D'Alembertian Solutions of Linear Differential and Difference Equations

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Abstract

D'Alembertian solutions are those expressible as nested indefinite integrals resp. sums of hyperexponential functions. They are a subclass of Liouvillian solutions, and can be constructed by recursively finding hyperexponential solutions and reducing the order. Knowing d'Alembertian solutions of $Ly = 0$, one can write down the corresponding solutions of $Ly = f$ and $L^*y = 0$.

1 Introduction

Let k be a field of characteristic zero, X an indeterminate over k, σ a nonzero field-endomorphism of k, and $\delta : k \to k$ a map satisfying

$$\delta(a+b) = \delta a + \delta b \quad \text{and} \quad \delta(ab) = \sigma(a)\,\delta b + \delta a\, b \qquad \text{for any } a, b \in k. \tag{1}$$

The *left skew polynomial ring* given by σ and δ is the ring $(k[X], +, \cdot)$ of polynomials in X over k with the usual polynomial addition, and the multiplication given by

$$Xa = \sigma(a)X + \delta a \qquad \text{for any } a \in k.$$

To avoid confusing it with the usual commutative polynomial ring $k[X]$, the left skew polynomial ring is denoted $k[X; \sigma, \delta]$. We will denote by $\mathrm{Const}_{\sigma,\delta}(k)$ the *constant subfield* of k (i.e., the set of all $a \in k$ such that $\sigma(a) = a$ and $\delta a = 0$).

Example 1 Let k be a differential field of characteristic 0 with derivation D. Let σ be the identity on k and take $\delta = D$. Then $k[X; \sigma, \delta] = k[D]$, the ring of linear differential operators with coefficients in k. $\square$

Example 2 Let k be a difference field of characteristic 0 with transform (shift) E. Take $\sigma = E$ and $\delta = 0$. Then $k[X; \sigma, \delta] = k[E]$, the ring of linear difference operators with coefficients in k. $\square$

The ring $k[X;\sigma,\delta]$ has no zero divisors and possesses a right Euclidean division algorithm. For any two nonzero polynomials $R, S \in k[X;\sigma,\delta]$, one can compute their greatest common right divisor $\mathrm{gcrd}(R,S)$ and their least common left multiple $\mathrm{lclm}(R,S)$. For details, see [Ore33] or [Bro&Pet94].

For $R, S \in k[X;\sigma,\delta] \setminus \{0\}$, let $T \in k[X;\sigma,\delta]$ be such that $\mathrm{lclm}(R,S) = TS$. Following [Koo91] we write $T = R/S$.

Definition 1 A polynomial $R \in k[X;\sigma,\delta]$ of degree $d \geq 1$ is *completely factorable* if there are first-order polynomials $R_1, R_2, \ldots, R_d \in k[X;\sigma,\delta]$ and $a \in k$ such that $R = aR_1R_2\cdots R_d$.

Theorem 1 *Let* $R, S \in k[X;\sigma,\delta] \setminus \{0\}$*. Then* $\deg R/S = \deg R - \deg \mathrm{gcrd}(R,S)$*. If* $R = R_1R_2\cdots R_k$ *where* $R_1, R_2, \ldots, R_k \in k[X;\sigma,\delta]$ *are irreducible polynomials then* $R/S = (R_1/S_1)(R_2/S_2)\cdots(R_k/S_k)$ *where* $S_k = S$ *and* $S_{j-1} = S_j/R_j$*, for* $j = 2, 3, \ldots, k$*.*

For a proof, see [Ore33, Thm. I/16] and [Koo91, pp. 16–20].

Corollary 1 *Let* $R, S \in k[X;\sigma,\delta] \setminus \{0\}$*. If* R *is completely factorable then so is* R/S*.*

Definition 2 Let $R, S \in k[X;\sigma,\delta]$. R is *similar* to S if $R = S/T$ for some $T \in k[X;\sigma,\delta]$ such that $\deg \mathrm{gcrd}(S,T) = 0$.

Theorem 2 [Ore33, Thm. II/1] *Let* $L \in k[X]$ *be monic. If* $L = R_1R_2\cdots R_k = S_1S_2\cdots S_n$ *are two factorizations of* L *into irreducible factors then* $k = n$ *and the factors are similar in pairs.*

Corollary 2 *A monic factor of a completely factorable polynomial is completely factorable.*

2 Homogeneous equations

Henceforth we limit attention to the rings of differential resp. difference operators $k[X]$ where X is D resp. E. If K is a differential resp. difference extension of k then any element $R \in k[X]$ can be viewed as acting on K, and its kernel $\mathrm{Ker}\, R$ is a linear space over $\mathrm{Const}_X(K)$. Call K *adequate* for R if $\dim \mathrm{Ker}\, R = \mathrm{ord}\, R$. It is well known that for any finite set of operators $\mathcal{R} \subseteq k[X]$ there is an extension K of k which is adequate for all $R \in \mathcal{R}$. If K is adequate for R and S then $\mathrm{Ker}\, R/S = S\, \mathrm{Ker}\, R$ (cf. [Koo91]).

Definition 3 Let k be a differential resp. difference field of characteristic 0 and y an element of a differential resp. difference extension K of k. Then y is *hyperexponential* over k if $y \neq 0$ and $Ry = 0$ for some first-order operator $R \in k[X]$; and *d'Alembertian* over k if $Ry = 0$ for some completely factorable operator $R \in k[X]$.

We will denote the set of all hyperexponential elements over k by $\mathcal{H}_k$, and the set of all d'Alembertian elements over k by $\mathcal{A}_k$.

Theorem 3 $\mathcal{A}_k$ *is a linear space over* $\mathrm{Const}_X(K)$*, for any extension* K *of* k*.*

Proof: Let $a, b \in \mathcal{A}_k$ and $\lambda, \mu \in \mathrm{Const}_X(K)$. Then there are completely factorable operators R and S such that $Ra = Sb = 0$. Since $\mathrm{lclm}(R, S)\,(\lambda a + \mu b) = 0$ and $\mathrm{lclm}(R, S) = (R/S)S$ is completely factorable by Corollary 1, it follows that $\lambda a + \mu b \in \mathcal{A}_k$. □

Lemma 1 *Let $L \in k[X]$. If y is hyperexponential over k then $Ly = ay$ for some $a \in k$. In particular, if $Ly \neq 0$ then Ly is hyperexponential. If y is d'Alembertian over k then so is Ly.*

Proof: If $Ry = 0$ then $(R/L)Ly = \mathrm{lclm}(R, L)y = (L/R)Ry = 0$. As $\mathrm{ord}\,R/L \leq \mathrm{ord}\,R$ and since by Corollary 1, R/L is completely factorable when R is, the Lemma follows. □

Theorem 4 *Let $L \in k[X]$. If the equation $Ly = 0$ has a nonzero d'Alembertian solution then it also has an hyperexponential solution.*

Proof: Let $a \neq 0$ and $La = Ra = 0$ where $R \in k[X]$ is completely factorable. If M is a minimal operator annihilating a then M is a right-hand factor of both L and R. By Corollary 2, M is completely factorable. It follows that L has a first-order right-hand factor, or equivalently, that the equation $Ly = 0$ has an hyperexponential solution. □

Assume that there is an algorithm **H** which takes an operator $L \in k[X]$ as input and returns an hyperexponential solution of the equation $Ly = 0$ if it exists. In the case when k is the field of rational functions $F(x)$ over some field F of characteristic 0 such algorithms are given, e.g., in [Sin91] for differential operators, and in [Pet92] for difference operators. We show that algorithm **H**, used recursively in combination with reduction of order, will construct a basis for the space of all d'Alembertian solutions of $Ly = 0$. We call this algorithm **A**.

Algorithm A

INPUT: A nonzero linear operator $L \in k[X]$.

OUTPUT: A basis for the space $\mathrm{Ker}\,L \cap \mathcal{A}_k$.

1. Call **H** on $Ly = 0$.
   If no hyperexponential solution was found then return $\emptyset$ and stop.
   Otherwise let y_1 be an hyperexponential solution of $Ly = 0$.
2. Let R_1 be the monic first-order operator such that $R_1 y_1 = 0$.
   Divide L by R_1 to obtain $L = L_1 R_1$.
3. Recursively call **A** on L_1. Let the output be $\{z_2, z_3, \ldots, z_m\}$.
4. For $i = 2, 3, \ldots, m$ construct solutions y_i of $R_1 y_i = z_i$.
   Return $\{y_1, y_2, \ldots, y_m\}$ and stop.

Theorem 5 *Algorithm* **A** *returns a basis for the space $Ker\,L \cap \mathcal{H}_1$.*

Proof: Obviously **A** finds a factorization $L = L_m R_m R_{m-1} \cdots R_1$ where $\operatorname{ord} R_i = 1$ and the equation $L_m y = 0$ has no hyperexponential solution, hence by Theorem 4 no nonzero d'Alembertian solution.

We claim that $\operatorname{Ker} L \cap \mathcal{A}_k = \operatorname{Ker} R_m R_{m-1} \cdots R_1$. If $R_m R_{m-1} \cdots R_1 a = 0$ then $La = 0$. Conversely, assume that $a \in \mathcal{A}_k$ and $La = 0$ but $b := R_m R_{m-1} \cdots R_1 a \neq 0$. By Lemma 1, $b \in \mathcal{A}_k$. Since $L_m b = La = 0$, it follows from the previous paragraph that $b = 0$, proving the claim.

It is easy to show that $R_j R_{j-1} \cdots R_1 y_j = 0$ and $R_{j-1} \cdots R_1 y_j \neq 0$ for $1 \leq j \leq m$. Hence $y_j \in \operatorname{Ker} R_m R_{m-1} \cdots R_1 = \operatorname{Ker} L \cap \mathcal{A}_k$ for $1 \leq j \leq m$. Let $\sum_{i=1}^m \alpha_i y_i = 0$ where $\alpha_i \in \operatorname{Const}_X(K)$ for some extension K of k. Successively applying $R_j R_{j-1} \cdots R_1$ for $j = m-1, \ldots, 1$ to both sides shows that $\alpha_m = \ldots = \alpha_1 = 0$. Since $\dim \operatorname{Ker} L \cap \mathcal{A}_k = \dim \operatorname{Ker} R_m R_{m-1} \cdots R_1 = m$, the set $\{y_1, y_2, \ldots, y_m\}$ is a basis for $\operatorname{Ker} L \cap \mathcal{A}_k$. □

Let $L_1 \in k[D]$ be a monic differential operator of order 1. Then the general solution of equation $L_1 y = f$ can be written as

$$y = h_1 \int \frac{f}{h_1} \tag{2}$$

where $h_1 \in \mathcal{H}_k$ is such that $L_1 h_1 = 0$. We write $\int u$ to denote the set of all v such that $Dv = u$. Using (2) repeatedly we see that the general solution of equation $L_n L_{n-1} \cdots L_1 y = f$ where $L_n, L_{n-1}, \ldots, L_1 \in k[D]$ are monic first-order operators can be written as

$$y = h_1 \int \frac{h_2}{h_1} \int \frac{h_3}{h_2} \cdots \int \frac{f}{h_n} \tag{3}$$

where $h_i \in \mathcal{H}_k$ are such that $L_i h_i = 0$.

Let $L_1 \in k[E]$ be a monic difference operator of order 1. Then the general solution of equation $L_1 y = f$ can be written as

$$y = h_1 \sum \frac{f}{E h_1} \tag{4}$$

where $h_1 \in \mathcal{H}_k$ is such that $L_1 h_1 = 0$. We write $\sum u$ to denote the set of all v such that $\Delta v = u$ where $\Delta = E - 1$. Using (4) repeatedly we see that the general solution of equation $L_n L_{n-1} \cdots L_1 y = f$ where $L_n, L_{n-1}, \ldots, L_1 \in k[E]$ are monic first-order operators can be written as

$$y = h_1 \sum \frac{h_2}{E h_1} \sum \frac{h_3}{E h_2} \cdots \sum \frac{f}{E h_n} \tag{5}$$

where $h_i \in \mathcal{H}_k$ are such that $L_i h_i = 0$.

Consider the homogeneous equation

$$Ly = 0 \tag{6}$$

where $L \in k[D]$ and $\operatorname{ord} L = n$. It is well known that if $\varphi \in \mathcal{H}_k$ is a nonzero solution of (6) then the substitution

$$y = \varphi \int u \tag{7}$$

leads to an $(n-1)$-st order equation for u with coefficients in k. The substitution (7) is called *d'Alembert substitution* connected with φ. Similarly, in the difference case the substitution

$$y = \varphi \sum u \tag{8}$$

leads to an $(n-1)$-st order equation for u with coefficients in k.

Assume that the new equation is reduced again using its solution $\varphi_2 \in \mathcal{H}_k$ and so on, until the last d'Alembert substitution connected with $\varphi_r \in \mathcal{H}_k$ produces an equation with no solution in $\mathcal{H}_k$. Then a subspace of solutions of (6) has been found, namely

$$\varphi_1 \int \varphi_2 \ldots \int \varphi_r \int 0 \tag{9}$$

in the differential case, and

$$\varphi_1 \sum \varphi_2 \ldots \sum \varphi_r \sum 0 \tag{10}$$

in the difference case. In [Abr91, Abr93a] it was shown that this subspace is independent of the choice of particular solutions φ_i, and that it is equal to the space of d'Alembertian solutions of (6). Furthermore, (9) and (10) correspond to a factorization

$$L = L_r R_r R_{r-1} \cdots R_1 \tag{11}$$

of L where L_r has no first-order right-hand factor, R_i are first-order operators, and if

$$h_i = \varphi_1 \varphi_2 \cdots \varphi_i \tag{12}$$

in the differential case, or

$$h_i = E^{i-1} \varphi_1 E^{i-2} \varphi_2 \cdots \varphi_i \tag{13}$$

in the difference case, then $R_i h_i = 0$.

Example 3 Consider the equation

$$xy^V - 2y^{IV} + x(x-1)y''' - (x-2)y'' - x^2 y' + xy = 0. \tag{14}$$

Repeated d'Alembert substitution, or algorithm **A**, produces the space of d'Alembertian solutions

$$e^x \int e^{-2x} \int 0 \tag{15}$$

of (14). Alternatively, one can express the *general* solution of the original equation as

$$e^x \int e^{-2x} \int g \tag{16}$$

where g is the general solution of the third-order equation

$$xy''' - (3x+2)y'' + (x^2+3x+4)y' - (x^2+2x+2)y = 0. \tag{17}$$

which has no solution in $\mathcal{H}_k$. Nevertheless, we will demonstrate in Section 4 that by means of d'Alembertian theory, the general solution of (17) and hence of (14) can be described in terms of solutions of a second-order equation. □

If the equation (6) was reduced by means of $\varphi_1, \varphi_2, \ldots, \varphi_r$ and the last d'Alembert substitution connected with φ_r produced an equation $L_r y = 0$, then the general solution of (6) can be written as

$$\varphi_1 \int \varphi_2 \ldots \int \varphi_r \int g \qquad \text{resp.} \qquad \varphi_1 \sum \varphi_2 \ldots \sum \varphi_r \sum g \tag{18}$$

where g is the general solution of equation $L_r y = 0$.

3 Nonhomogeneous equations

Consider the equation

$$Ly = f \tag{19}$$

where $L \in k[X]$ and $\operatorname{ord} L = n$. Let a_n be the leading coefficient of L.

Theorem 6 *If L is completely factorable, and $\varphi_1 \int \varphi_2 \ldots \int \varphi_n \int 0$ (in the differential case), resp. $\varphi_1 \sum \varphi_2 \ldots \sum \varphi_n \sum 0$ (in the difference case) is the general solution of the homogeneous equation $Ly = 0$ then*

$$\varphi_1 \int \varphi_2 \int \cdots \int \varphi_n \int \frac{f}{a_n \varphi_1 \cdots \varphi_n} \tag{20}$$

respectively

$$\sum \varphi_1 \sum \varphi_2 \ldots \sum \varphi_n \sum \frac{1}{a_n E^n \varphi_1 E^{n-1} \varphi_2 \ldots E \varphi_n}$$

is the general solution of (19).

Proof: According to (3), the general solution of (19) is

$$y = h_1 \int \frac{h_2}{h_1} \int \frac{h_3}{h_2} \cdots \int \frac{f}{a_n h_n}$$

where, according to (12), $h_i = \varphi_1 \varphi_2 \cdots \varphi_i$. This implies (20). In the difference case, we use (5) instead of (3), and (13) instead of (12). □

This approach does not require solving systems of linear algebraic equations (as does the method of variation of constants).

Example 4 Let $(x^2 - 1)y'' + 4xy' + 2y = \sin x$. The general solution

$$\frac{x}{x^2 - 1} \int \frac{1}{x^2} \int 0$$

of the corresponding homogeneous equation leads immediately to the general solution of the original equation:

$$\frac{x}{x^2 - 1} \int \frac{1}{x^2} \int x \sin x = \frac{C_1 x + C_2 - \sin x}{x^2 - 1}.$$

□

Now consider the problem of finding all d'Alembertian solutions of (19) assuming that f satisfies $Mf = 0$ for some $M \in k[X]$. Then $MLy = 0$ for any solution y of (19). Use algorithm **A** to write $ML = RA$ where $A = L_r L_{r-1} \cdots L_1$ and $\operatorname{ord} L_i = 1$.

Proposition 1 *With the above notation, equation (19) has a d'Alembertian solution if and only if* $(A/L)f = 0$.

Proof: If $La = f$ and $a \in \mathcal{A}_k$ then $Aa = 0$. Hence $(A/L)f = (A/L)La = (L/A)Aa = 0$. – Conversely, if $(A/L)f = 0$ then $f \in \operatorname{Ker} A/L = L \operatorname{Ker} A$, hence there is an $a \in \operatorname{Ker} A$ such that $f = La$. □

When $(A/L)f = 0$ it is possible to construct a particular d'Alembertian solution a of (19) by expanding it into a power series around some point which is not a singularity of the equation $MLy = 0$.

4 Adjoint equations

The adjoint of a linear differential operator is defined by

$$\left(\sum_{i=0}^{n} a_i D^i\right)^* = \sum_{i=0}^{n} (-1)^i D^i a_i \,.$$

It can be shown that for any $L_1, L_2 \in k[D]$,

$$(L_1^*)^* = L_1, \quad (L_1 L_2)^* = L_2^* L_1^* \,. \tag{21}$$

The adjoint of a linear difference operator is defined by

$$\left(\sum_{i=0}^{n} a_i E^i\right)^* = \sum_{i=0}^{n} E^i a_{n-i}$$

when $a_n, a_0 \neq 0$. Direct computation shows that for any $L_1, L_2 \in k[E]$,

$$(L_1^*)^* = E^n L_1^* E^{-n}, \quad (L_1 L_2)^* = (E^n L_2^* E^{-n}) L_1^* \tag{22}$$

where $n = \deg(L_1)$. Hence in both cases left-hand factors of L correspond to right-hand factors of L^*, and vice versa.

Let $R_1 \in k[X]$ be a monic operator of order 1, and $h_1 \in \mathcal{H}_k$ such that $R_1 h_1 = 0$. Then it is easy to see that

$$L_1^* \frac{1}{h_1} = 0 \tag{23}$$

in the differential case, and

$$L_1^* \frac{1}{E h_1} = 0 \tag{24}$$

in the difference case.

Theorem 7 *Let $L \in k[X]$, $\operatorname{ord} L = n$, and let a_n be the leading coefficient of L. If $\varphi_1 \int \varphi_2 \dots \int \varphi_n \int 0$ (in the differential case), resp. $\varphi_1 \sum \varphi_2 \dots \sum \varphi_n \sum 0$ (in the difference case) is the general solution of equation $Ly = 0$ then*

$$\frac{1}{a_n \varphi_n \varphi_{n-1} \dots \varphi_1} \int \varphi_n \int \varphi_{n-1} \dots \int \varphi_2 \int 0. \tag{25}$$

respectively

$$\frac{1}{a_n E\varphi_n E^2\varphi_{n-1} \dots E^n\varphi_1} \sum E\varphi_n \sum E^2\varphi_{n-1} \dots \sum E^{n-1}\varphi_2 \sum 0 \tag{26}$$

is the general solution of the adjoint equation $L^* y = 0$.

Proof: According to (11),

$$L = a_n R_n R_{n-1} \cdots R_1$$

where R_i are monic first-order operators such that $R_i h_i = 0$. In the differential case, let $R_i^* h_i^* = 0$ where $h_i^* = 1/h_i$ as in (23). Since $L^* = R_1^* R_2^* \cdots R_n^* a_n$, the general solution of $L^* y = 0$ is, by (3),

$$\frac{h_n^*}{a_n} \int \frac{h_{n-1}^*}{h_n^*} \int \dots \frac{h_1^*}{h_2^*} \int 0 = \frac{1}{a_n h_n} \int \frac{h_n}{h_{n-1}} \int \dots \frac{h_2}{h_1} \int 0$$

which, using (12), gives (25).

In the difference case, $L^* = (E^{n-1} R_1^* E^{-(n-1)})(E^{n-2} R_2^* E^{-(n-2)}) \cdots (R_n^*) a_n$. Here we use (24) instead of (23) and (5) instead of (3). We also take into account the fact that R_i^* is not monic but rather has leading coefficient equal to $-E^2 h_i / E h_i$. □

Example 5 Consider again equation (14). The adjoint equation of (17)

$$xy''' + (3x+5)y'' + (x^2+3x+10)y' + (x^2+4x+5)y = 0$$

has general solution

$$\frac{e^{-x}}{x^3} \int \xi_1 \int \xi_2 \int 0, \tag{27}$$

where $\int \xi_1 \int \xi_2 \int 0$ is the general solution of equation

$$x^2 y'' + 4xy' + (x^3 + 6)y = 0 \tag{28}$$

which has no Liouvillian solution. By Theorem 7,

$$\frac{x^2 e^x}{\xi_1 \xi_2} \int \xi_1 \int \xi_2 \int 0$$

is the general solution of (17), hence

$$e^x \int e^{-2x} \int \frac{x^2 e^x}{\xi_1 \xi_2} \int \xi_1 \int \xi_2 \int 0$$

is the general solution of (14). □

Note that this approach is also suitable in the case of a nonhomogeneous equation.

Adjoint equations give convenient possibilities to parallelize the solving process. Consider the original equation $Ly = 0$ together with the adjoint equation $L^*y = 0$ and begin to search for particular solutions of both of them in parallel. If a particular solution of one of them is found then we reduce the order of the equation and begin to search for particular solutions of the new equation and its adjoint. If some branch leads to a an equation of order zero we use formulae (25) and (18) to obtain the general solution of the original equation.

5 Concluding remarks

Call an operator $L \in k[X]$ *rational* if $\operatorname{Ker} L$ has a basis in k. If in the definition of complete factorability the factors are required to be both first-order *and* rational, then all our results remain valid provided that hyperexponential solutions are everywhere replaced by solutions in k, and that in algorithm **A**, in place of algorithm **H** an algorithm for finding solutions in k is used. In the case when $k = F(x)$ such algorithms are given in [Abr89b] both for differential and for difference equations.

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