COMPUTER ALGEBRA LIBRARY FOR THE CONSTRUCTION OF THE MINIMAL TELESCOPERS

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This paper is an exposition of various recently-developed results related to the construction of the minimal telescopers for hypergeometric terms. A Maple library which includes an implementation of these results is described. A comparison between this implementation and other well-known implementations is also given.

1 Introduction

The sequences that are named *hypergeometric terms* (or simply *terms*) are very often involved in various combinatorial sums. The characteristic property of a term $T(k)$ is that the ratio $T(k+1)/T(k)$ is a rational function in k. This rational function, denoted by $\mathcal{C}_k(T)$, is the *certificate* of $T(k)$. A term $T(n,k)$ in two variables n and k has two certificates $\mathcal{C}_n(T) = T(n+1,k)/T(n,k)$ and $\mathcal{C}_k(T) = T(n,k+1)/T(n,k)$. They are named the n-certificate and the k-certificate, respectively. These certificates are rational functions in n and k. By using the notations E_n, E_k for the shift operators w.r.t. n and k, we obtain $\mathcal{C}_n(T) = E_nT/T$ and $\mathcal{C}_k(T) = E_kT/T$.

Given a term $T(n,k)$ as input, Zeilberger's algorithm [11,13], also known as the method of creative telescoping, tries to construct for $T(n,k)$ a *Z*-pair (L,G) which consists of a linear recurrence operator with coefficients which are polynomials in n over $\mathbb{C}$

$$L = a_\rho(n)E_n^\rho + \cdots + a_1(n)E_n^1 + a_0(n)E_n^0, \tag{1}$$

i.e., $L \in \mathbb{C}[n, E_n]$, and a term $G(n,k)$ such that

$$LT(n,k) = (E_k - 1)G(n,k). \tag{2}$$

It is proven in [13] that if there exist Z-pairs for $T(n,k)$ then Zeilberger's algorithm terminates with one of the Z-pairs and the telescoper L in the returned Z-pair is of minimal order. The minimal-order telescoper is unique up to a left-hand factor $P(n) \in \mathbb{C}[n]$, and is called the minimal telescoper. The Z-pair (L, G) where L is the minimal telescoper is called the minimal Z-pair.

Zeilberger's algorithm, named hereafter as $\mathcal{Z}$, has a wide range of applications which include finding closed forms of definite sums of hypergeometric terms, verification of combinatorial identities, and asymptotic estimate [11,13,10].

For a given term $T(n,k)$, it was for a quite long period of time that the question whether there exists a Z-pair for T (or whether $\mathcal{Z}$ terminates in finite time given T as input) was not conclusively answered, although a sufficient condition was known via the "fundamental theorem" [11,12]. It states that a Z-pair for T exists if T is a proper term (see Section 3 for a definition).

$\mathcal{Z}$ uses an item-by-item examination on the order ρ of the telescopers L. It starts with the value of 0 for ρ and increases ρ until it is successful in finding a Z-pair (L, G) for T, provided that such a pair exists. It is easy to observe that this strategy has two deficiencies. First, $\mathcal{Z}$ tries to compute a Z-pair for T when such a pair might not exist (Examples 2, 5). It is well-known that the "fundamental theorem" does not provide a necessary condition for the existence of a Z-pair. Secondly, let ρ be the order of the minimal telescoper for T, then $\mathcal{Z}$ simply wastes resources trying to compute a Z-pair where the guessed orders of the telescopers are less than ρ (Examples 4, 7).

The recent results [1,5,9] supply a theoretical foundation and algorithms to overcome the aforementioned problems. The main focus of this paper is a complete Maple implementation of these algorithms, in particular of a function which constructs the minimal Z-pairs by combining all these algorithms.

We first summarize the results that help guarantee the existence and expedite the construction of the minimal telescopers.

2 Additive Decomposition of Terms

We describe in this section the result on additive decomposition of terms in one variable [2]. This decomposition is used as a basis for the follow-up algorithms.

We consider an additive decomposition of a term $T(k)$ (see Theorem 1 below) assuming that the certificate of $T(k)$ is a rational function in k over an arbitrary field K of characteristic 0.

Definition 1 [3] *Let $R \in K(k)$ be a nonzero rational function. If there exist nonzero polynomials $f_1, f_2, v_1, v_2 \in K[k]$ such that*

(i) $R = F \cdot \frac{E_k V}{V}$ *where* $F = \frac{f_1}{f_2}$, $V = \frac{v_1}{v_2}$, *and* $\gcd(v_1, v_2) = 1$,

(ii) $\gcd(f_1, E_k^h f_2) = 1$ *for all* $h \in \mathbb{Z}$,

then $F \cdot \frac{E_k V}{V}$ *is a* rational normal form (RNF) *of* R.

Note that every rational function has an RNF [3] which in general is not unique. See [3] for a description of a construction of such a form.

As presented in [2], the algorithm to solve the additive decomposition problem for a term $T(k)$ constructs two terms $T_1(k)$, $T_2(k)$ such that

$$T(k) = (E_k - 1)\, T_1(k) + T_2(k), \tag{3}$$

and either T_2 vanishes or $\mathcal{C}_k(T_2)$ has an RNF

$$\frac{f_1}{f_2} \frac{E_k(v_1/v_2)}{(v_1/v_2)} \tag{4}$$

with v_2 of minimal degree. Any RNF of $\mathcal{C}_k(T_2)$ of the form (4) has $v_2 \in K[k]$ of the same minimal degree.

Theorem 1 [2] *Let $T(k)$ be a term and equality (3) be valid for some terms $T_1(k), T_2(k)$. Suppose that $T_2(k) \neq 0$. Let (4) be an RNF of $\mathcal{C}_k(T_2)$. Then (3) is an additive decomposition of $T(k)$ iff for each irreducible p from $K[k]$ such that $p \,|\, v_2$, the following three properties hold:*

$$\mathbf{Pa}: E_k^h p \,|\, v_2 \Rightarrow h = 0, \quad \mathbf{Pb}: E_k^h p \,|\, f_1 \Rightarrow h < 0, \quad \mathbf{Pc}: E_k^h p \,|\, f_2 \Rightarrow h > 0. \tag{5}$$

When working with terms in two variables n and k over $\mathbb{C}$, we can consider n as a parameter, and hence can construct an additive decomposition w.r.t. k:

$$T(n,k) = (E_k - 1)\, T_1(n,k) + T_2(n,k). \tag{6}$$

If (4) is an RNF w.r.t k of $\mathcal{C}_k(T_2)$ with $f_1, f_2, v_1, v_2 \in \mathbb{C}[n,k]$, then for each irreducible $p \in \mathbb{C}[n,k]$ such that $p \,|\, v_2$, properties (5) hold. Here K is $\mathbb{C}(n)$, and $K(k)$ is $\mathbb{C}(n,k)$.

Example 1 Consider the term

$$T(n,k) = -\frac{nk}{nk+1}\binom{n+1}{k} + \binom{n+1}{k+1}.$$

An additive decomposition (T_1, T_2) of $T(n,k)$ w.r.t. k that satisfies (6) is

$$\left(\frac{2nk+2n-1}{2n} \prod_{w=0}^{k-1} \frac{n-w+1}{w+2}, \frac{n^2k+5nk-2k+5n-1}{4n(nk+1)} \prod_{w=0}^{k-1} \frac{n-w+1}{w+3} \right).$$

It follows from (6) that if the term T_2 vanishes, then $(1, T_1)$ is the minimal Z-pair for T. For the remainder of this paper, we assume that T_2 is not identically zero. Also note that a telescoper for T exists iff a telescoper for T_2 exists, and the minimal telescopers for these two terms are equal to each other.

3 Applicability of Zeilberger's Algorithm

This section provides a description of the result related to the applicability of $\mathcal{Z}$ to terms [1].

Definition 2 *A polynomial $\alpha(n,k) \in \mathbb{C}[n,k]$ is* integer-linear *if it has the form $a\,n + b\,k + c$ where $a, b \in \mathbb{Z}$ and $c \in \mathbb{C}$.*

Definition 3 [11,12] *A term $T(n,k)$ is* proper *if it can be written in the form*

$$P(n,k) \frac{\prod_{i=1}^{l} \Gamma(\alpha_i(n,k))}{\prod_{i=1}^{m} \Gamma(\beta_i(n,k))} u^n v^k \quad , \tag{7}$$

where $\alpha_i(n,k), \beta_i(n,k)$ are integer-linear polynomials, $l, m \in \mathbb{N}$, $u, v \in \mathbb{C}$.

The following theorem provides a necessary and sufficient condition for the termination of $\mathcal{Z}$.

Theorem 2 [1] *Let $T(n,k)$ be a term in n and k, and (6) be an additive decomposition of T. Let (4) be an RNF w.r.t. k of $\mathcal{C}_k(T_2)$ with $v_2 \in K[n,k]$. Then a telescoper for $T(n,k)$ exists iff each factor of $v_2(n,k)$ irreducible in $\mathbb{C}[n,k]$ is an integer-linear polynomial, i.e., iff $T_2(n,k)$ is a proper term.*

For a given polynomial $f(n,k) \in \mathbb{C}[n,k]$, a decision procedure for the factorability of f into integer-linear polynomials is described in [6]. This procedure does not require a complete factorization of f into irreducible factors.

Example 2 Consider the term $T(n,k)$ in Example 1. It follows from the computed additive decomposition of T that an RNF of $\mathcal{C}_k(T_2)$ of the form (4) has $v_2 = (n^2 + 5\,n - 2)(nk+1)$ which cannot be written as a product of integer-linear polynomials. This is an example where the input term T is not a proper term, and $\mathcal{Z}$ is not applicable to T either.

Example 3 Consider the term

$$T(n,k) = \frac{1}{nk+n+1}\binom{n}{k+1} + \frac{nk-2k-n+2}{n^2k+2nk^2-nk+2k+n-1}\binom{n}{k}.$$

An additive decomposition of $T(n,k)$ has $v_2 = (2\,k + n - 3)$ which is integer-linear. Hence, even though $T(n,k)$ itself is not proper [4], $\mathcal{Z}$ is applicable to T.

4 Efficient Algorithms to Compute the Minimal Telescopers

Let $T(n,k)$ be a term. In this section we assume that $\mathcal{Z}$ is proven applicable to T. For the case where T is also a rational function in n and k (the class of rational functions is a proper subset of the class of terms), there exists a direct algorithm [9] which constructs the minimal telescoper for T efficiently without using item-by-item examination. For the case where T is a non-rational term, there exists an algorithm [5] which computes a lower bound μ for the order of the telescopers for T. This helps save the time to compute a telescoper of order less than μ.

4.1 Rational Function Case: a Direct Algorithm

Let $T(n,k) \in \mathbb{C}(n,k)$. Consider an additive decomposition of T of the form (6). First one constructs a special form of representation for T_2 as stated in the following theorem.

Theorem 3 [9] *Set*

$$T_2 = \sum_{i=1}^{t}\sum_{j=1}^{m_i} \frac{r_{ij}(n)}{(a_i n + b_i k + c_i)^j}, \quad a_i, b_i \in \mathbb{Z}, b_i > 0, c_i \in \mathbb{C}, \gcd(a_i, b_i) = 1, \tag{8}$$

$r_{ij}(n) \in \mathbb{C}(n)$. Then $T_2(n,k)$ can be represented in the form

$$M_1 F_1 + \cdots + M_s F_s, \tag{9}$$

where each $M_i \in \mathbb{C}(n)[E_n, E_k, E_k^{-1}]$, each $F_i = 1/(a_i n + b_i k + c_i)^{m_i}$ is such that

$$a_i, b_i \in \mathbb{Z},\ b_i > 0,\ c_i \in \mathbb{C},\ \gcd(a_i, b_i) = 1,\ m_i \in \mathbb{N} \setminus \{0\}, \tag{10}$$

and for all $i \neq j$, at least one of the following four relations is not satisfied:

$$m_i = m_j, a_i = a_j, b_i = b_j, c_i - c_j \in \mathbb{Z} \setminus \{0\}. \tag{11}$$

T_2 can be written in the form (8) since $\mathcal{Z}$ is assumed to be applicable to T. Once the representation (9) is constructed, one can compute the minimal telescopers for each member $M_i F_i \in \mathbb{C}(n,k)$ directly and efficiently [9]. The minimal Z-pair for $T_2(n,k)$, and subsequently for $T(n,k)$, can then be constructed using Least Common Left Multiple (LCLM) computation. This direct algorithm is in general more efficient than the original $\mathcal{Z}$. See Example 6 for a result of our experimentation.

4.2 Hypergeometric Case: a Lower Bound

Let $T(n,k)$ be a non-rational term. Consider an additive decomposition of T of the form (6). Since the minimal telescopers for T and T_2 are the same, the focus can be shifted to computing a lower bound for the order of the telescopers for the term T_2. Let an RNF w.r.t. k of $\mathcal{C}_k(T_2)$ be of the form (4). For each irreducible p such that $p \mid v_2$, the three properties **Pa, Pb, Pc** in (5) hold.

Definition 4 [5] *Let $M \in \mathbb{C}[n, E_n]$ be such that $MT_2 \neq 0$, and there exists an RNF $F' \frac{E_k V'}{V'}$, $V' = \frac{v_1'}{v_2'}$ of $\mathcal{C}_k(MT_2)$ such that each of the irreducible factors of v_2' does not have at least one of the three properties* **Pa, Pb, Pc**. *Then M is a* crushing operator *for T_2. The minimal crushing operator is a crushing operator of minimal order.*

It is simple to show that if L is a telescoper for T_2, then L is also a crushing operator for T_2. Hence, the problem of computing a lower bound for the order of the telescopers for T_2 is reduced to the problem of computing a lower bound for the order of the minimal crushing operator for T_2.

Theorem 4 [5] *Let $\mathcal{C}_k(T_2)$ have an RNF w.r.t. k $F(E_k V)/V$ of the form (4), $f_1, f_2, v_1, v_2 \in \mathbb{C}[n,k]$, and $D = d_1(n,k)/d_2(n,k)$ be such that $\mathcal{C}_n(T_2) = D(E_n V)/V$. Let there exist a crushing operator for T_2 of order ρ. Then for each integer-linear factor p of v_2, $\deg_k\, p = 1$, there exists an integer h such that*

$$E_k^h\, p \mid E_n v_2 \cdot E_n^2 v_2 \cdots E_n^{\rho} v_2 \cdot d_2 \cdot E_n d_2 \cdots E_n^{\rho-1} d_2. \tag{12}$$

As a consequence, if ρ_p is the minimal positive value of ρ such that there exists an h satisfying (12), then the order of any crushing operator for T_2 is not less than $\mu = \max_{p \mid v_2} \rho_p$.

Since $\mathcal{Z}$ is assumed to be applicable to the input $T(n,k)$, it follows from Theorem 2 that the polynomial $v_2 \in K[n,k]$ factors into integer-linear polynomials. By [4], the polynomial $d_2 \in K[n,k]$ in Theorem 4 also factors into integer-linear polynomials. An algorithm, called *LowerBound*, which realizes Theorem 4 is described in [5]. Once each of the two polynomials v_2, d_2 is written as a product of integer-linear polynomials (this does require a complete factorization of monic univariate polynomials into irreducible factors, see [9]), the algorithm is reduced to solving bivariate linear diophantine equations, a very inexpensive operation.

Example 4 Consider the term

$$T(n,k) = \frac{(n+k+2)!}{(n^2+k+2)(k+3)!} - \frac{(n+k+1)!}{(n^2+k+1)(2+k)!} + \frac{(n+k)!}{(n+7\,k-2)\,k!}.$$

The computed lower bound for T as the result of applying *LowerBound* to $T(n,k)$ is 7 which is also equal to the order of the minimal telescoper for T.

5 Implementation

We have implemented the algorithms described in Sections 2, 3, 4 in the computer algebra system Maple 7. They are grouped together into a module named `Telescopers`.

```
> eval(Telescopers);
```

module()
export AdditiveDecomposition, IsZApplicable, ZpairDirect,
        LowerBound, Zeilberger, MinimalZpair;
option package;
description "Algorithms to compute the minimal telescopers for
                hypergeometric terms";
end module

5.1 *Functionalities*

The exported local variables indicate the functions that are accessible to users. Due to space limitations, we only briefly describe the functions in the module. See Section 6 for information on how to obtain the specifications for these functions.

- AdditiveDecomposition(T, k) computes an additive decomposition of the term $T(k)$. The output is a list of two elements $[T_1, T_2]$ representing the two terms T_1, T_2 in the decomposition (6);
- IsZApplicable(T, n, k) returns *true* if $\mathcal{Z}$ is applicable to the term $T(n, k)$, *false* otherwise;
- ZpairDirect(T, n, k, E_n) computes the minimal Z-pair for the rational function $T(n, k)$. The output is a list of two elements $[L, G]$ representing the minimal Z-pair (L, G) for T, or an error message if it is proven that $\mathcal{Z}$ is not applicable to T;
- LowerBound(T, n, k) returns $\mu \in \mathbb{N}$ which is the computed lower bound for the order of the telescopers for the term $T(n, k)$, or an error message if it is proven that $\mathcal{Z}$ is not applicable to T;
- Zeilberger(T, n, k, E_n) returns a list of two elements $[L, G]$ representing the minimal Z-pair (L, G) for the input term $T(n, k)$. Note that an upper bound ρ for the order of the telescopers for $T(n, k)$ needs to be specified in advance (the default value is 6). The function returns an error message if no telescoper of order less than or equal to ρ exists.

The main function of the module is MinimalZpair. It has the calling sequence "MinimalZpair(T, n, k, E_n)" where T is a term in n and k, and E_n denotes the shift operator which acts on n. This function combines the functionalities of all functions in the above list. For an input term $T(n,k)$, the execution steps can be described as follows.

1. determine the applicability of $\mathcal{Z}$ to T;
2. if it is proven in step 1 that a Z-pair for T does not exist, return the conclusive error message "There does not exist a Z-pair for T"; Otherwise,
   a. if T is a rational function in n and k, apply the direct algorithm to compute the minimal Z-pair;
   b. T is a non-rational term. First compute a lower bound μ for the order of the telescopers for T. Then compute the minimal Z-pair using $\mathcal{Z}$ with μ as the starting value for the guessed orders.

For case 2b, since the term T_2 in the additive decomposition (6) is "simpler" than T in some sense, we first apply $\mathcal{Z}$ to T_2 to obtain the minimal Z-pair (L, G) for T_2. It is easy to show that $(L, LT_1 + G)$ is the minimal Z-pair for the input term T (Example 7).

Example 5 Consider the term

$$T(n,k) = \frac{1}{nk+1}\binom{2\,n}{2\,k}.$$

Apply MinimalZpair and Zeilberger to $T(n,k)$, and record the time required [a].

MinimalZpair first checks for the applicability of $\mathcal{Z}$ to T (step 1). It recognizes that $\mathcal{Z}$ is not applicable to T and returns the conclusive answer "Error, (in MinimalZpair) There does not exist a Zpair for T" in 0.56 seconds. Zeilberger, on the other hand, does not know if a Z-pair (L, G) for T exists. It tries to compute one and returns the inconclusive answer "Error, (in Zeilberger) No telescoper of order 6 was found" in 30.55 seconds. Since there does not exist a Z-pair for T, the higher the value of the upper bound for the order of L is set, the more time and memory are wasted.

Example 6 (rational function case) This example is a comparison between the original Zeilberger's algorithm and the direct algorithm (case 2a of MinimalZpair). The test samples are the same as those used in Example 5 in [9]. Three set of tests each of which consists of 20 rational functions in n

[a] All the reported timings were obtained on a 400Mhz SUN SPARC SOLARIS with 1Gb RAM.

and k were randomly generated. Each rational function where the numerator and the denominator are in expanded forms is of the form (9). We ran MinimalZpair ($\mathcal{M}$), Zeilberger ($\mathcal{Z}$) on these tests, and collected resource requirements. We also enforced a limit of 2,000 seconds on each input rational function in the tests. Note that we only recorded the time and space requirements for the tests that ran under this time limit.

Table 1 shows the time and space requirements to run the three sets of tests S_1, S_2 and S_3.

Table 1. Time and space requirements for $\mathcal{M}$ and $\mathcal{Z}$ (Example 6).

	Completed		Timing (seconds)		Memory (kilobytes)	
	$\mathcal{M}$	$\mathcal{Z}$	$\mathcal{M}$	$\mathcal{Z}$	$\mathcal{M}$	$\mathcal{Z}$
S_1	20	15	12.15	3127.84	54,159	8,095,930
S_2	20	18	12.43	2635.94	54,653	7,873,146
S_3	20	0	959.07	–	3,864,026	–

Example 7 (hypergeometric case) For $b \in \mathbb{N} \setminus \{0\}$, $j \in \{1, 3\}$, let

$$T_1 = \frac{1}{(nk-1)(n-bk-2)^j(2n+k+3)!}, \; T_2 = \frac{1}{(n-bk-2)(2n+k+3)!}.$$

Consider the term $T(n,k) = (E_k - 1)\, T_1(n,k) + T_2(n,k)$. This example is a comparison between the original Zeilberger's algorithm and case 2b of MinimalZpair. The computed lower bound for the order of the telescopers is b, while the order of the minimal telescoper is $b+1$. Let $\mu \in \mathbb{N}$ be the starting value for the guessed order of the telescopers. Recall that the function Zeilberger applies $\mathcal{Z}$ to the input term T with $\mu = 0$, while MinimalZpair applies $\mathcal{Z}$ to the term T_2 in the decomposition (6) with $\mu = b$. Table 2 shows the time and space requirements. As one can easily notice, as b and/or j increase, the relative performance of Zeilberger (compared to MinimalZpair) quickly worsens.

5.2 A Comparison

There exist different Maple implementations of $\mathcal{Z}$ such as `Zeil` in the EKHAD package [11], `sumrecursion` in the sumtools package [8], `SummandToRec` in the HYPERG package [7]. A Mathematica implementation (the function `Zb`) is described in [10]. Due to the lack of a criterion for the applicability of $\mathcal{Z}$ at the time these programs were implemented, the item-by-item examination strategy is employed (these programs are in principal equivalent to the program

Table 2. Time and space requirements for $\mathcal{M}$ and $\mathcal{Z}$ (Example 7).

		Timing (seconds)		Memory (kilobytes)	
j	b	$\mathcal{M}$	$\mathcal{Z}$	$\mathcal{M}$	$\mathcal{Z}$
1	1	6.49	5.35	27,838	24,702
	2	8.34	34.64	33,066	142,889
	3	11.13	124.53	44,233	535,736
	4	14.46	570.02	56,410	1,882,730
	5	25.79	2999.22	97,506	6,536,309
3	1	14.64	16.40	62,566	73,830
	2	17.24	228.59	68,304	770,529
	3	20.15	1286.51	78,701	3,074,051
	4	24.08	8771.08	91,844	10,766,646
	5	38.60	77663.68	139,823	33,423,168

`Zeilberger` in our package). This strategy leads to the two deficiencies which are discussed in Section 1, and which are illustrated by the examples in this paper.

For the case where the input is a rational function, a program such as `Zb` "accepts an input if the irreducible factors of the denominator are integer-linear" [10]. This is equivalent to the condition that the input be a proper term. By Theorem 2, such a program prevents the computation of a Z-pair when such a pair exists. Note that we also implemented in the program `MinimalZpair` a direct and efficient algorithm to compute the minimal Z-pairs.

For the case where the input $T(n, k)$ is a non-rational term, all the aforementioned programs apply $\mathcal{Z}$ directly to T. On the other hand, MinimalZpair first computes a lower bound μ for the order of the telescopers (a fairly low-cost operation), and then applies $\mathcal{Z}$ to the term T_2 in the additive decomposition (6) using μ as the starting value for the guessed orders of the telescopers (note that the existence of a Z-pair is guaranteed). The minimal Z-pair for T can then be easily obtained. Experimentation shows that this proposed approach helps expedite the construction of the minimal Z-pairs.

6 Availability

Information on the availability of the library archive, functional specifications, test samples used in this paper can be found at the URL

`www.scg.math.uwaterloo.ca/~hqle/code/Telescopers/Telescopers.html`

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