those series may appear as inputs for certain mathematical problems. In order to be able to discuss the corresponding algorithms, we must agree on representation of the infinite series (algorithm inputs are always objects represented by specific finite words in some alphabet). This paper examines two possible solutions to the problem of representation of power series.

In Section 2, we consider the algorithmic representation. For each series in x, an algorithm is specified that, given an integer i, finds the coefficient of x^i. Any deterministic algorithms are allowed (any such algorithm defines a so called computable, or constructive series). Here there is a dissimilarity with the publications [14], [15, Ch. 10], where some specific case of input (mainly the hypergeometric type) is considered, and the coefficients of the power series which are returned by the corresponding algorithms can be given "in closed form".

For example, suppose that a linear ordinary differential system S of arbitrary order with infinite formal power series coefficients is given, decide whether the system has non-zero Laurent series, regular, or formal exponential-logarithmic solutions, and find all such solutions if they exist. If the coefficients of the original systems are arbitrary formal power series represented algorithmically (thus, we are not able, in general, to recognize whether a given series is equal to zero or not) then these three problems are algorithmically undecidable, and this can be deduced from the classical results of A. Turing [21]. But, it turns out that the first two problems are decidable in the case when we know in advance that a given system S is of full rank [5]. However, the third problem (finding formal exponential-logarithmic solutions) is not decidable even in this case [3]. It is shown that, despite the fact that such a system has a basis of formal exponential-logarithmic solutions involving only computable (i.e., algorithmically represented) series, there is no algorithm to construct such a basis. But, it is possible to specify a limited version of the third

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problem, for which there is an algorithm of the desired type: namely, if S and a positive integer d are such that for the system S the existence of at least d linearly independent solutions is guaranteed, then we can construct such d solutions [20].

It is shown also that the algorithmic problems connected with the ramification indices of irregular formal solutions of a given system are mostly undecidable even if we fix a conjectural value ρ of the ramification index [2]. However, there is nearby an algorithmically decidable problem: if a system S of full rank and positive integers ρ, d are such that for S the existence at least of d linearly independent formal solutions of ramification index ρ is guaranteed then one can compute such d solutions of S.

Thus, when we use the algorithmic way of power series representation, a neighborhood of algorithmically solvable and unsolvable problems is observed.

For the solvable problems mentioned above, a Maple implementation is proposed [9]. In Section 2.2, we report some experiments.

Note that the ring of computable formal power series is smaller than the ring of all formal power series because not every sequence of coefficients can be represented algorithmically. Indeed, the set of elements of the constructive formal power series is countable (each of the algorithms is a finite word in some fixed alphabet) while the set of all power series is uncountable.

In Section 3, we consider an "approximate" representation. A well-known example is the results [16] related to the number of terms of entries in A that can influence some components of formal exponential-logarithmic solutions of a differential system $x^s y' = Ay$, where s is a given non-negative integer, A is a matrix whose entries are power series. As a further example we consider matrices with infinite power series entries and suppose that those series are represented in an approximate form, namely, in a truncated form. Thus, it is assumed that a polynomial matrix P which is the l-truncation (l is a non-negative integer, $\deg P = l$) of a power series matrix A is given, and P is non-singular, i.e., $\det P \neq 0$. In [4], it is proven that the question of strong non-singularity, i.e., the question whether P is not the l-truncation of a singular matrix having power series entries, is algorithmically decidable. Assuming that a non-singular power series matrix A (which is not known to us) is represented by a strongly non-singular polynomial matrix P, we give a tight lower bound for the number of initial terms of A^{-1} which can be determined from P^{-1}.

We discuss the possibility of applying the proposed approach to "approximate" linear higher-order differential systems: if a system is given in the approximate truncated form and the leading matrix is strongly non-singular then the results [16, 18] and their generalization can be used, and the number of reliable terms of Laurent series solution can be estimated by the algorithm proposed in [6].

Theorems are known that if a system has a solution in the form of a series, then this system also has a solution in the form of a series with some specific properties such that the initial terms of these series coincide (and estimates of the number of coinciding terms are given), see, e.g., [13]. To avoid misunderstandings, note that this is a different type of task. We are considering a situation where a truncated system is initially given, and we do not know the original system. We are trying to establish, whether it is possible to get from the solutions of this system an information on solutions of any system obtained from this system by a prolongation of the polynomial coefficients to series.

The information that can be extracted from truncated series, matrices, systems, etc. may be sufficient to obtain certain characteristics of the original (untruncated) objects. Naturally, these characteristics are incomplete, but may suffice for some purposes.

In Section 4, we discuss the fact that the width of a given full-rank system S with computable formal power series coefficients can be found, where the width of S is the smallest non-negative integer w such that any l-truncation of S with $l \geqslant w$ is a full-rank system. It is shown also that the above-mentioned value w exists for any full-rank system [5]. We introduce also the notion of the s-width. This is done on the base of the notion of the strong non-singularity.

2 Algorithmic Representation

Definition 1 We suppose that for each series $a(x) = \sum_{i=0}^{\infty} a_i x^i$ under consideration, an algorithm Ξ_a (a procedure, terminating in finitely many steps) such that $a(x) = \sum_{i=0}^{\infty} \Xi_a(i) x^i$, i.e., such that $a_i = \Xi_a(i)\ \forall i$ is given. We will call such series *computable* (or *constructive*).

2.1 Computable Infinite Power Series in the Role of Coefficients of Linear Differential Systems

Let K be a field of characteristic 0. We will use the standard notation $K[x]$ for the ring of *polynomials* in x and $K(x)$ for the field of *rational functions* of x with coefficients in K. Similarly, we denote by $K[[x]]$ the ring of *formal power series* and $K((x)) = K[[x]][x^{-1}]$ its quotient field (the field of *formal Laurent series*) with coefficients in K. The ring of $n \times n$-matrices with entries belonging to a ring (a field) R is denoted by $\mathrm{Mat}_n(R)$.

Definition 2 A ring (field) is said to be *constructive* if there exist algorithms for performing the ring (field) operations and an algorithm for zero testing in the ring (field).

We suppose that the ground field K is a constructive field of characteristic 0.
We write θ for $x\frac{d}{dx}$ and consider differential systems of the form

$$A_r(x)\theta^r y + A_{r-1}(x)\theta^{r-1} y + \cdots + A_0(x) y = 0 \tag{1}$$

where $y = (y_1, \ldots, y_m)^T$ is a column vector of unknown functions, and $y_1, \ldots, y_m$ are the *components* of y.

For the matrices

$$A_0(x), A_1(x), \ldots, A_r(x) \tag{2}$$

we have $A_i(x) \in \mathrm{Mat}_m(K[[x]])$, $i = 0, 1, \ldots, r$, and $A_r(x)$ (the *leading* matrix of the system) is non-zero.

We call elements of the matrices $A_i(x)$ *system coefficients*. As the system coefficients will appear computable series.

It can be deduced from the classical results of A. Turing [21] that

We are not able, in general, to test whether a given computable series is equal to zero or not; for a square matrix whose entries are computable series - to test, whether this matrix is non-singular or not.

However, it turns out that the problems of finding solutions of some types are decidable in the case when we know in advance that a given differential system S is of full rank, i.e., that the equations of the system are linearly independent over $K[\theta]$. Algorithms for constructing local solutions of certain types can be proposed (the components of local solutions either are series in x, or contain such series as constituents). All the involved series are supposed to be formal.

Definition 3 The solutions whose components are formal Laurent series are *Laurent* solutions. The components of a *regular* solution are of the form

$$y_i(x) = \sum_{i=1}^{u} x^{\lambda_i} \sum_{s=0}^{k_i} g_{i,s}(x) \frac{\ln^s x}{s!}, \tag{3}$$

where $u, k_i \in \mathbb{N}$, $\lambda_i \in \bar{K}$ and $g_{i,s}(x) \in \bar{K}((x))^m$ ($\bar{K}$ denotes the *algebraic closure* for K.)

Definition 4 A *proper formal (exponential-logarithmic) solution* of a system is a solution of the form

$$e^{Q(\frac{1}{t})} t^{\lambda} \Phi(t), \quad x = t^{\rho}, \tag{4}$$

where

$\lambda \in \bar{K}$;

$Q(\frac{1}{t})$ is a polynomial in $\frac{1}{t}$ over $\bar{K}$ and the constant term of this polynomial is equal to zero;

ρ is a positive integer;

$\Phi(t)$ is a column vector with components in the form $\sum_{i=0}^{k} g_i(t) \log^i(t)$, and all $g_i(t)$ are power series over $\bar{K}$.

If ρ has the minimal possible value in representation (4) of a proper formal solution then ρ is the *ramification index* of that solution.

A *formal (exponential-logarithmic) solution* is a finite linear combination with coefficients from $\bar{K}$ of proper formal solutions.

Formal exponential-logarithmicis solutions are of a special interest since, e.g., any system of the form $y' = Ay$, where A is an $m \times m$-matrix whose entries are formal Laurent series, has m linearly independent (over $\bar{K}$) formal solutions [19].

The main problems which are considered in this section are the following. Suppose that a linear ordinary differential system S of arbitrary order having the form (1) with computable formal power series coefficients (entries of the matrices $A_i(x)$) is given, test whether the system has

1) non-zero Laurent series,
2) regular, or
3) formal exponential-logarithmic solutions,

and find all such solutions if they exist.

Theorem 1 (i) ([5, 8]) *The first two problems are decidable in the case when we know in advance that a given system S is of full rank, i.e., in the case where the equations of the given system are linearly independent over the ring $K[\theta]$.*

(ii) ([3]) *Despite the fact that such a system has a basis of formal exponential-logarithmic solutions involving only computable series, there is no algorithm to construct such a basis.*

However, it is possible to specify a limited version of the third problem, for which there is an algorithm of the desired type:

Theorem 2 ([20]) *If S and a positive integer d are such that for the system S the existence of at least d linearly independent solutions is guaranteed, we can construct such d solutions.*

It is shown also that the algorithmic problems connected with the ramification indices of irregular formal solutions of a given system are mostly undecidable even if we fix a conjectural value of the ramification index:

Theorem 3 ([2]) *There exists no algorithm which, given a system S with computable power series coefficients and a positive integer ρ, tests the existence of a proper formal solution of ramification index ρ for the system S*

Thus,

When we use the algorithmic way of power series representation, a neighborhood of algorithmically solvable and unsolvable problems is observed.

2.2 Procedures for Constructing Local Solutions

For the solvable problems mentioned above, a Maple [17] implementation as procedures of the package EG was proposed [9]. The package is available from

http://www.ccas.ru/ca/eg

We report some experiments (Figures 1 – 3). The degree of the truncation of the series involved in the solutions returned by our procedures is not less than it is required by the user. That degree can be even bigger: in any case, it is big enough to represent the dimension of the space of the solutions under consideration.

Let $m = 3$ and the system be of the form presented in Fig. 1.

```
> sys := Matrix([[Sum(f(k)*x^k,k=0..infinity),-x^2,-1-x],
                 [-x^2,-Sum(x^k,k=1..infinity),-1-x^3],
                 [-x^3,x,-Sum(x^k,k=0..infinity)]]).y(x)+
         Matrix([[x^2,0,0],[0,1,0],[0,0,1]]).theta(y(x),x,1)+
         Matrix([[x+Sum(x^k,k=3..infinity),0,0],
                 [0,Sum(x^k,k=1..infinity),0],
                 [0,0,Sum(x^k,k=1..infinity)]]).theta(y(x),x,2);
```

$$sys := \begin{bmatrix} \sum_{k=0}^{\infty} f(k)\,x^k & -x^2 & -1-x \\ -x^2 & -\left(\sum_{k=1}^{\infty} x^k\right) & -x^3-1 \\ -x^3 & x & -\left(\sum_{k=0}^{\infty} x^k\right) \end{bmatrix} . \, y(x) + \begin{bmatrix} x^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \, \theta(y(x), x, 1) + \begin{bmatrix} x + \sum_{k=3}^{\infty} x^k & 0 & 0 \\ 0 & \sum_{k=1}^{\infty} x^k & 0 \\ 0 & 0 & \sum_{k=1}^{\infty} x^k \end{bmatrix} . \, \theta(y(x), x, 2)$$

Fig. 1.

Suppose that we define the procedure for computing coefficients of the series $\sum_{k=0}^{\infty} f(k)x^k$ as presented in Fig. 2

```
> f := proc(k)
        if k::integer then
            piecewise(k<0, 0, k <= 1, -1, k = 2, 1, -k^2+k)
        else
            'procname(k)'
        end if
     end proc:
>
```

Fig. 2.

(Thus, $\sum_{k=0}^{\infty} f(k)x^k = -1 - x + x^2 + \sum_{k=3}^{\infty}(-k^2 + k)x^k$.) The results of the search for Laurent, regular and formal solutions are presented on Fig. 3.

The procedure of construction of all formal solutions constructs also all regular, in particular, Laurent solutions. Actually, one procedure EG[FormalSolution] is sufficient in order to obtain solutions of all three types. However, if it is required to construct, say, only Laurent solutions, then it is advantageous to use procedure EG[LaurentSolution], because it will construct them considerably faster, even if the original system has no formal solutions but the Laurent ones. For this reason, we propose three procedures for searching solutions of various types.

In conclusion of this section note that the ring of computable formal power series is smaller than the ring of all formal power series because not every sequence of coefficients can be represented algorithmically. Indeed, the set of elements of the computable formal power series is countable (each of the algorithms is a finite word in some fixed alphabet) while the set of all power series is uncountable.

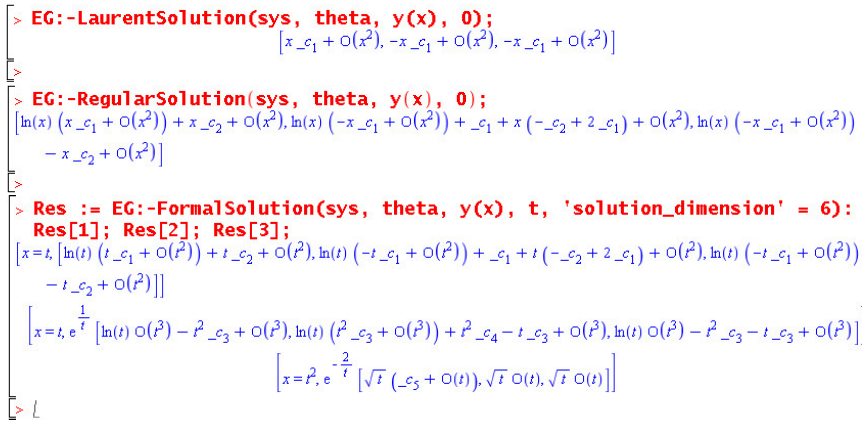

Fig. 3.

3 Approximate (Truncated) Representation

Now, we consider an "approximate" representation of series.

A well-known example [16] is the result by Lutz and Schäfke. It is related to the number of terms of entries of a power series matrix A that can influence initial terms of some constituents of formal exponential-logarithmic solutions of a differential system $x^s y' = Ay$, where s is a non-negative integer.

As a further example ([4]), we consider matrices with infinite power series entries and suppose that those series are represented in an approximate form, namely, in a truncated form.

We start with introducing some notions.

If $l \in \mathbb{Z}, a \in K((x))$ then we define the *l-truncation* $a^{\langle l \rangle}$ which is obtained by omitting all the terms of degree larger than l in a. For a non-zero element $a = \sum a_i x^i$ of $K((x))$, we denote by $\operatorname{val} a$ the *valuation* of a defined by $\operatorname{val} a = \min \{i \text{ such that } a_i \neq 0\}$; by convention, $\operatorname{val} 0 = \infty$.

For $A \in \operatorname{Mat}_n(K((x)))$, we define $\operatorname{val} A$ as the minimum of the valuations of the entries of A. We define the *leading coefficient* of a non-zero matrix $A \in \operatorname{Mat}_n(K((x)))$ as $\operatorname{lc} A = (x^{-\operatorname{val} A} A)|_{x=0}$. For $A \in \operatorname{Mat}_n(K[x])$, we define $\deg A$ as the maximum of the degrees of the entries of A.

The notation A^T is used for the transpose of a matrix (vector) A. I_n is the *identity* $n \times n$-matrix.

Given $A \in \operatorname{Mat}_n(K((x)))$, we define the matrix $A^{\langle l \rangle} \in \operatorname{Mat}_n(K[x, x^{-1}])$ obtained by replacing the entries of A by their l-truncations (if $A \in \operatorname{Mat}_n(K[[x]])$ then $A^{\langle l \rangle} \in \operatorname{Mat}_n(K[x])$).

If $P \in \operatorname{Mat}_n(K[x])$ then any $\hat{P} \in \operatorname{Mat}_n(K[[x]])$ such that $(\hat{P})^{\langle \deg P \rangle} = P$ is a *prolongation* of P.

3.1 Strongly Non-singular Matrices

Definition 5 *A polynomial matrix P which is non-singular, i.e.,* $\det P \neq 0$*, is* strongly non-singular *if P is not the l-truncation ($l = \deg P$) of a singular matrix having power series entries; in other words, P is strongly non-singular if* $\det \hat{P} \neq 0$ *for any prolongation $\hat{P}$ of P.*

It is proven that the question of strong non-singularity is algorithmically decidable. For the answer to this question, the number

$$h = \deg P + \operatorname{val} P^{-1} \tag{5}$$

plays the key role.

Theorem 4 ([4]) *P is strongly non-singular if and only if*

$$\deg P + \operatorname{val} P^{-1} \geqslant 0, \tag{6}$$

i.e., $h \geqslant 0$.

Example 1 If P is a non-singular constant matrix then P is a strongly non-singular due to the latter proposition. However, the matrix

$$\begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix}, \tag{7}$$

is not strongly non-singular:

$$\det \begin{pmatrix} x & x^2 \\ 1 & x \end{pmatrix} = 0. \tag{8}$$

This could be recognized in advance: for (7) we have $\deg P = 1, \operatorname{val} P^{-1} = -2$ (since $\det P = x^2$), and the inequality $h \geqslant 0$ does not hold: $1 - 2 = -1$. □

Assuming that a non-singular power series matrix A (which is not known to us) is represented by a strongly non-singular polynomial matrix P, we give a tight lower bound for the number of initial terms of entries of A^{-1} which can be determined from P^{-1}.

Theorem 5 ([4]) *Let P be a polynomial matrix. If the inequality $h \geqslant 0$ holds then first, for any prolongation $\hat{P}$, the valuations of the determinant and the inverse matrix of the approximate matrix and, resp., of the determinant and the inverse of the prolonged matrix coincide. Second, in the determinants of the approximate and prolonged matrices, the coefficients coincide for $x^{\operatorname{val} \det P}$, as well as h subsequent coefficients (for larger degrees of x). A similar statement holds for the inverse matrix. The bound h is tight.*

Example 2 Let

$$P = \begin{pmatrix} 1+x & 0 \\ 1 & 1-x \end{pmatrix}.$$

Here $h = 1$. The matrix P is strongly non-singular.

Let

$$\hat{P} = \begin{pmatrix} 1+x+x^2+\dots & 0 \\ 1 & 1-x \end{pmatrix}.$$

We have

$$\det P = 1 - x^2 = \underline{1 + 0 \cdot x} - 1 \cdot x^2, \ \ \det \hat{P} = \underline{1 + 0 \cdot x} + 0 \cdot x^2 + \dots$$

We have also:

$$P^{-1} = \begin{pmatrix} 1/(1+x) & 0 \\ -1/(1-x^2) & 1/(1-x) \end{pmatrix} = \begin{pmatrix} \overline{1-x} + x^2 + \dots & \overline{0 + 0 \cdot x} \\ \underline{-1 + 0 \cdot x} - x^2 - \dots & \underline{1+x} + x^2 + \dots \end{pmatrix},$$

$$\hat{P}^{-1} = \begin{pmatrix} 1-x & 0 \\ -1 & 1/1-x \end{pmatrix} = \begin{pmatrix} \overline{1-x} & \overline{0+0\cdot x} \\ \underline{-1 + 0 \cdot x} & \underline{1+x} + x^2 + \dots \end{pmatrix}.$$

□

As a consequence of Theorem 5, if $\operatorname{val} \det P = e$ then $\operatorname{val} \det \hat{P} = e$ and

$$\det P - \det \hat{P} = O(x^{e+h+1}).$$

Similarly, if $\operatorname{val} P^{-1} = e$ then $\operatorname{val} (\hat{P})^{-1} = e$ and

$$P^{-1} - \hat{P}^{-1} = O(x^{e+h+1}).$$

3.2 When Only a Truncated System is Known

In this section, we are interested in the following question. Suppose that for a system S of the form (1) only a finite number of terms of the entries of $A_0(x), A_1(x), \dots, A_r(x)$ is known, i.e., we know not the system S itself but the system $S^{\langle l \rangle}$ for some non-negative integer l. Suppose that we also know that

- $\operatorname{ord} S^{\langle l \rangle} = \operatorname{ord} S$,
- $A_r(x)$ is invertible.

How many terms of Laurent series solutions of S can be determined from the given "approximate" system $S^{\langle l \rangle}$?

We first recall the following result:

Proposition 1 ([6, Prop. 6]) *Let S be a system of the form (1) and*

$$\gamma = \min_i \operatorname{val} \left(A_r^{-1}(x) A_i(x) \right), \quad q = \max\{-\gamma, 0\}.$$

There exists an algorithm that uses only the terms of degree less than

$$rmq + \gamma + \operatorname{val} \det A_r(x) + 1 \tag{9}$$

of the entries of the matrices $A_0(x), A_1(x), \dots, A_r(x)$, and computes a non-zero polynomial (the so called indicial *polynomial* [12, Ch. 4, §8], [10, Def. 2.1], [6, Sect. 3.2]*) $I(\lambda)$ such that:*

- *if $I(\lambda)$ has no integer root then (1) has no solution in $K((x))^m \setminus \{0\}$,*
- *otherwise, let e_*, e^* be the minimal and maximal integer roots of $I(\lambda)$; then the sequence*

$$a_k = rmq + \gamma + \operatorname{val} \det A_r(x) + \max\{e^* - e_* + 1,\ k + (rm - 1)q\}, \tag{10}$$

  $k = 1, 2, \dots,$ is such that for any $e \in \mathbb{Z}$, $k \in \mathbb{Z}^+$ and column vectors

$$c_e, c_{e+1}, \dots, c_{e+k-1} \in K^m,$$

  the system S possesses a solution $y(x) \in K((x))^m$ of the form

$$y(x) = c_e x^e + c_{e+1} x^{e+1} + \dots + c_{e+k-1} x^{e+k-1} + O(x^{e+k}),$$

  if and only if, the system $S^{\langle a_l \rangle}$ possesses a solution $\tilde{y}(x) \in K((x))^m$ such that $\tilde{y}(x) - y(x) = O(x^{e+k})$.

Using the latter proposition we prove

Theorem 6 ([4]) *Let Σ be a system of the form*

$$P_r(x)\theta^r y + P_{r-1}(x)\theta^{r-1} y + \dots + P_0(x) y = 0$$

with polynomial matrices $P_0(x), P_1(x), \dots, P_r(x)$. Let its leading matrix $P_r(x)$ be strongly non-singular. Let

$$d = \deg P_r, \quad p = -\operatorname{val} P_r^{-1}, \quad h = d - p, \quad \gamma = \min_{0 \leqslant i \leqslant r-1} (\operatorname{val} (P_r^{-1} P_i))$$

be such that the inequality

$$h - p - \gamma \geqslant 0$$

holds. Let $I(\lambda)$ be the indicial polynomial of Σ. Let the set of integer roots of $I(\lambda)$ be non-empty, and e_*, e^* be the minimal and maximal integer roots of $I(\lambda)$. Let a non-negative integer k satisfy the equality

$$\max\{e^* - e_* + 1,\, k + (rm-1)q\} = l - rmq - \gamma - \operatorname{val}\det P_r(x). \tag{11}$$

Let $\hat{\Sigma}$ be an arbitrary system of the form (1) such that $\hat{\Sigma}^{\langle l\rangle} = \Sigma$ for $l = \deg \Sigma$ (i.e., $\hat{\Sigma}$ is an arbitrary prolongation of Σ). Then for any $e \in \mathbb{Z}$, the system $\hat{\Sigma}$ possesses a solution

$$\hat{y}(x) \in K((x))^m, \ \ \operatorname{val}\hat{y}(x) = e,$$

if and only if, the system Σ possesses a solution $y(x) \in K((x))^m$ such that

$$y(x) - \hat{y}(x) = O(x^{e+k+1}) \tag{12}$$

(evidently, the equalities $\operatorname{val}\hat{y}(x) = e$ and (12) imply that $\operatorname{val} y(x) = e$).

Example 3 Let

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1-x \end{pmatrix}, \ \ P_0 = \begin{pmatrix} 0 & -1 \\ -x + 2x^2 + 2x^3 + 2x^4 & -2+4x \end{pmatrix}.$$

For the first-order differential system Σ

$$P_1(x)\theta y + P_0(x) y = 0$$

we have

$$d = 1, \ \ p = 0, \ \ h = 1, \ \ \gamma = 0, \ \ I(\lambda) = \lambda(\lambda - 2), \ \ e^* - e_* + 1 = 3.$$

The conditions of Theorem 6 are satisfied.

The general solution of Σ is

$$y_1 = C_1 - C_1 x + C_2 x^2 - C_2 x^3 + 0x^4 + \frac{2C_1}{15}x^5 + \frac{C_1}{30}x^6 + \left(\frac{C_1}{210} + \frac{2C_2}{35}\right)x^7 + \dots,$$

$$y_2 = \qquad - C_1 x + 2C_2 x^2 - 3C_2 x^3 + 0x^4 + \frac{2C_1}{3}x^5 + \frac{C_1}{5}x^6 + \left(\frac{C_1}{30} + \frac{2C_2}{5}\right)x^7 + \dots,$$

where C_1 and C_2 are arbitrary constants.

Equation (11) has the form $\max\{3,\, k\} = 4$, thus

$$k = 4.$$

This means that all Laurent series solutions of any system $\hat{\Sigma}$ of the form

$$A_1(x)\theta y + A_0(x) y = 0 \tag{13}$$

with non-singular matrix A_1 and such that $\hat{\Sigma}^{\langle 4\rangle} = \Sigma$ (we have $\deg \Sigma = 4$) are power series solutions having the form

$$\hat{y}_1 = C_1 - C_1 x + C_2 x^2 - C_2 x^3 + O(x^5),$$

$$\hat{y}_2 = \qquad - C_1 x + 2C_2 x^2 - 3C_2 x^3 + O(x^5),$$

where C_1, C_2 are arbitrary constants. Consider, e.g., the first-order differential system $\hat{\Sigma}$ of the form (13) with

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1-x \end{pmatrix},$$

$$A_0 = \begin{pmatrix} 0 & -1 \\ -x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 + x^7 + x^8 + \dots & -2 + 4x \end{pmatrix}.$$

Its general solution is

$$\hat{y}_1 = \underline{C_1 - C_1 x + C_2 x^2 - C_2 x^3 + 0x^4} + 0x^5 + 0x^6 + \frac{C_1}{35}x^7 + \dots,$$

$$\hat{y}_2 = \quad \underline{-C_1 x + 2C_2 x^2 - 3C_2 x^3 + 0x^4} + 0x^5 + 0x^6 + \frac{C_1}{5}x^7 + \dots,$$

what corresponds to the forecast and expectations. □

Remark 1 The latter example shows that Theorem 6 gives a tight bound for possible value of k: in that example that we cannot take $k+1$ instead of k. Indeed, y_1 contains the term $\frac{2C_1}{15}x^5$, while $\hat{y}_1$ has factually no term of degree 5.

We see that the information that can be extracted from truncated series, matrices, systems, etc. may be sufficient to obtain certain characteristics of the original (untruncated) objects. Naturally, these characteristics are incomplete, but may suffice for some purposes.

In the context of truncated systems we considered only the problem of testing the existence and constructing Laurent series solutions, but we did not discuss similar problems related to regular and formal exponential-logarithmic solutions. We will continue to investigate this line of enquiry.

4 The Width

In conclusion, we discuss a plot which connects both thematic lines of the paper.

Definition 6 ([4, 5]) Let S be a system of full rank over $K[[x]][\theta]$. The minimal integer w such that $S^{\langle l \rangle}$ is of full rank for all $l \geqslant w$ is called the *width* of S The minimal integer w_s such that any system S_1 having power series coefficients and satisfying the condition $S_1^{\langle w_s \rangle} = S^{\langle w_s \rangle}$, is of full rank, is called the *s-width* (the *strong width*) of S.

We will use the notations $w(S), w_s(S)$ when it is convenient.

Any linear algebraic system can be considered as a linear differential system of zero order. This lets us state using the following example that for an arbitrary differential system S we have $w_s(M) \neq w(M)$ in general, however, the inequality

$$w_s(S) \geqslant w(S)$$

holds.

Example 4 Let A be

$$\begin{pmatrix} x & x^3 \\ 1 & x \end{pmatrix}, \tag{14}$$

then

$$w(A) = 1,$$

since $\det A^{\langle 0 \rangle} = 0$ and

$$A^{\langle 1 \rangle} = A^{\langle 2 \rangle} = \begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix}, \ \det \begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix} \neq 0,$$

and $A^{\langle l \rangle} = A$ when $l \geqslant 3$, $\det A \neq 0$. However, $w_s(A) > 1$, due to $\det \begin{pmatrix} x & x^2 \\ 1 & x \end{pmatrix} = 0$. It is easy to check that $w_s(A) = 2$. □

It was proven in [5, Thm 2] that if a system S of the form (1) is of full rank then there exists the width w of S. The value w may be computed if the coefficients of S are represented algorithmically.

As for the idea of the proof from [5], it is shown that the rank-preserving EG-eliminations [1, 7] give a confirmation for the fact that S is of full rank. That confirmation uses only a finite number of the terms of power series which are coefficients of S. For this, the induced recurrent system R is considered (such R is a specific recurrent system for the coefficients of Laurent series solutions of S). This system has polynomial coefficients of degree less than or equal to $r = \operatorname{ord} S$. The system S is of full rank if and only if R is of full rank as a recurrent system. A recurrent system of this kind can be transformed by a special version of the EG-eliminations ([5, Sect.3]) into a recurrent system $\tilde{R}$ whose leading matrix is non-singular. This gives the confirmation mentioned above. It is important that only a finite number of the coefficients of R are involved in the obtained leading matrix of $\tilde{R}$ (due to some characteristic properties of the used version of the EG-eliminations). Each of polynomial coefficients of R is determined from a finite number (bounded by a non-negative integer N) of the coefficients of the power series involved in S. This proves the existence of the width and of the s-width as well. The mentioned number N can be computed algorithmically when all power series are represented algorithmically; thus, in this case we can compute the width of S since we can test ([1, 7, 11]) whether a finite order differential system with polynomial coefficients is of full rank or not. From this point we can consider step-by-step $S^{\langle N-1\rangle}, S^{\langle N-2\rangle}, \dots, S^{\langle 1\rangle}, S^{\langle 0\rangle}$ until appearing the first which is not of full rank. If all the truncated systems are of full rank then $w = 0$.

Concerning the s-width, we get the following theorem

Theorem 7 ([4]) *Let S be a full rank system of the form (1). Then the s-width $w_s(S)$ is defined. If the power series coefficients of S are represented algorithmically then we can compute algorithmically a non-negative integer N such that $w_s(S) \leqslant N$.*

However, it is not exactly clear how to find the minimal value N, i.e., $w_s(S)$. Is this problem algorithmically solvable? The question is still open.

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