Rational Solutions of First Order Linear q-Difference Systems

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Abstract

We propose an algorithm to compute rational function solutions for a first order system of linear q-difference equations with rational coefficients. We make use of the fact that q-difference equations bear similarity with differential equations at the point 0 and with difference equations at other points. This allows combining known algorithms for the differential and the difference cases.

1 Introduction

Let K be a computable field of characteristic zero, $q \in K$ a nonzero element which is not a root of unity, and x transcendental over K.

A system of first order linear q-difference equations with rational coefficients over the field K is a system of the form :

$$p_1(x)y_1(qx) = a_{11}(x)y_1(x) + \cdots + a_{1m}(x)y_m(x) + b_1(x)$$

$$\cdots \tag{1}$$

$$p_m(x)y_m(qx) = a_{m1}(x)y_1(x) + \cdots + a_{mm}(x)y_m(x) + b_m(x)$$

with polynomial coefficients $p_i(x), a_{ij}(x), b_j(x)$ over the field K.

q-Calculus and the theory and algorithms for linear q-difference equations are of interest in combinatorics, especially in the theory of partitions ([8, 9]). In this paper we solve the problem of computing all the rational solutions $y(x) = (y_1(x), \dots, y_m(x)) \in K(x)^m$

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of (1). Algorithms for solving this problem in the scalar case (that is the case of a single scalar linear q-difference equation of arbitrary order) have been proposed in [4, 1]. The algorithmic study of systems is, generally, less well-developed.

The traditional computer algebra approach to solve linear functional systems is via the cyclic vector method, or other similar elimination methods [12, 7], that convert the systems to scalar equations (such a procedure is called the *uncoupling*). Gröbner bases technique also can be used to reduce a recurrent system to the uncoupled form [13]. The major, and well-known, problem of this approach is the increasing size of the coefficients of equations, that makes those approaches applicable only to systems of very small dimension. In this paper we propose an alternative approach (a direct method) to solve the problem for the q-difference case.

It should be noted that there is some progress in solving the analogous problem in the differential and the difference cases: direct methods have been proposed in [11] and in [3, 15]. The methods [3, 15] are applicable to the q-difference case, excepting the situation where the denominators of some of $y_i(x)$ are divisible by x.

We will show below that by combining both differential and difference approaches it is possible to solve completely the problem in the q-difference case. A characteristic feature of q-difference equations is that they are similar to differential equations near the point 0 and are similar to difference equations near other points. It was used in the scalar case in [1].

Similarly to the differential and difference cases the construction proceeds in two steps. In the first step we construct a *universal denominator*. We mean a polynomial $U(x) \in K[x]$ such that: for all $y(x) \in K(x)^m$, if $y(x)$ is a solution of (1), then $U(x)y(x)$ is a polynomial vector. Then the substitution

$$y(x) = \frac{1}{U(x)} z(x) \tag{2}$$

into (1) reduces the problem to finding polynomial solutions of a system in $z(x)$ of the same type as (1). The second step of our method deals with this last problem.

From time to time we will need to find the largest non-negative n such that q^n is a root of a given polynomial with coefficients in K. Therefore we assume that K is a *q-suitable field*, meaning that there exists an algorithm which given $p \in K[x]$ finds all non-negative integer n such that $p(q^n) = 0$. For instance, if $K = k(q)$ where q is transcendental over k we can proceed as follows: Let $p(x) = \sum_{i=0}^{d} c_i x^i$ where $c_i \in k[q]$. Compute $s = \min\{i;\ c_i \neq 0\}$ and $t = \max\{j;\ q^j \mid c_s\}$. Then $p(q^n) = 0$ only if $n \leq t$, and the set of all such n can be found by testing the values $n = t, t-1, \ldots, 0$ ([5]).

2 Universal Denominators

2.1 Factors other than x

Here we demonstrate the q-modification of the algorithm [3]. The algorithm [15] also allows such a modification. In this paper we do not compare the algorithms [3] and [15], and take the former because its description is shorter.

We can assume the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mm} \end{pmatrix}, \tag{3}$$

which corresponds to (1) and whose elements are polynomials over K, to be invertible (otherwise either the system is incompatible or the number of unknowns can be reduced).

We name the modification qUD. First of all we set $A(x)$ to be equal to

$$\mathrm{lcm}(p_1(q^{-1}x), \dots, p_m(q^{-1}x))$$

and compute $B(x)$ as lcm of denominators of the elements of the matrix inverse of (3). Then compute qUD(A, B), considering the polynomial $V(x)$ as the result of the computation:

$V(x) := 1;$
$R(n) := Res_x(A(x), B(q^n x));$
`if` $R(n)$ has some nonnegative integer roots `then`
  $N :=$ the largest nonnegative integer root of $R(n)$;
  `for` $i = N, N-1, \dots, 0$ `do`
    $d(x) := gcd(A(x), B(q^i x));$
    $A(x) := A(x)/d(x);$
    $B(x) := B(x)/d(q^{-i}x);$
    $V(x) := V(x)d(x)d(q^{-1}x)\cdots d(q^{-i}x)$
  `od`
`fi.`

If rational functions $F_1(x), \dots, F_m(x)$ are such that

$$\begin{gathered} p_1(x)F_1(qx) - a_{11}(x)F_1(x) - \dots - a_{1m}(x)F_m(x) \\ \dots \\ p_m(x)F_m(qx) - a_{m1}(x)F_1(x) - \dots - a_{mm}(x)F_m(x) \end{gathered} \tag{4}$$

are polynomials and $A(x), B(x)$ are not divisible by x then the denominators of $F_1(x), \dots, F_m(x)$ (taken in the reduced form) divide $V(x)$.

The proof is similar to the proof of Theorems 1, 2 in [3]. All arguments that were given in [3] will hold if we replace the shifts

$$x \to x + i, \ x \to x - i,$$

where i is a nonnegative integer, by

$$x \to q^i x, \ x \to q^{-i}x,$$

and if we ignore the factor x when considering irreducible factors of polynomials. The factor x has to be considered separately since the polynomial x and qx are not relatively prime over K, though any other irreducible polynomial $r(x)$ is relatively prime with $r(qx)$.

2.2 A bound for the exponent of x

In the general case the components of any rational solution $y_i(x), i = 1, \ldots, m$, of (1) can be represented in the form

$$y_i(x) = \frac{f_i(x)}{g_i(x)} + \frac{l_{i1}}{x} + \frac{l_{i2}}{x^2} + \cdots + \frac{l_{ih_i}}{x^{h_i}},\ g_i(0) \neq 0, \tag{5}$$

$i = 1, \ldots, m$. The substitutions of

$$F_i^{(1)}(x) = \frac{f_i(x)}{g_i(x)}$$

and

$$F_i^{(2)}(x) = \frac{l_{i1}}{x} + \frac{l_{i2}}{x^2} + \cdots + \frac{l_{ih_i}}{x^{h_i}}$$

into expressions (4) for F_i, $i = 1, \ldots m$, give for each of those expressions two rational functions with the relatively primes denominators. Thereby the rational functions are polynomials. Freeing $A(x)$ and $B(x)$ from the factor x (denote the result by $\widetilde{A}(x), \widetilde{B}(x)$) we compute $\mathrm{qUD}(\widetilde{A}(x), \widetilde{B}(x))$ and get $V(x)$ which is divisible by all $g_i(x)$, $i = 1, \ldots, m$.

Now it is sufficient to find an upper bound H for all $h_1, \ldots, h_m$ and then it will be possible to use $U(x) = x^H V(x)$ as a universal denominator for all rational solutions of (1). To obtain such a bound one can use the technique of indicial equations. (This technique is well known in the theory of linear ordinary differential equations.) The main computational problem connected with the construction of the indicial equation in the differential case is reducing the given system to the super-irreducible form ([14, 10]). To the author's knowledge, the super-irreducible form for q-difference systems has not been considered yet. But in any case we can use a universal approach which called *EG-eliminations* ([2]). This approach allows one, in particular, to construct the indicial equations for differential and q-difference equations.

In other words, any rational solution (5) of system (1) can be considered as a solution in the class of Laurent series: $F_i^{(1)}(x)$ and $F_i^{(2)}(x)$, $i = 1, \ldots, m$, are, resp., regular and singular parts of the Laurent solution. Thus an upper bound for its pole order at $x = 0$ can be taken as H. EG-eliminations allow one to find such a bound.

The general scheme of using EG-method is the following (see [2] for details). Rewrite system (1) in the operator form:

$$\begin{pmatrix} p_1 Q - a_{11} & -a_{12} & \ldots & -a_{1m} \\ -a_{21} & p_2 Q - a_{22} & \ldots & -a_{2m} \\ \vdots & \vdots & & \vdots \\ -a_{m1} & -a_{m2} & \ldots & p_m Q - a_{mm} \end{pmatrix} y(x) = b(x), \tag{6}$$

where the operator Q is such that $Qf(x) = f(qx)$ for any function $f(x)$. Consider $y_i(x), b_i(x), i = 1, \ldots, m$, as Laurent series. Let $z_i(n), c_i(n), i = 1, \ldots, m$, be the sequences of coefficients of these series, $-\infty < n < \infty$. Consider the mapping $\mathcal{R}_q : K[x, Q] \to K[q^n, E^{-1}]$ defined by

$$\mathcal{R}_q Q = q^n,\ \ \mathcal{R}_q x = E^{-1}, \tag{7}$$

where the operator E is such that $Ef(x) = f(x+1)$ for any function $f(x)$. (In [6] it has been shown that $\mathcal{R}_q$ is an isomorphism from $K[x, Q]$ onto $K[q^n, E^{-1}]$.) Applying $\mathcal{R}_q$ to the elements of the operator matrix of (6) we get the operator matrix of the recurrent system for the column of sequences $z(n) = (z_1(n), \dots, z_m(n))^T$. This system can be rewritten in the form

$$P_l z(n+l) + P_{l-1} z(n+l-1) + \cdots + P_t z(n+t) = c(n), \tag{8}$$

where l, t are integer, $l \geq t$; $c(n) = (c_1(n), \dots, c_m(n))^T$; $P_l, \dots, P_t$ are $m \times m$-matrices over $K[n]$ with P_l and P_t (the *leading* and *trailing* matrices of the system) being nonzero but the determinants of P_l, P_t can vanish for all n. The process of EG-eliminations in the *explicit* matrix $P = (P_l | P_{l-1} | \dots | P_t)$ allows one to transform (8) into an equivalent system S with nonsingular leading (or analogously, trailing) matrix. Suppose that using EG-eliminations we constructed the corresponding system S of the form (8) with non-singular $P_l(n)$. Then $\det P_l(n) = 0$ is the indicial equation for (1) at the point 0. The following theorem can be easily proven in the usual way.

Theorem 1 *Let $p(n) = \det P_l(n)$ be a nonzero polynomial in q^n. Let n_0 be the smallest integer root of $p(n) = 0$ if such roots exist and $n_0 = 1$ otherwise. Then the pole orders of a Laurent series solution $y(x) = (y_1(x), \dots, y_m(x))$ of (1) do not exceed $|\min\{n_0 + l, 0\}|$.*

Proof : All $z_i(n)$ are equal to 0 for all negative integer n with large enough $|n|$. Besides, $c_i(n) = 0$ for all negative n, $i = 1, \dots, m$. The matrix $P_l(n)$ is invertible for all $n < n_0$ and we can use the recurrence (8) to get $z_i(n) = 0$, $i = 1, \dots, m$, for all $n < n_0 + l$. □

3 Polynomial solutions

After substitution (2) and cleaning denominators we have to solve the problem of finding polynomial solutions of a system of the form (1). It is sufficient to find an upper bound for degrees of all $y_1(x), \dots, y_m(x)$. Then all polynomial solutions of (1) can be found by the method of undetermined coefficients (the problem can be reduced to a system of linear algebraic equations). Less costly approaches such as those proposed for difference system in [3] are also possible.

We can construct the corresponding recurrent system for (1). Suppose that using EG-eliminations we transformed the recurrent system to the system S of the form (8) with non-singular $P_t(n)$. Then $\det P_t(n) = 0$ is the indicial equation for (1) at ∞. The following theorem can be easily proven in the usual way.

Theorem 2 *Let $p(n) = \det P_t(n)$ be a nonzero polynomial in q^n. Let n_1 be the largest integer root of $p(n) = 0$ if such roots exist and $n_1 = -1$ otherwise. Let $d = \max_{i=1}^m \deg b_i$ for (1). Then the degrees of the components of polynomial solution $y(x) = (y_1(x), \dots, y_m(x))$ of (1) do not exceed $\max\{n_1 + t, d + t\}$.*

Proof : For polynomial solutions all $z_i(n)$ are equal to 0 for all large enough n. Besides, $c_i(n) = 0$ for all $n > d$, $i = 1, \dots, m$. The matrix $P_t(n)$ is invertible for all $n > n_1$ and we can use the recurrence (8) to get $z_i(n) = 0$, $i = 1, \dots, m$, for all $n > \max\{n_1 + t, d + t\}$. $\square$

4 Example

Let's consider the following system of q-difference equations:

$$\begin{array}{l} (-qx + q^3x)\, y_1(qx) + (x - q^4x)y_1(x) + (q^4x + 100q^4 - q^2x - 100q^2)y_2(x) = 0 \\ (qx + 100)y_2(qx) - xy_1(x) = 0. \end{array}$$

Here

$$p_1(x) = -qx + q^3x,\ p_2(x) = x\,q + 100$$

and corresponing matrix (3) is

$$\begin{pmatrix} x - q^4x & q^4x + 100q^4 - q^2x - 100q^2 \\ -x & 0 \end{pmatrix}.$$

The inverse matrix is

$$\begin{pmatrix} 0 & -\frac{1}{x} \\ \frac{1}{q^2(q^2x+100q^2-x-100)} & -\frac{q^2+1}{(x+100)q^2} \end{pmatrix}.$$

Thus

$$A(x) = q^4x^2 + 100q^4x - q^2x^2 - 100q^2x,\ B(x) = -q^2x^2 - 100xq + q^4x^2 + 100q^3x.$$

Freeing from the factor x:

$$\widetilde{A}(x) = q^2(q^2x + 100q^2 - x - 100),\ \widetilde{B}(x) = q(q^3x - xq - 100 + 100q^2)$$

we get $V = \mathrm{qUD}(\widetilde{A}(x), \widetilde{B}(x)) = x + 100$.

Now we should find the exponent of x in the universal denominatior. The corresponding recurrent system has the explicit matrix

$$\begin{pmatrix} 0 & q^2(100q^2 - 100) & q^{n+2} - q^n - q^4 + 1 & q^2(q^2 - 1) \\ 0 & 100\,q^n & -1 & q^n \end{pmatrix}$$

with $l = 0$, $t = -1$. EG-eliminations lead to the system with the following leading matrix:

$$\begin{pmatrix} 100(q^{2n+4} - q^{2n+2} + q^{n+1} - q^{n+5} + q^4 - q^2) & 0 \\ 0 & 100(q^4 - q^2) \end{pmatrix}.$$

The equation $10000(q^{2n+4} - q^{2n+2} + q^{n+1} - q^{n+5} + q^4 - q^2)(q^4 - q^2) = 0$ has the integer roots -1, 1. So the degree of the pole is $\le |-1 + l| = 1$ and the universal denominator is

$$U(x) = xV(x) = x(x + 100).$$

After the substitution of the found denominator the system is

$$
\begin{aligned}
&((q^2-1)x^2+100(q^2-1)x)z_1(qx)-(q(q^4+1)x^2+100(1-q^4)x)z_1(x)-\\
&(q^3(1+q^2)x^2-100q^2((q^3+q^2-1)x+100q^2-100)z_2(x)=0\\
&(x+100)z_2(qx)-qxz_1(x)=0.
\end{aligned}
$$

The corresponding recurrent system has the explicit matrix

$$
\begin{pmatrix}
0 & 0\\
q^2(10000q^2-10000) & 100q^n\\
100q^{n+1}-100q^{n-1}-100q^4+100 & -q\\
q^2(100q^3+100q^2-100q-100) & q^{n-1}\\
-q^5+q+q^n-q^{n-2} & 0\\
q^2(q^3-q) & 0
\end{pmatrix}^T .
$$

with $l=0$, $t=-2$. EG-eliminations lead to the system with the following trailing matrix:

$$
\begin{pmatrix}
0 & q^{2n-2}-q^{2n-4}-q^{n+3}+q^{n-1}+q^6-q^4\\
-q & q^{n-2}
\end{pmatrix}.
$$

The polynomial $-q(q^{2n-2}-q^{2n-4}-q^{n+3}+q^{n-1}+q^6-q^4)$ has the integer roots 3, 5. So the polynomial solutions of the system has the degree $\leq 5+t=3$. It leads us to the following solution of the system for finding numerator

$$
\left[100\,\_c_1+x\,\_c_1+x^2\,\_c_2+\frac{x^3\,\_c_2}{100},\, x\,\_c_1+\frac{1}{100}\,\frac{x^3\,\_c_2}{q^2}\right]
$$

and correspondingly to the solution of the given system

$$
[\frac{x^2\,\_c_2+100\,\_c_1}{x},\ \frac{100\,\_c_1\,q^2+x^2\,\_c_2}{(x+100)\,q^2}].
$$

5 Implementation

This algorithm is implemented in Maple V by D.Khmelnov.

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