Hypergeometric summation revisited

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Abstract

We consider hypergeometric sequences, i.e., the sequences which satisfy linear first-order homogeneous recurrence equations with relatively prime polynomial coefficients. Some results related to necessary and sufficient conditions are discussed for validity of discrete Newton-Leibniz formula $\sum_{k=v}^{w} t(k) = u(w+1) - u(v)$ when $u(k) = R(k)t(k)$ and $R(k)$ is a rational solution of Gosper's equation.

1 Introduction

Let K be a field of characteristic zero ($K = \mathbb{C}$ in all examples). If $t(k) \in K(k)$ then the *telescoping equation*

$$u(k+1) - u(k) \;=\; t(k) \tag{1}$$

may or may not have a rational solution $u(k)$, depending on the type of $t(k)$. Here the telescoping equation is considered as an equality in the rational-function field, regardless of the possible integer poles that $u(k)$ and/or $t(k)$ might have.

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An algorithm for finding rational $u(k)$ was proposed in 1971 (see [1]). It follows from that algorithm that if $t(k)$ has no integer poles, then a rational $u(k)$ satisfying (1), if it exists, has no integer poles either, and the *discrete Newton-Leibniz formula*

$$\sum_{k=v}^{w} t(k) = u(w+1) - u(v) \tag{2}$$

is valid for any integer bounds $v \leq w$. Working with polynomial and rational functions we will write $f(k) \perp g(k)$ for $f(k), g(k) \in K[k]$ to indicate that $f(k)$ and $g(k)$ are coprime; if $R(k) \in K(k)$, then $\mathrm{den}(R(k))$ is the monic polynomial from $K[k]$ such that $R(k) = \frac{f(k)}{\mathrm{den}(R(k))}$ for some $f(k) \in K[k], f(k) \perp \mathrm{den}(R(k))$.

The problem of solving equation (1) can be considered for sequences. If $t(k)$ is a sequence, we use the symbol E for the shift operator w.r. to k, so that $Et(k) = t(k+1)$. In the rest of the paper we assume that the sequences under consideration are defined on an infinite interval I of integers and either $I = \mathbb{Z}$, or

$$I = \mathbb{Z}_{\geq l} = \{k \in \mathbb{Z} \mid k \geq l\}, \;\; l \in \mathbb{Z}.$$

If a sequence $t(k)$ defined on I is given, and a sequence $u(k)$, which is also defined on I and satisfies (1) for all $k \in I$, is found (any such sequence is a *primitive* of $t(k)$), then we can use formula (2) for any $v \leq w$ with $v, w \in I$.

Gosper's algorithm [6], which we denote hereafter by $\mathcal{GA}$, discovered in 1978, focuses on the case where a given $t(k)$ and an unknown $u(k)$ are hypergeometric sequences.

Definition 1 *A sequence $y(k)$ defined on an infinite interval I is hypergeometric if it satisfies the equation $Ly(k) = 0$ for all $k \in I$, with*

$$L = a_1(k)E + a_0(k) \;\in\; K[k, E], \quad a_1(k) \perp a_0(k). \tag{3}$$

$\mathcal{GA}$ starts by constructing the operator L for a given concrete hypergeometric sequence $t(k)$, and this step is not formalized. On the next steps $\mathcal{GA}$ works with L only, while the sequence $t(k)$ itself is ignored (more precisely, in the case of $L = a_1(k)E + a_0(k)$, $\mathcal{GA}$ works with the *certificate* of $t(k)$, i.e., with the rational function $-\frac{a_0(k)}{a_1(k)}$, but this is not essential). The algorithm tries to construct a rational function $R(k)$, which is a solution in $K(k)$ of *Gosper's equation*

$$a_0(k)R(k+1) + a_1(k)R(k) = -a_1(k) \tag{4}$$

(such $R(k)$, when it exists, can also be found by general algorithms from [2, 3]). If such $R(k)$ exists then

$$R(k+1)t(k+1) - R(k)t(k) = t(k)$$

is valid for *almost* all integers k. The fact is that even when $t(k)$ is defined everywhere on I, it can happen that $R(k)$ has some poles belonging to I, and $u(k) = R(k)t(k)$ cannot be defined in such a way as to make (1) valid for all integers from I. One can encounter the situation where formula (2) is not valid even when all of

$$t(v), t(v+1), \ldots, t(w), \quad u(v), u(w+1)$$

are well-defined. The reason is that (1) may fail to hold at certain points k of the summation interval. However, sometimes it is possible to define the values of $u(k) = R(k)t(k)$ appropriately for all integers k, even though $R(k)$ has some integer poles. In such well-behaved cases (2) can be used to compute $\sum_{k=v}^{w} t(k)$ for any $v \leq w$, $v, w \in I$.

Example 1 Gosper's equation, corresponding to $L = kE - (k+1)^2$, has a solution $R = \frac{1}{k}$. The sequences

$$t_1(k) = \begin{cases} 0, & \text{if } k < 0, \\ k \cdot k!, & \text{if } k \geq 0 \end{cases}$$

and

$$t_2(k) = \begin{cases} \frac{(-1)^k k}{(-k-1)!}, & \text{if } k < 0, \\ 0, & \text{if } k \geq 0 \end{cases}$$

both satisfy $Ly = 0$ on $I = \mathbb{Z}$.

Generally speaking, (2) is not applicable to $t_1(k)$, but is applicable to $t_2(k)$. We can illustrate this as follows. Applying (2) to $t_1(k)$ with $v = -1, w = 1$, we have

$$t_1(-1) + t_1(0) + t_1(1) = \frac{1}{k} t_1(k) \,|_{k=2} \; - \frac{1}{k} t_1(k) \,|_{k=-1} \; = \frac{1}{2} \cdot 4 - 0 = 2$$

which is wrong, because $t_1(-1) + t_1(0) + t_1(1) = 0 + 0 + 1 = 1$. Applying (2) to t_2 with the same v, w, we have

$$t_2(-1) + t_2(0) + t_2(1) = \frac{1}{k} t_2(k) \,|_{k=2} \; - \frac{1}{k} t_2(k) \,|_{k=-1} \; = 0 - (-1) = 1$$

which is correct, because $t_2(-1) + t_2(0) + t_2(1) = 1 + 0 + 0 = 1$. ∎

In this paper we discuss some results related to necessary and sufficient conditions for validity of formula (2) when $u(k) = R(k)t(k)$, and $R(k)$ is a rational solution of corresponding Gosper's equation. If such $R(k)$ exists, then we describe the linear space of all hypergeometric sequences $t(k)$ that are defined on I and such that formula (2) is valid for $u = Rt$ and any integer bounds $v \leq w$ such that $v, w \in I$. The dimension of this space is always positive (it can be even bigger than 1). We will denote

- by $\mathcal{H}_I$ the set of all hypergeometric sequences defined on I;
- by $\mathcal{L}$ the set of all operators of type (3);
- by $V_I(L)$, where $L \in \mathcal{L}$, the K-linear space of all sequences $t(k)$ defined on I for which $Lt(k) = 0$ for all $k \in I$;
- by $W_I(R(k), L)$, where $L \in \mathcal{L}$ and $R(k) \in K(k)$ is a solution of the corresponding Gosper's equation, the K-linear space of all $t(k) \in V_I(L)$ such that (2) with $u(k) = R(k)t(k)$ is valid for all $v \leq w$ with $v, w \in I$.

The paper is a summary of the results that have been published in [4, 5]. In addition we consider the case where Gosper's equation has non-unique rational solution (Section 3.2). In Section 2 we consider individual hypergeometric sequences while in Section 3 we concentrate on spaces of the type $W_I(R(k), L)$.

2 Validity conditions of the discrete Newton-Leibniz formula

2.1 A criterion

Theorem 1 *([4, 5]) Let $L \in \mathcal{L}$, $t(k) \in V_I(L)$, and let Gosper's equation corresponding to L have a solution $R(k) \in K(k)$, with* $\operatorname{den}(R) = g(k)$*. Then $t(k) \in W_I(R(k), L)$ iff there exists a $\bar{t}(k) \in \mathcal{H}_I$ such that $t(k) = g(k)\bar{t}(k)$ for all $k \in I$.*

Example 2 Consider again the sequences $t_1(k), t_2(k)$ on $I = \mathbb{Z}$ from Example 1. We have $t_2(k) = k\bar{t}_2(k)$, where

$$\bar{t}_2(k) = \begin{cases} \frac{(-1)^k}{(-k-1)!}, & \text{if } k < 0, \\ 0, & \text{if } k \geq 0 \end{cases}$$

is a hypergeometric sequence defined everywhere:

$$E\bar{t}_2(k) - (k+1)\bar{t}_2(k) = 0.$$

On the other hand, if $t_1(k) = k\bar{t}_1(k)$ for some sequence $\bar{t}_1(k)$, then

$$\bar{t}_1(k) = \begin{cases} 0, & \text{if } k < 0, \\ \zeta, & \text{if } k = 0, \\ k!, & \text{if } k > 0 \end{cases}$$

where $\zeta \in \mathbb{C}$. Notice that the sequence $\bar{t}_1(k)$ is not hypergeometric on $\mathbb{Z}$, for any $\zeta \in \mathbb{C}$. ∎

2.2 Summation of proper hypergeometric sequences

Definition 2 *Following conventional notation, the* rising factorial power $(\alpha)_k$ *and its* reciprocal $1/(\beta)_k$ *are defined for $\alpha, \beta \in K$ and $k \in \mathbb{Z}$ by*

$$(\alpha)_k = \begin{cases} \prod_{m=0}^{k-1} (\alpha + m), & k \geq 0; \\ \prod_{m=1}^{|k|} \frac{1}{\alpha - m}, & k < 0, \ \alpha \neq 1, 2, \ldots, |k|; \\ \text{undefined}, & \text{otherwise}; \end{cases}$$

$$\frac{1}{(\beta)_k} = \begin{cases} \prod_{m=0}^{k-1} \frac{1}{\beta + m}, & k \geq 0, \ \beta \neq 0, -1, \ldots, 1-k; \\ \prod_{m=1}^{|k|} (\beta - m), & k < 0; \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

Note that if $(\alpha)_k$ resp. $1/(\beta)_k$ is defined for some $k \in \mathbb{Z}$, then $(\alpha)_{k+1}$ resp. $1/(\beta)_{k-1}$ is defined for that k as well. Thus $(\alpha)_k$ and $1/(\beta)_k$ are hypergeometric sequences which satisfy

$$(\alpha)_{k+1} = (\alpha + k)(\alpha)_k, \quad (\beta + k)/(\beta)_{k+1} = 1/(\beta)_k \tag{5}$$

whenever $(\alpha)_k$ and $1/(\beta)_{k+1}$ are defined.

Example 3 Let $t(k) = (k-2)(-1/2)_k/(4k!)$. This hypergeometric sequence is defined for all $k \in \mathbb{Z}$ (note that $t(k) = 0$ for $k < 0$) and satisfies $Lt(k) = 0$ for all $n \in \mathbb{Z}$ where $L = a_1(k)E + a_0(k)$ with $a_0(k) = -(k-1)(2k-1)$ and $a_1(k) = 2(k-2)(k+1)$. Gosper's equation, corresponding to L, has a rational solution

$$R(k) = \frac{2k(k+1)}{k-2}. \tag{6}$$

Equation (1) indeed fails at $k = 1$ and $k = 2$ because $u(k) = R(k)t(k)$ is undefined at $k = 2$. But if we cancel the factor $k - 2$ and replace $u(k)$ by the sequence

$$\bar{u}(k) \;=\; k(k+1)\frac{(-1/2)_k}{2k!},$$

then equation

$$\bar{u}(k+1) - \bar{u}(k) = t(k) \tag{7}$$

holds for all $k \in \mathbb{Z}$, and

$$\sum_{k=v}^{w} t(k) \;=\; \bar{u}(w+1) - \bar{u}(v). \tag{8}$$

■

The sequence $t(k)$ from Example 3 is an instance of a *proper hypergeometric* sequence which we are going to define now. As it turns out, there are no restrictions on the validity of the discrete Newton-Leibniz formula for proper sequences (Theorem 2).

Definition 3 *A hypergeometric sequence* $t(k)$ *defined on an infinite interval* I *of integers is* proper *if there are*

- *a constant* $z \in K$,
- *a polynomial* $p(k) \in K[k]$,
- *nonnegative integers* q, r,
- *constants* $\alpha_1, \ldots, \alpha_q$, $\beta_1, \ldots, \beta_r \in K$

such that

$$t(k) \;=\; p(k) z^k \frac{\prod_{i=1}^{q} (\alpha_i)_k}{\prod_{j=1}^{r} (\beta_j)_k} \tag{9}$$

for all $k \in I$.

Theorem 2 *([4]) Let $t(k)$ be a proper hypergeometric sequence defined on I and given by (9). Denote $a(k) = z\prod_{i=1}^{q}(k+\alpha_i)$ and $b(k) = \prod_{j=1}^{r}(k+\beta_j)$. If a polynomial $y(k) \in K[k]$ satisfies*

$$a(k)y(k+1) - b(k-1)y(k) \;=\; p(k) \tag{10}$$

and if

$$\bar{u}(k) \;=\; y(k)z^k \frac{\prod_{i=1}^{q}(\alpha_i)_k}{\prod_{j=1}^{r}(\beta_j)_{k-1}}$$

for all $k \in I$, then equation (7) holds for all $k \in I$, and the discrete Newton-Leibniz formula (8) is valid for all $v \leq w$, when $v, w \in I$.

Notice that (10) has a solution in $K[k]$ iff Gosper's equation, corresponding to the operator from $\mathcal{L}$, annihilating $t(k)$, has a solution in $K(k)$.

Example 4 The hypergeometric sequence

$$t(k) = \frac{\binom{2k-3}{k}}{4^k}, \tag{11}$$

which is defined for all $k \in \mathbb{Z}$ can be written as

$$t(k) = \begin{cases} 2s(k), & k < 2, \\ s(k), & k \geq 2, \end{cases}$$

where

$$s(k) = (2-k)\frac{(-1/2)_k}{4(1)_k}$$

is the proper sequence from Example 3. For $w \geq 1$, one should first split summation range in two

$$\sum_{k=0}^{w} t(k) = \frac{3}{4} + \sum_{k=2}^{w} s(k),$$

then the discrete Newton-Leibniz formula can be safely used to evaluate the sum on the right. However, applying directly (2) to (11) with (6) we obtain

$$\sum_{k=0}^{w} t(k) \;=(?)\; u(w+1) - u(0) \;=\; \frac{(w+1)(w+2)\binom{2w-1}{w+1}}{2(w-1)4^w}. \tag{12}$$

If we assume that the value of $\binom{2k-3}{k}$ is 1 when $k=0$ and -1 when $k=1$ (that is natural from combinatorial point of view) then the expression on the right gives the true value of the sum only at $w=0$. ∎

2.3 When the interval I contains no leading integer singularity of L

Definition 4 *For a linear difference operator (3) we call* $M = \max(\{k \in \mathbb{Z};\, a_1(k-1) = 0\} \cup \{-\infty\})$ *the* maximal leading integer singularity *of* L*,*

Proposition 1 *([4]) Let* $R(k)$ *be a rational solution of (4). Then* $R(k)$ *has no poles larger than* $M - 1$*.*

Theorem 3 *([4]) Let* $L \in \mathcal{L}$*,* M *be the maximal integer singularity of* L*,* $l \geq M$*,* $I = \mathbb{Z}_{\geq l}$ *and* $t(k) \in V_I(L)$*. Let Gosper's equation, corresponding to* L*, have a solution* $R(k)$ *in* $K(k)$*. Then* $t(k) \in W_I(R(k), L)$*.*

Example 5 For the sequence (11) we have $a_0(k) = -(2k-1)(k-1)$, $a_1(k) = 2(k+1)(k-2)$, $R(k) = 2k(k+1)/(k-2)$, and $u(k) = 2k(k+1)\binom{2k-3}{k}/((k-2)4^k)$. Thus $M = 3$, and the only pole of $R(k)$ is $k = 2$. As predicted by Theorem 3, the discrete Newton-Leibniz formula is valid when, e.g., $3 \leq v \leq w$. ∎

3 The spaces $V_I(L)$ and $W_I(R(k), L)$

3.1 The structure of $W_I(R(k), L)$

Theorem 4 *([5]) Let* $L \in \mathcal{L}$ *and Gosper's equation, corresponding to* L*, have a solution* $R(k) \in K(k)$*,* $\mathrm{den}(R) = g(k)$*. Then*

$$W_I(R(k), L) = g(k) \cdot V_I(\mathrm{pp}(L \circ g(k))),$$

where the operator $\mathrm{pp}(L \circ g(k))$ *is computed by removing from* $L \circ g$ *the greatest common polynomial factor of its coefficients.*

In addition, if $R = \frac{f(k)}{g(k)}$*,* $f(k) \perp g(k)$*, then the space of the corresponding primitives of the elements of* $W_I(R(k), L)$ *can be described as* $f(k) \cdot V_I(\mathrm{pp}(L \circ g(k)))$*.*

We will denote by $\bar{L}$ the operator $\mathrm{pp}(L \circ g(k))$.

Example 6 Consider again the operator $L = kE - (k+1)^2$ from Example 1 with $I = \mathbb{Z}$. We have $R = \frac{1}{k}$, and

$$L \circ k = kE \circ k - (k+1)^2 k = k(k+1)E - (k+1)^2 k = k(k+1)(E - k - 1),$$

$$\bar{L} = E - (k+1).$$

The space $W_I(R(k), \bar{L})$ is generated by $\bar{t}_2$, and, resp., the space $k \cdot W_I(R(k), \bar{L})$ is generated by $k\bar{t}_2$. In accordance with Theorem 4 the space $W_I(R(k), L)$ coincides with $k \cdot V_I(\bar{L})$. ■

It is possible to give examples showing that in some cases $\dim W_I(R(k), L) > 1$.

Example 7 Let $L = 2(k^2-4)(k-9)E - (2k-3)(k-1)(k-8)$, $I = \mathbb{Z}$. Then Gosper's equation, corresponding to L, has the rational solution

$$R(k) = -\frac{2(k-3)(k+1)}{k-9}.$$

Here $g(k) = k-9$ and $\bar{L} = 2(k^2-4)E - (2k-3)(k-1)$. Any sequence $\bar{t}$ which satisfies the equation $\bar{L}\bar{t} = 0$ has $\bar{t}(k) = 0$ for $k = 2$ or $k \leq -2$. The values of $\bar{t}(1)$ and $\bar{t}(3)$ can be chosen arbitrarily, and all the other values are determined uniquely by the recurrence $2(k^2-4)\bar{t}(k+1) = (2k-3)(k-1)\bar{t}(k)$. Hence $\dim V_I(\bar{L}) = 2$.

At the same time, $\dim V_I(L) = 3$. Indeed, if $Lt = 0$, then $t(-2) = t(2) = t(9) = 0$. The value $t(k) = 0$ from $k = -2$ propagates to all $k \leq -2$, but on each of the integer intervals $[-1, 0, 1]$, $[3, 4, 5, 6, 7, 8]$ and $[10, 11, \ldots)$ we can choose one value arbitrarily, and the remaining values on that interval are then determined uniquely. A sequence $t(k) \in V_I(L)$ belongs to $W_I(R(k), L)$ iff $22t(10) - 13t(8) = 0$. So $\dim W_I(R(k), L) = 2$. ■

3.2 When a rational solution of Gosper's equation is not unique

We give an example showing that if $L \in \mathcal{L}$ and Gosper's equation, corresponding to L, has different solutions $R_1(k), R_2(k) \in K(k)$, then it is possible that $W_I(R_1(k), L) \neq W_I(R_2(k), L)$. Moreover, these two spaces can have different dimensions.

Example 8 If $L = kE - (k+1)$, then Gosper's equation, corresponding to L, is

$$-(k+1)R(k+1) + kR(k) = -k,$$

and its general rational solution is

$$\frac{k-1}{2} + \frac{c}{k} = \frac{k^2-k+2c}{2k}.$$

Consider the solutions

$$R_1(k) = \frac{k-1}{2} \quad (g_1(k) = 1), \quad \text{and} \quad R_2(k) = \frac{k^2-k+2}{2k} \quad (g_2(k) = k).$$

We have $L \circ g_1(k) = L$, and $W_I(R_1(k), L) = V_I(L)$. This space has a basis that consists of two linearly independent sequences:

$$t_1(k) = \begin{cases} k, & \text{if } k \le 0, \\ 0, & \text{if } k > 0 \end{cases}$$

and

$$t_2(k) = \begin{cases} 0, & \text{if } k \le 0, \\ k, & \text{if } k > 0. \end{cases}$$

So this space contains, e.g., the sequence $t(k) = |k|$.

We have $L \circ g_2(k) = k(k+1)(E-1)$, therefore $W_I(R_2(k), L)$ is generated by the sequence $t(k) = k$. ■

If Gosper's equation, corresponding to $L \in \mathcal{L}$, has non-unique solution in $K(k)$, then the equation $Ly = 0$ has a non-zero solution in $K(k)$.

3.3 If Gosper's equation has a rational solution $R(k)$ then $W_I(R, L) \neq 0$

Theorem 5 *([5]) Let $L \in \mathcal{L}$ and let Gosper's equation, corresponding to L, have a solution $R(k) \in K(k)$. Then $W_I(R(k), L) \neq 0$ (i.e., $\dim W_I(R(k), L) \ge 1$).*

Example 9 Let $L = (k+2)E - k$. The rational function $\frac{1}{k(k+1)}$ is a solution in $K(k)$ of the equation $Ly = 0$. Here $R(k) = -k-1$, and $-1/k$ is a solution of the corresponding telescoping equation:

$$-\frac{1}{k+1} + \frac{1}{k} = \frac{1}{k(k+1)}.$$

The rational functions

$$\frac{1}{k(k+1)} \quad \text{and} \quad -\frac{1}{k}$$

have integer poles. Nevertheless, by Theorem 5 it has to be $W_I(R(k), L) \neq 0$ even when $I = \mathbb{Z}$. The space $W_I(R(k), L)$ is generated by the sequence

$$t(k) = \begin{cases} 1, & \text{if } k = -1, \\ -1, & \text{if } k = 0, \\ 0, & \text{otherwise,} \end{cases}$$

while the primitive of $t(k)$ is

$$(-k-1)t(k) = \begin{cases} 1, & \text{if } k=0, \\ 0, & \text{otherwise.} \end{cases}$$

If $I = \mathbb{Z}_{\geq 1}$, then $W_I(R(k), L)$ is generated by the sequence $t'(k) = \frac{1}{k(k+1)}$. ∎

By Theorem 3, if M is the maximal integer singularity of L, $l \geq M$, $I = \mathbb{Z}_{\geq l}$, and Gosper's equation, corresponding to L, has a solution $R(k)$ in $K(k)$, then $V_I(L) = W_I(R(k), L)$. As a consequence, $\dim V_I(L) = \dim W_I(R(k), L) = 1$.

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