Indefinite sums of rational functions *

S.A.Abramov
Computer Center of the Russian Academy of Science
Vavilova 40, Moscow 117967, Russia
abramov@sms.ccas.msk.su

Abstract

We propose a new algorithm for indefinite rational summation which, given a rational function $F(x)$, extracts a rational part $R(x)$ from the indefinite sum of $F(x)$:

$$\sum F(x) = R(x) + \sum H(x).$$

If $H(x)$ is not equal to 0 then the denominator of this rational function has the lowest possible degree. We then solve the same probleme in the q-difference case.

1 The decomposition problem

We discuss here the problem of indefinite summation of rational functions. This problem is equivalent to the problem of solving the difference equation

$$y(x+1) - y(x) = F(x) \tag{1}$$

where $F(x)$ is a rational function over a field K of characteristic 0. The *decomposition problem* is to find whether (1) has a rational solution, and if it does not, then to extract an additive rational part $R(x)$ from the solution s.t. the remaining part satisfies a simpler difference equation, where the denominator of the new right-hand side has the lowest possible degree. This gives an equality

$$\sum F(x) = R(x) + \sum H(x) \tag{2}$$

where $H(x)$ is a rational function whose denominator has the lowest possible degree. Similar algorithms are well known in integration theory: we have in mind especially the algorithms of M.V.Ostrogradsky [Ost1845] and Ch.Hermite [Her1872]. (Some people mistakenly attribute the algorithm of Ostrogradsky to E. Horowitz.)

Note that by $\sum F(x)$ we denote the set of all solutions of (1), and the same for $\sum H(x)$ etc... This is an analogue of the indefinite integral. If those functions take on integer values then we can use some integer bounds $s \le t$ for our sums:

$$\sum_{x=s}^{t} F(x) = R(t+1) - R(s) + \sum_{x=s}^{t} H(x).$$

It is probable that the publication [Abr75] was the first in which the rational and nonrational parts were computed in an algorithmic way. In [Pau93], P.Paule introduced the concept of shift-saturated extension which allows one to give some useful explicit formulas, and presented two new algorithms to construct (2). One works iteratively and is similar to Hermite's algorithm. The other is an analogue of the algorithm of Ostrogradsky. Neither of them requires full factorization of polynomials.

Our old algorithm [Abr75] works iteratively. In the next paragraph we discuss a new algorithm which is an analogue of the algorithm of Ostrogradsky. We will compare it with Paule's algorithm.

2 A new algorithm to solve the decomposition problem

We can assume that $F(x)$ is a proper rational function (the degree of its numerator is lower than the degree of the denominator). If the degree of the numerator of $F(x)$ is not lower than the degree of its denuminator, then one can extract the polynomial part $p(x)$ from $F(x)$:

$$F(x) = p(x) + F^*(x),$$

where $F^*(x)$ is a proper rational function. A polynomial $q(x)$ s.t. $q(x+1) - q(x) = p(x)$ can be found, so our main goal will be finding a solution of the decomposition problem when $F(x)$ is a proper rational function. From here on we will consider only this case.

Consider (1) again. We temporarily replace the coefficient field K by its algebraic closure $\overline{K}$. The partial fraction decomposition of $F(x)$ has the form

$$F(x) = \sum_{i=1}^{m} \sum_{j=1}^{t_i} \frac{\beta_{ij}}{(x-\alpha_i)^j}. \tag{3}$$

Write $\alpha_i \sim \alpha_j$ if $\alpha_i - \alpha_j$ is an integer. Obviously, $\sim$ is an equivalence relation in the set $\{\alpha_1, \ldots, \alpha_m\}$. Each of the corresponding equivalence classes has a largest element in the sense that the other elements of the class are obtained by subtracting positive integers from it. Let $\alpha_1, \ldots, \alpha_k$ be

*Work reported herein was supported in part by the Russian Fund for Fundamental Research under Grant 95-01-01138a.

the largest elements of all the classes ($k \le m$). Then (3) can be rewritten as

$$F(x) = \sum_{i=1}^{k}\sum_{j=1}^{l_i} M_{ij}(E)\frac{1}{(x-\alpha_i)^j}. \tag{4}$$

Here E is the shift operator: $Ef(x) = f(x+1)$ for any function $f(x)$, and $M_{ij}(E)$ is a linear difference operator with constant coefficients (a polynomial in E over $\overline{K}$).

Let $F(x)$ have the form (4) and suppose that (1) possesses a solution $R(x) \in K(x)$. The rational fuction $R(x)$ can be written in a form analogous to (4):

$$\sum_{i=1}^{k}\sum_{j=1}^{l_i} L_{ij}(E)\frac{1}{(x-\alpha_i)^j}. \tag{5}$$

This presentation is unique and therefore

$$L_{ij}(E)(E-1) = M_{ij}(E). \tag{6}$$

So, a necessary and sufficient condition for existence of a rational solution of (1) is that for all $i = 1,\dots,k$; $j = 1,\dots,l_i$ there is an operator $L_{ij}(E)$ s.t. (6) holds. Then, (1) has the solution (5) and all the other rational solutions of (1) can be obtained by adding arbitrary constants.

Example 1 Let

$$F(x) = \frac{-3x+4}{x^3-3x^2+2x} = \frac{-3x+4}{x(x-1)(x-2)} =$$

$$= (2E^2-E-1)\frac{1}{x-2}.$$

We have $2E^2-E-1 = (2E+1)(E-1)$. Therefore (1) in this case has the rational solution

$$R(x) = (2E+1)\frac{1}{x-2} = \frac{3x-5}{(x-1)(x-2)}.$$

Example 2 If the denominator of $F(x)$ is equal to $x(x-1)^2(x-2)(x-4)^3$ then (1) has no rational solution because we have in (4) the term $M(E)\frac{1}{(x-4)^3}$ with $\deg M(E) = 0$.

Let

$$L_{ij}(E) = \tilde{L}_{ij}(E)E^{\gamma_{ij}}$$

where $\tilde{L}_{ij}$ is not divisible by $E, j = 1,\dots,l_i$, and

$$\sigma_{ij} = \min\{\gamma_{ij}, \gamma_{i,j+1},\dots,\gamma_{il_i}\},$$

$$\tau_{ij} = \max\{\deg L_{ij}, \deg L_{i,j+1},\dots,\deg L_{il_i}\};$$

$i = 1,\dots,k; j = 1,\dots,l_i$.

It is easy to see that the polynomial

$$u(x) = \prod_{i=1}^{k}\prod_{j=1}^{l_i}\prod_{m=0}^{\tau_{ij}-\sigma_{ij}}(x-\alpha_i+\sigma_{ij}+m) \tag{7}$$

is divisible by the denominator of (5). The polynomial (7) can be found without full factorization of polynomials. To show that, we note the following: it is obvious that

$$\tau_{i_1}-\sigma_{i_1} = \max\{\tau_{i_1}-\sigma_{i_1},\dots,\tau_{i_{l_i}}-\sigma_{i_{l_i}}\},$$

$i = 1,\dots,k$. Let

$$\rho = \max\{\tau_{11}-\sigma_{11},\dots,\tau_{k_1}-\sigma_{k_1}\}.$$

Let $a(x)$ be the denominator of $F(x)$. Then ρ can be computed as the largest nonnegative integer root h_1 of the polynomial $r(h) = Res_x(a(x-1), a(x+h))$ and if there is no such root then (1) has no solution in $K(x)$; in such a case, as will be made clear below, we may set $R(x) = 0$ and $H(x) = F(x)$ in (2). Let now h_1 be such a root. We can easy find the product of all polynomials of the form

$$\prod_{m=0}^{\tau_{i1}-\sigma_{i1}}(x-\alpha_i+\sigma_{i1}+m)$$

for all i s.t. $\tau_{i1}-\sigma_{i1} = \rho$. Indeed, it is enough to find

$$d(x) = gcd(a(x-1), a(x+h_1))$$

and to compute

$$d(x)d(x-1)\dots d(x-h_1).$$

The process can be continued, yielding the following algorithm to compute (7):

```
A(x) := a(x−1); B(x) := a(x); u(x) := 1;
find all nonnegative integers h_1 > ... > h_m
s.t. deg gcd(A(x), B(x+h_i)) > 0, i = 1,...,m;
let d_i(x) = gcd(A(x), B(x+h_i)), i = 1,...,m;
k := 0;
while m > k do
    k := k+1;
    s_k(x) := d_k(x) d_k(x−1) ... d_k(x−h_k);
    u(x) := u(x) s_k(x);
    j := k;
    for i = k+1, k+2, ..., m do
      d_i(x) := d_i(x)/gcd(d_i(x), s_k(x));
      if deg d_i(x) > 0
        then j := j+1; d_j(x) := d_i(x); h_j := h_i
      fi
    od;
    m := j
od.
```

Thus, we need neither full factorization nor the use of elements of $\overline{K}$ in our computation.

The denominator found by Paule's algorithm [Pau93] is equal to

$$\prod_{i=1}^{k}\prod_{m=0}^{\tau_{i1}-\sigma_{i1}}(x-\alpha_i+\sigma_{i1}+m)^{l_i}. \tag{8}$$

Comparing (7) with (9) shows that the former divides the latter and is of lower degree in general.

Example 3 Let

$$F(x) = -\frac{1}{x} - \frac{2}{(x+1)^2} + \frac{1}{x+1} + \frac{2}{(x+2)^2} - \frac{1}{x+2} +$$

$$+\frac{1}{x+3} + \frac{1}{(x+4)^2} - \frac{1}{(x+5)^2}$$

(the function is taken in the decomposed form for clarity only; the algorithm deals with the denominator

$$a(x) = x(x+1)^2(x+2)^2(x+3)(x+4)^2(x+5)^2 \tag{9}$$

taken in nonfactorized form). In this case

$$R(x) = \frac{1}{x} + \frac{2}{(x+1)^2} + \frac{1}{x+2} - \frac{1}{(x+4)^2},$$

and the denominator of $R(x)$ is

$$x(x+1)^2(x+2)(x+4)^2.$$

All the α_i's in our case are equivalent and $\alpha_1 = 0$ is the largest element:

$$F(x) = (-E^5 + E^4 + 2E^2 - 2E)\frac{1}{x^2} + (E^3 - E^2 + E - 1)\frac{1}{x},$$

$$R(x) = (-E^4 + 2E)\frac{1}{x^2} + (E^2 + 1)\frac{1}{x}.$$

We have $\sigma_{11} = 0, \sigma_{12} = 1, \tau_{11} = \tau_{12} = 4$. Both the above algorithm and formula (7) give the same polynomial $u(x)$:

$$x(x+1)^2(x+2)^2(x+3)^2(x+4)^2.$$

Paule's algorithm gives the denominator

$$(x(x+1)(x+2)(x+3)(x+4))^2.$$

Note that the same polynomial $u(x)$ can be computed by this simpler algorithm (see [Abr95]):

> $A(x) := a(x-1); B(x) := a(x); u(x) := 1;$
> $R(h) := Res_x(A(x), B(x+h));$
> **if** $R(h)$ has some nonnegative integer root **then**
> $N :=$ largest nonnegative integer root of $R(h)$;
> **for** $i = 0, 1, ..., N$ **do**
> $d(x) := gcd(A(x), B(x+i));$
> $A(x) := A(x)/d(x);$
> $B(x) := B(x)/d(x-i);$
> $u(x) := u(x)d(x)d(x-1)\dots d(x-i)$
> **od**
> **fi**.

But the algorithm given earlier is more adequate for our goal of computing not only the denominator of $R(x)$, but the corresponding denominator of $H(x)$ as well: in the earlier algorithm, every product

$$s_k(x) := d_k(x)d_k(x-1)\dots d_k(x-h_k)$$

is such that $d_k(x), \dots, d_k(x-h_k)$ are pairwise relatively prime (we emphasize however, that the polynomial $u(x)$ is the same in both cases). In addition, in our algorithm for constructing the denominator of $H(x)$ together with the denominator of $R(x)$ (see below) we use the polynomials $s_k(x)$ in the order

$$s_m(x), \dots, s_1(x).$$

It is not surprising, then, that all the s_k are stored.

It is likely that this algorithm gives the smallest denominator $u(x)$ that it is possible to predict without taking the numerator into account.

The algorithm [AbrKva93] can be used for computing $d_1(x), \dots, d_k(x)$ efficiently.

Going back to (4) and (6), if at least one polynomial $M_{ij}(E)$ is not divisible by $E-1$ then (1) has no rational solution. We want then to construct (2). Let (4) have only one term, and let us write this term for simplicity in the form

$$M(E)\frac{1}{(x-\alpha)^j},\ j \geq 1.$$

Then our problem can be solved by computing the quotient $L(E)$ and the remainder u:

$$M(E) = L(E)(E-1) + w,\ w \in \overline{K}. \tag{10}$$

We can write the right-hand side of (2) in the form

$$L(E)\frac{1}{(x-\alpha)^j} + \sum \frac{w}{(x-\alpha)^j}.$$

The denominator of the rational function under the sign of the indefinite sum has obviously the lowest possible degree.

Note that instead of (10) one can consider the reduction modulo $E-1$ of the form

$$M(E) = V(E)(E-1) + vE^c, v \in \overline{K},$$

where c is some convenient nonnegative integer. It is easy to see that if $c < \deg M(E)$ then $\deg V(E) \leq \deg L(E)$. If $M(E) = M_{ij}(E)$ then we can take $c = \delta_i$, where δ_i is s.t. M_{i,l_i} is divisible by E^{δ_i} and is not divisible by E^{δ_i+1}. Thus (7) can be used as a denominator of the rational part, i.e. of $R(x)$, and

$$\prod_{i=1}^{k}(x - \alpha_i + \delta_i)^{l_i} \tag{11}$$

as a denominator of $H(x)$. If we compute the numerator of $H(x)$ corresponding to the denominator (11) then we can obtain a reducible fraction. We will get the lowest possible degree denominator after reducing this fraction.

Example 4 Let $F(x)$ have the denominator (7) then formula (11) gives us $(x+1)^2$. The numerator of $F(x)$ can be chosen s.t. $H(x) = 0$ in (2) (as in Example 3). If $H(x)$ is nonzero and if its denominator has lowest possible degree then its degree is 1 or 2. We can choose the presentation (2) s.t. the denominator is equal to $x+1$ in the first case, and to $(x+1)^2$ in the second case. If

$$F(x) = (-E^5 + E^4 + 2E^2 - 2E)\frac{1}{x^2} + (E^3 - E^2 + E + 1)\frac{1}{x}$$

then since

$$-E^5 + E^4 + 2E^2 - 2E = (-E^4 + 2E)(E-1),$$

$$E^3 - E^2 + E + 1 = (E^2 - 1)(E-1) + 2E$$

we can take

$$R(x) = (-E^4 + 2E)\frac{1}{x^2} + (E^2 - 1)\frac{1}{x},$$

$$H(x) = 2E\frac{1}{x}.$$

If

$$F(x) = (E^5 + E^4 + 2E^2 - 2E)\frac{1}{x^2} + (E^3 - E^2 + E + 1)\frac{1}{x}$$

then since

$$E^5 + E^4 + 2E^2 - 2E = (E^4 + 2E^3 + 2E^2 + 4E)(E-1) + 3E$$

we can take

$$R(x) = (E^4 + 2E^3 + 2E^2 + 4E)\frac{1}{x^2} + (E^2 - 1)\frac{1}{x},$$

$$H(x) = 3E\frac{1}{x^2} + 2E\frac{1}{x}.$$

If we already have expressions (7) and (11) for some concrete $F(x)$ then the numerators of $R(x)$ and $H(x)$ can be found by the method of unknown coefficients. To construct a system of linear algebraic equations we use that $R(x)$ and $H(x)$ are proper rational functions and that

$$F(x) = R(x+1) - R(x) + H(x).$$

The functions $R(x), H(x)$ must be reduced after their construction.

How we can construct (11) without the full factorization of $a(x)$, the denominator of $F(x)$? Note that (11) includes all the factors of $a(x)$ which are relatively prime with the least common multple (lcm) of the denominators of $R(x)$ and $R(x+1)$. The lcm is equal to

$$d_1(x+1)\ldots d_m(x+1)u(x).$$

Additionaly, (11) includes some factors of

$$d_1(x-h_1), \ldots, d_m(x-h_m),$$

taken to particular powers.

The computation of (11) can be combined with the computation of (7). But first we have to know how to do the following: Let $g(x), d(x) \in K[x]$, h be a nonnegative integer and

$$s(x) = d(x)d(x-1)\ldots d(x-h).$$

Let

$$d(x), d(x-1), \ldots, d(x-h)$$

be pairwise relatively prime. Let

$$g(x) = p_1^{\alpha_1}(x)p_2^{\alpha_2}(x)\ldots \tag{12}$$

be the full factorization of $g(x)$ over K. Assume that for no pair $p_i(x), p_j(x)$ in (12) $p_i(x)$ equals $p_j(x+l)$ for an integer l. We must extract from $s(x)$ all the irreducible factors which are of the form $p_i(x)$ and from $d(x-h)$ all the irreducible factors which are not of the form $p_i(x-l)$, $0 < l \leq h$. Finally, $g(x)$ must be multiplied by these factors.

All this can be achieved by the procedure, which uses only $g(x)$ itself rather than its factored form (11):

```
procedure nonrational (g(x), d(x), h, s(x));
  f(x) := gcd(g(x), s(x));
  while deg f(x) > 0  do
    g(x) := g(x)f(x);
    s(x) := s(x)/f(x);
    f(x) := gcd(g(x), s(x))
  od;
  e(x) := d(x);
  for l = 1, 2, ..., h  do
    e(x) := e(x)/gcd(e(x), g(x));
    e(x) := e(x - 1)
  od;
  e(x) := e(x)/gcd(e(x), g(x));
  g(x) := g(x)e(x).
```

The complete algorithm can now be given as follows.

input: $a(x)$ is the denominator of the right-hand side of (1);
output: $u(x), g(x)$ are polynomials that can be used as the denominators of $R(x)$ and $H(x)$ (see (2));

```
    A(x) := a(x - 1); B(x) := a(x); u(x) := 1; g(x) := a(x);
    find all nonnegative integers h_1 > ... > h_m
    s.t. deg gcd(A(x), B(x + h_i)) > 0,  i = 1, ..., m;
    let d_i(x) = gcd(A(x), B(x + h_i)), i = 1, ..., m;
    k := 0;
    while m > k  do
      k := k + 1;
      s_k(x) := d_k(x)d_k(x - 1)...d_k(x - h_k);
      u(x) := u(x)s_k(x);
      g(x) := g(x)/gcd(g(x), d_k(x + 1)s_k(x));
      j := k;
      for i = k + 1, k + 2, ..., m  do
        d_i(x) := d_i(x)/gcd(d_i(x), s_k(x));
        if deg d_i(x) > 0
          then j := j + 1; d_j(x) := d_i(x); h_j := h_i
        fi
      od;
      m := j
    od;
    for k = m, m - 1, ..., 1  do
      call nonrational (g(x), d_k(x), h_k, s_k(x))
    od.
```

Remark that we need neither full factorization nor the use of elements of $\overline{K}$ in our computation.

3 The decomposition problem in the q-difference case

A q-analogue of (1) is the equation

$$y(qx) - y(x) = F(x) \tag{13}$$

where $F(x) \in K(x)$, $q \in K$ is not zero and not a root of unity.

The q-analogue of the decomposition problem is to find whether (13) has a rational solution, and if it does not, then to extract an additive rational part $R(x)$ from the solution s.t. the remaining part satisfies a simpler q-difference equation, where the denominator of the new right-hand side has the lowest possible degree. This gives an equality

$$\sum\nolimits_q F(x) = R(x) + \sum\nolimits_q H(x) \tag{14}$$

where $H(x)$ is a rational function whose denominator has the lowest possible degree. Note that by $\sum_q F(x)$ we denote the set of all solutions of (13), and the same for $\sum_q H(x)$ etc... If those functions take on values in $\{1, q, q^2, \ldots\}$ then we can use some integer bounds $s \leq t$ for our sums:

$$\sum_{i=s}^{t} F(q^i) = R(q^{t+1}) - R(q^s) + \sum_{i=s}^{t} H(q^i).$$

Note that the equation

$$y(qx) - y(x) = x^m,$$

where m is a nonzero integer, has the rational solution

$$\frac{1}{q^m - 1}x^m.$$

It is easy to show that if $m = 0$ then that equation has no rational solution. We can write a given rational function $F(x)$ in the form

$$F(x) = a_v x^v + \ldots + a_1 x + a_0 + \frac{a_{-1}}{x} + \ldots + \frac{a_{-w}}{x^w} + F^*(x)$$

where $a_{-w}, \ldots, a_{-1}, a_0, a_1, \ldots, a_v \in K, F^*(x) \in K(x)$ and $F^*(x)$ is a proper rational function which does not have a pole at $x = 0$. After solving the q-decomposition problem (the presentation

$$\sum\nolimits_q F^*(x) = R^*(x) + \sum\nolimits_q H^*(x)$$

will be obtained) we will be able to go from $F^*(x)$ to $F(x)$ by adding the rational function

$$\frac{a_v}{q^v - 1}x^v + \ldots + \frac{a_1}{q-1}x + \frac{qa_{-1}}{(1-q)x} + \ldots + \frac{q^w a_{-w}}{(1-q^w)x^w}$$

to $R^*(x)$ and by adding a_0 to $H^*(x)$. Hence, we can suppose that $F(x)$ is a proper rational function which does not have a pole at $x = 0$. This supposition allows us to transform the algorithm described in §2 to an analogous algorithm for the q-difference case. It can be done simply replacing any shift $x + n$ by $q^n x$.

The complete algorithm can now be given as follows.

input: $a(x)$ is the denominator of the right-hand side of (13);
output: $u(x), g(x)$ are polynomials that can be used as the denominators of $R(x)$ and $H(x)$ (see (14));

```
A(x) := a(q^{-1}x); B(x) := a(x); u(x) := 1; g(x) := a(x);
find all nonnegative integers h_1 > ... > h_m
s.t. deg gcd(A(x), B(q^{h_i}x)) > 0,  i = 1,...,m;
let d_i(x) = gcd(A(x), B(q^{h_i}x)), i = 1,...,m;
k := 0;
while m > k  do
    k := k + 1;
    s_k(x) := d_k(x)d_k(q^{-1}x)...d_k(q^{-h_k}x);
    u(x) := u(x)s_k(x);
    g(x) := g(x)/gcd(g(x), d_k(qx)s_k(x));
    j := k;
    for i = k+1, k+2, ..., m  do
      d_i(x) := d_i(x)/gcd(d_i(x), s_k(x));
      if deg d_i(x) > 0
        then j := j + 1; d_j(x) := d_i(x); h_j := h_i
      fi
    od;
    m := j
od;
for k = m, m-1, ..., 1  do
    call nonrational (g(x), d_k(x), h_k, s_k(x))
od.
```

```
procedure nonrational (g(x), d(x), h, s(x));
  f(x) := gcd(g(x), s(x));
  while deg f(x) > 0  do
    g(x) := g(x)f(x);
    s(x) := s(x)/f(x);
    f(x) := gcd(g(x), s(x))
  od;
  e(x) := d(x);
  for l = 1, 2, ..., h  do
    e(x) := e(x)/gcd(e(x), g(x));
    e(x) := e(q^{-1}x)
  od;
  e(x) := e(x)/gcd(e(x), g(x));
  g(x) := g(x)e(x).
```

The search for the nonnegative integers $h_1, \ldots, h_m$ can be done, for instance, by computing the following polynomial r in q^h:

$$r(q^h) = Res_x(A(x), B(q^h x))$$

and by finding of all $h_1, \ldots, h_m$ s.t. $r(q^{h_i}) = 0$. If $K = K'(q)$, q is an undetermined parameter, then after simplifications we get an algebraic equation whose coefficients are polynomials in q and the constant term of the equation can be assumed nonzero (we are interested in nonzero roots only). All roots of the form q^{h_i} divide the constant term. The search for $h_1, \ldots, h_m$ is also easy when q is a rational number, $K = \mathbf{Q}$, or when $K = \mathbf{Q}(q)$, q is a Gaussian number s.t. $|q| \neq 1$.

A proof of correctness of the above algorithm can be obtained from the proof of the algorithm given in §2: it is enough to change the operator E by the operator Q s.t. $Qf(x) = f(qx)$ for any rational function $f(x)$. The principle of the construction of the analogue of (4) is that we write $\alpha_i \sim \alpha_j$ if $\alpha_i = q^l \alpha_j$, where l is an integer. Each of the corresponding equivalence classes has a largest element in the sense that the other elements of the class are obtained by multiplying it by q to a negative integer power. (The motivation is: $Q^l(x - \alpha_i) = q^l x - \alpha_i = q^l(x - q^{-l}\alpha_i)$.) The analogues of the formulas (4),(5) are formulas in which the operators M_{ij}, L_{ij} are polynomials in Q. The presentation $F(x), R(x)$ in the form is unique because no α_i is equal to zero. Hense $L_{ij}(Q)(Q-1) = M_{ij}(Q)$. The analogues of the key formulas (7),(11) are

$$\prod_{i=1}^{k}\prod_{j=1}^{l_i}\prod_{m=0}^{\tau_{ij}-\sigma_{ij}} (q^{\sigma_{ij}+m}x - \alpha_i)$$

and

$$\prod_{i=1}^{k}(q^{\delta_i}x - \alpha_i)^{l_i}.$$

Example 5 Let

$$F(x) = \frac{(1-3q)x^2 - (2+q)x - 1}{qx^3 + (1+q)x^2 + x}.$$

Let $K = \mathbf{Q}(q)$, q is an undetermined parameter. Then

$$F(x) = -\frac{1}{x} + F^*(x),$$

where

$$F^*(x) = \frac{(1-2q)x - 1}{qx^2 + (1+q)x + 1}.$$

We get

$$R^*(x) = \frac{1}{x+1}, H^*(x) = -\frac{1}{x+1}.$$

For $F(x)$ we have

$$R(x) = R^*(x) - \frac{q}{(1-q)x} = \frac{(1-2q)x - q}{(1-q)x^2 + (1-q)x},$$

$$H(x) = H^*(x) = -\frac{1}{x+1}.$$

Therefore

$$\sum_{i=s}^{t} \frac{(1-3q)q^{2i} - (2+q)q^i - 1}{q^{3i+1} + (1+q)q^{2i} + q^i} =$$

$$= \frac{(1-2q)q^{t+1} - q}{(1-q)q^{2t+2} + (1-q)q^{t+1}} - \frac{(1-2q)q^s - q}{(1-q)q^{2s} + (1-q)q^s} - $$

$$-\sum_{i=s}^{t} \frac{1}{q^i + 1}.$$

References

[Abr75] Abramov, S.A. The rational component of the solution of a first- order linear recurrence relation with a rational right-hand side. U.S.S.R. Comput. Maths. Math. Phys. 15. Transl. from Zh. vychisl. mat. mat. fiz., 15, 1035-1039.

[Abr95] Abramov, S.A. Rational solutions of linear difference and q-difference equations with polynomial coefficients. These Proceedings.

[AbrKva93] Abramov, S.A., Kvashenko, K.Yu. On the greatest common divisor of polynomials which depend on a parameter. Proc. ISSAC'93, 152-156.

[Her1872] Hermite, Ch. Sur L'integration des fractions rationelles. Nouvelles Annales de Mathematiques (2 serie) 11, 145-148.

[Ost1845] Ostrogradsky, M. De l'integration des fractions rationelles. Bull. de la classe physico-mathematique de l'Academie Imperiale des Sciences de Saint-Petersburg. Tome IV, 147-168, 286-300.

[Pau93] Paule, P. Greatest-factorial factorization and symbolic summation I. RISC-Linz Report Series No. 93-02, p.71.