Gosper's Algorithm, Accurate Summation, and the Discrete Newton-Leibniz Formula*

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ABSTRACT

Sufficient conditions are given for validity of the discrete Newton-Leibniz formula when the indefinite sum is obtained either by Gosper's algorithm or by Accurate Summation algorithm. It is shown that sometimes a polynomial can be factored from the summand in such a way that the safe summation range is increased.

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1. INTRODUCTION

Let K be a field of characteristic zero. A function $t : I \mapsto K$ defined on an interval of integers $I \subseteq \mathbb{Z}$ is a

- *hypergeometric term* if there are nonzero polynomials $a_0, a_1 \in K[n]$ such that $a_1(n)t(n+1) + a_0(n)t(n) = 0$ for all $n \in \mathbb{Z}$ such that $n, n+1 \in I$;

- *P-recursive sequence* if there are polynomials a_0, a_1, $\ldots$, $a_\rho \in K[n]$ such that $a_0a_\rho \neq 0$ and $a_\rho(n)t(n+\rho) + \cdots + a_1(n)t(n+1) + a_0(n)t(n) = 0$ for all $n \in \mathbb{Z}$ such that $n, n+1, \ldots, n+\rho \in I$.

Each hypergeometric term is, of course, a P-recursive sequence.

If $t(n)$ is a hypergeometric term, one can use the well-known Gosper's algorithm [6] to find (if it exists) another hypergeometric term $u(n)$ which satisfies the *key equation*

$$u(n+1) - u(n) \;=\; t(n) \tag{1}$$

for all $n \in I \setminus S$ where S is a finite set. Summing this equation on n from v to w we get the discrete analog of the Newton-Leibniz formula

$$\sum_{n=v}^{w} t(n) \;=\; u(w+1) - u(v) \tag{2}$$

provided that $[v, w] \cap \mathbb{Z} \subseteq I \setminus S$.

In many existing implementations of Gosper's algorithm, however, indiscriminate use of (2) sometimes results in wrong answers. Here is a case in point.

Example 1. *Consider the sequence*

$$t(n) = \frac{\binom{2n-3}{n}}{4^n}, \tag{3}$$

which is defined for all $n \in \mathbb{Z}$. This is a hypergeometric term which satisfies

$$2(n+1)(n-2)t(n+1) = (2n-1)(n-1)t(n) \tag{4}$$

for all $n \in \mathbb{Z}$. Gosper's algorithm succeeds with input $t(n)$ and returns

$$u(n) = \frac{2n(n+1)\binom{2n-3}{n}}{(n-2)4^n}.$$

Summing equation (1) on n from 0 to m the left-hand side telescopes, and we obtain

$$\sum_{n=0}^{m} t(n) \;=(?)\; u(m+1) - u(0) \;=\; \frac{(m+1)(m+2)\binom{2m-1}{m+1}}{2(m-1)4^m}. \tag{5}$$

But the expression on the right gives the true value of the sum only at $m = 0$. At $m = 1$ it is undefined, while at each $m \geq 2$ its value is 3/8 less than the actual value of the sum. The problem here is that $u(n)$ is undefined at $n = 2$, hence equation (1) does not hold for $n \in \{1, 2\}$, and summing it over a range including 1 or 2 may give a wrong answer.

This is not an isolated example: a similar phenomenon seems to occur with the sum

$$\sum_{n=0}^{m} \frac{\binom{2n-p}{n}}{4^n}$$

for each positive integer p. ■

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If t is a P-recursive sequence, then one can use Accurate Summation algorithm from [3], or its generalization in [5], to solve equation (1) (we discuss this algorithm in Section 5). Problems similar to those arising in Example 1 are possible when one uses the resulting Newton-Leibniz formula. Notice that one can apply Accurate Summation algorithm in the case $\rho = 1$ as an alternative to Gosper's algorithm; then the incorrect formula (5) will appear again.

This common error is the discrete analogon of a well-known error in definite integration committed by some of the early symbolic integrators: when attempting to evaluate $I = \int_a^b f(x)dx$ by computing first an antiderivative $F(x)$ such that $F'(x) = f(x)$, and then using the Newton-Leibniz formula $I = F(b) - F(a)$, we may obtain an incorrect answer unless $F(x)$ is continuous on $[a, b]$. For example, the actual value of

$$\int_{-1}^{1} \frac{x^2+1}{x^4 - x^2 + 1}\, dx$$

is π, but using the antiderivative $\arctan(x - 1/x)$ in the Newton-Leibniz formula gives 0.

The obvious solution is to split the summation interval into several subintervals that do not contain the exceptional points from S. In this paper we analyze the exceptional set S that appears in Gosper's algorithm when summing hypergeometric terms, and more generally, in the Accurate Summation algorithm [3] when summing P-recursive sequences.

Section 3 provides sufficient conditions for the Newton-Leibniz formula (2) to hold when the indefinite sum $u(n)$ is obtained by Gosper's algorithm, and Section 5 does the same for Accurate Summation. These conditions provide a bounding interval for the exceptional set S, and are of two kinds: *a priori*, which are weaker but readily available even before running the algorithms, as they are based on the singularities of the operator annihilating the summand; and *a posteriori*, which are stronger but available only after running the algorithms, as they are based on their output. On the other hand, in Section 4 we prove that for proper hypergeometric terms the discrete Newton-Leibniz formula is valid without restrictions. For general P-recursive sequences Section 6 shows that sometimes a polynomial can be factored from the summand in such a way that the size of the bounding interval in the *a priori* condition is decreased.

A thorough analysis of the relationship between hypergeometric terms as syntactic objects and their analytic meaning in the context of summation has been provided by M. Schorn in [8]. The solution proposed there for evaluation of sums such as the one in Example 1 is by means of suitably chosen limiting processes.

2. PRELIMINARIES

Definition 1. *Following conventional notation, the* rising factorial power $(\alpha)_n$ *and its* reciprocal $1/(\beta)_n$ *are defined for $\alpha, \beta \in K$ and $n \in \mathbb{Z}$ by*

$$(\alpha)_n = \begin{cases} \prod_{k=0}^{n-1} (\alpha + k), & n \geq 0, \\ \prod_{k=1}^{|n|} \frac{1}{\alpha - k}, & n < 0,\ \alpha \neq 1, 2, \ldots, |n|, \\ \textit{undefined}, & \textit{otherwise}; \end{cases}$$

$$\frac{1}{(\beta)_n} = \begin{cases} \prod_{k=0}^{n-1} \frac{1}{\beta + k}, & n \geq 0,\ \beta \neq 0, -1, \ldots, 1-n, \\ \prod_{k=1}^{|n|} (\beta - k), & n < 0, \\ \textit{undefined}, & \textit{otherwise}. \end{cases}$$

Note that if $(\alpha)_n$ resp. $1/(\beta)_n$ is defined for some $n \in \mathbb{Z}$, then $(\alpha)_{n+1}$ resp. $1/(\beta)_{n-1}$ is defined for that n as well. More precisely, if $\alpha \in \mathbb{Z}$ and $\alpha \geq 1$ then $(\alpha)_n$ is defined on $[-\alpha + 1, \infty) \cap \mathbb{Z}$, otherwise it is defined on all $\mathbb{Z}$. Similarly, if $\beta \in \mathbb{Z}$ and $\beta \leq 0$ then $1/(\beta)_n$ is defined on $(-\infty, -\beta] \cap \mathbb{Z}$, otherwise it is defined on all $\mathbb{Z}$. Thus $(\alpha)_n$ and $1/(\beta)_n$ are hypergeometric terms which satisfy

$$(\alpha)_{n+1} = (\alpha + n)(\alpha)_n, \quad (\beta + n)/(\beta)_{n+1} = 1/(\beta)_n \tag{6}$$

whenever $(\alpha)_n$ and $1/(\beta)_{n+1}$ are defined.

If $I \subseteq \mathbb{Z}$ is an infinite interval of integers we denote

$$I^+ = \begin{cases} (-\infty, a+1] \cap \mathbb{Z}, & \text{if } I = (-\infty, a] \cap \mathbb{Z}, \\ I, & \text{otherwise;} \end{cases}$$

$$I^- = \begin{cases} (-\infty, a-1] \cap \mathbb{Z}, & \text{if } I = (-\infty, a] \cap \mathbb{Z}, \\ I, & \text{otherwise.} \end{cases}$$

We use E to denote the shift operator w.r.t. n, so that $E\,t(n) = t(n+1)$. Since juxtaposition can mean not only operator application but also composition of operators, we use $\circ$ to denote the latter in case of ambiguity, so that, e.g., $E \circ t(n) = t(n+1) \circ E = t(n+1)E$. Sometimes we use parentheses to denote operator application, writing, e.g., $E(1) = E\,1 = 1$.

Definition 2. *For a linear difference operator*

$$L = a_\rho E^\rho + a_{\rho-1} E^{\rho-1} + \cdots + a_0 \tag{7}$$

where $\rho \geq 1$, $a_\rho, \ldots, a_0 \in K[n]$, $a_\rho a_0 \neq 0$ and $\gcd(a_0, \ldots, a_\rho) = 1$, *we define the sets S_{Ll} of* leading *and S_{Lt} of* trailing integer singularities *by*

$$\begin{aligned} S_{Ll} &= \{x \in \mathbb{Z};\, a_\rho(x - \rho) = 0\}, \\ S_{Lt} &= \{x \in \mathbb{Z};\, a_0(x) = 0\}. \end{aligned}$$

We call

- $m_{Ll} = \min(S_{Ll} \cup \{+\infty\})$ *the* minimal leading *singularity of L,*
- $M_{Ll} = \max(S_{Ll} \cup \{-\infty\})$ *the* maximal leading *singularity of L,*
- $m_{Lt} = \min(S_{Lt} \cup \{+\infty\})$ *the* minimal trailing *singularity of L,*
- $M_{Lt} = \max(S_{Lt} \cup \{-\infty\})$ *the* maximal trailing *singularity of L.*

Proposition 1. *Let L be as in (7) and $b \in K[n]$. If a rational function $y \in K(n)$ satisfies*

$$a_\rho(n) y(n+\rho) + \cdots + a_0(n) y(n) = b(n), \tag{8}$$

then $y(n)$ has no integer poles outside the interval (possibly empty) $[m_{Ll}, M_{Lt}]$.

For a proof, see [1].

3. WHEN CAN GOSPER'S ALGORITHM BE USED TO SUM HYPERGEOMETRIC TERMS?

We denote Gosper's algorithm hereafter by $\mathcal{GA}$. Consider the case when (8) has the form

$$a_1(n)t(n+1) + a_0(n)t(n) = 0 \tag{9}$$

and set $L = a_1(n)E + a_0(n)$. Let a hypergeometric term $t(n)$ satisfy equation (9). Given $a_0(n)$, $a_1(n)$ as input, $\mathcal{GA}$ tries to construct $r \in K(n)$ such that

$$a_0(n)r(n+1) + a_1(n)r(n) = -a_1(n) \tag{10}$$

(this can also be done by the algorithms from [1] or [2]). If such $r(n)$ exists then $u(n) = r(n)t(n)$ satisfies the key equation (1), possibly with finitely many exceptions. We now give two kinds of sufficient conditions for this $u(n)$ to satisfy equation (1) and for the discrete Newton-Leibniz formula in the form

$$\sum_{k=v}^{w} t(k) \;=\; u(w) - u(v) + t(w) \tag{11}$$

to be valid:

1. an *a posteriori* condition, depending on the poles of $r(n)$ (Proposition 2),
2. an *a priori* condition, depending only on the integer singularities of L (Theorem 1).

In both, we make the following assumptions:

- $L = a_1(n)E + a_0(n)$ is an operator of type (7) with $\rho = 1$,
- $r \in K(n)$ is a rational function which satisfies (10) as an equation in $K(n)$,
- v, w are integers such that $v \le w$,
- $I_1 := [v, w-1] \cap \mathbb{Z}$,
- $t(n)$ is a K-valued sequence which is defined for all $n \in [v, w] \cap \mathbb{Z}$ and satisfies (9) for all $n \in I_1$,
- $u(n)$ is a K-valued sequence such that $u(n) = r(n)t(n)$ whenever both $r(n)$ and $t(n)$ are defined.

REMARK 1. *Since $u(n) = r(n)t(n)$ it is clear that, in general, formula (11) should be used instead of (2), because the latter formula needs values of the summand lying outside the summation interval which however may be undefined. A nice example is provided by the sum*

$$\sum_{k=0}^{2n} (-1)^k \binom{4n}{2k} \Big/ \binom{2n}{k}$$

whose evaluation was posed as Problem 10494 in the Amer. Math. Monthly *in 1996. Here $\mathcal{GA}$ succeeds (see [7]), but the summand is undefined everywhere outside the summation interval.*

PROPOSITION 2. (a posteriori condition for $\mathcal{GA}$) *If $r(n)$ has no integer poles in $[v, w]$, then the key equation (1) holds for all $n \in I_1$, and the discrete Newton-Leibniz formula (11) is valid.*

Proof: By assumption, $t(n)$, $t(n+1)$, $r(n)$, $r(n+1)$, $u(n)$, $u(n+1)$ are defined for all $n \in I_1$, and (9), (10) are valid on I_1. Therefore, for all $n \in I_1$,

$$\begin{aligned} a_0(n)u(n+1) &= a_0(n)r(n+1)t(n+1) \\ &= -a_1(n)(1+r(n))t(n+1) \quad \text{(by (10))} \\ &= a_0(n)(1+r(n))t(n) \quad \text{(by (9))} \\ &= a_0(n)u(n) + a_0(n)t(n), \end{aligned}$$

or equivalently,

$$a_0(n)(u(n+1) - u(n)) \;=\; a_0(n)t(n). \tag{12}$$

Pick an $n \in I_1$. If $a_0(n) \neq 0$ then (12) implies (1). If $a_0(n) = 0$ then, since $\gcd(a_0, a_1) = 1$, we have $a_1(n) \neq 0$. Hence (9) implies $t(n+1) = 0$ and (10) implies $r(n)+1 = 0$. Therefore $u(n+1) = r(n+1)t(n+1) = 0$ and $u(n) + t(n) = (r(n)+1)t(n) = 0$, so (1) holds for all $n \in I_1$. Summing (1) over I_1 yields (11). $\square$

THEOREM 1. (a priori condition for $\mathcal{GA}$) *If $[v, w] \cap [m_{Lt}+1, M_{Ll}-1] = \emptyset$, then the key equation (1) holds for all $n \in I_1$, and the discrete Newton-Leibniz formula (11) is valid.*

Proof: Since $r(n)$ satisfies (10), Proposition 1 implies that $r(n)$ has no integer poles outside the interval $[\alpha, \beta]$ where

$$\begin{aligned} \alpha &= \min(\{x \in \mathbb{Z};\ a_0(x-1) = 0\} \cup \{+\infty\}) \\ &= m_{Lt} + 1, \\ \beta &= \max(\{x \in \mathbb{Z};\ a_1(x) = 0\} \cup \{-\infty\}) \\ &= M_{Ll} - 1. \end{aligned}$$

By assumption, the interval $[v, w]$ is disjoint from $[\alpha, \beta]$. Hence $r(n)$ has no poles in $[v, w]$, and the assertion follows from Proposition 2. $\square$

In practice, one would run $\mathcal{GA}$ and then check if the *a posteriori* condition of Proposition 2 is satisfied, i.e., if $r(n)$ has any integer poles in the summation interval. If yes, this interval would be split into several subintervals in order to guarantee correct evaluation. But it may be useful to check the *a priori* condition of Theorem 1 first, because this will, in general, restrict the relevant domain to check for poles of $r(n)$.

EXAMPLE 2. *For the hypergeometric term $t(n) = \binom{2n-3}{n}/4^n$ of Example 1, we have $L = 2(n+1)(n-2)E - (2n-1)(n-1)$, $r(n) = 2n(n+1)/(n-2)$, and $u(n) = 2n(n+1)\binom{2n-3}{n}/((n-2)4^n)$. Thus $S_{Lt} = \{1\}$, $S_{Ll} = \{0, 3\}$, $m_{Lt} = 1$, $M_{Ll} = 3$, $[m_{Lt}+1, M_{Ll}-1] = \{2\}$, and the only integer pole of $r(n)$ is $n = 2$. In this case both the* a priori *and the* a posteriori *conditions give the same point $n = 2$ to be avoided by the summation interval. As predicted by either condition, the key equation (1) fails at $n = 1$ and $n = 2$ because $u(n)$ or $u(n+1)$ are undefined there. One is tempted to absorb the denominator factor $n-2$ into the binomial coefficient, and replace $u(n)$ by, say, the sequence $\bar{u}(n) = n(n+1)\binom{2n-1}{n}/((2n-1)4^n)$ which is defined everywhere, and agrees with $u(n)$ for all $n \neq 1, 2$. But then equation*

$$\bar{u}(n+1) - \bar{u}(n) = t(n) \tag{13}$$

fails at $n = 0$ and $n = 1$. ■

EXAMPLE 3. *Let*

$$t(n) \;=\; \begin{cases} (n-2)(n-3)(n-5)(n-1)!, & n \geq 2 \\ (n-2)(n-3)(n-5)\frac{(-1)^n}{(-n)!}, & n \leq 1 \end{cases}$$

where we define as usual $1/k! = 0$ *when* k *is a negative integer. This is a hypergeometric term which satisfies*

$$(n-5)(n-3)\,t(n+1) \;=\; (n-4)(n-1)n\,t(n)$$

for all $n \in \mathbb{Z}$. *Here we have* $a_0(n) = -(n-4)(n-1)n$, $a_1(n) = (n-5)(n-3)$, $r(n) = (n-6)/((n-2)(n-3))$, *and* $u(n) = r(n)t(n)$. *Thus* $S_{Lt} = \{0,1,4\}$, $S_{Ll} = \{4,6\}$, $m_{Lt} = 0$, $M_{Ll} = 6$, $[m_{Lt}+1, M_{Ll}-1] = [1,5]$, *and the set of integer poles of* $r(n)$ *is* $\{2,3\}$. *The set to be avoided by the summation interval given by the* a priori *condition is* $\{1,2,3,4,5\}$, *while the analogous set given by the* a posteriori *condition is* $\{2,3\}$. *The key equation (1) fails at* $n = 1,2,3$ *as predicted by the* a posteriori *condition, because* $u(n)$ *or* $u(n+1)$ *are undefined there. One can try cancelling the factor* $(n-2)(n-3)$ *and replace* $u(n)$ *by the sequence*

$$\bar{u}(n) \;=\; \begin{cases} (n-5)(n-6)(n-1)!, & n \geq 2 \\ (n-5)(n-6)\frac{(-1)^n}{(-n)!}, & n \leq 1 \end{cases}$$

which is defined everywhere, and agrees with $u(n)$ *for all* $n \neq 2,3$. *But equation (13) still fails at* $n = 1$. ■

4. SUMMATION OF PROPER HYPERGEOMETRIC TERMS

It is clear that the *a priori* condition given in Theorem 1 is, in general, too cautious: e.g., if the summand is a polynomial sequence then the integer singularities of the corresponding recurrence present no obstacles to validity of the discrete Newton-Leibniz formula (11). The following example shows that even the *a posteriori* condition given in Proposition 2 can sometimes be too pessimistic.

EXAMPLE 4. *Let* $t(n) = (2-n)(-1/2)_n/(4n!)$. *This hypergeometric term is defined for all* $n \in \mathbb{Z}$ *(note that* $t(n) = 0$ *for* $n < 0$*) and satisfies* $Lt(n) = 0$ *for all* $n \in \mathbb{Z}$ *where* L *is the same operator as in Example 2. Thus both Theorem 1 and Proposition 2 require the point* $n = 2$ *to be excluded from the summation interval. Equation (1) indeed fails at* $n = 1$ *and* $n = 2$ *because* $u(n) = r(n)t(n)$ *is undefined at* $n = 2$. *But if we cancel the factor* $n-2$ *in the product* $r(n)t(n)$, *where* $r(n) = 2n(n+1)/(n-2)$, *and replace* $u(n)$ *by the resulting sequence*

$$\bar{u}(n) \;=\; -n(n+1)\frac{(-1/2)_n}{2n!},$$

then equation (13) holds for all $n \in \mathbb{Z}$, *and the discrete Newton-Leibniz formula*

$$\sum_{n=v}^{w} t(n) \;=\; \bar{u}(w+1) - \bar{u}(v) \tag{14}$$

is valid for all $v \leq w$.

This example also shows that, thanks to possible singularities, a hypergeometric term (or a P-recursive sequence) is, in general, not *uniquely defined by its annihilating operator and an appropriate number of initial values. In fact, it is shown in [4] that every positive integer is the dimension of the kernel of some operator of type (7) with* $\rho = 1$ *in the space of sequences* $t:\ \mathbb{Z} \to K$. ■

The hypergeometric term $t(n)$ from Example 4 is an instance of a *proper* term which we are going to define now. Then we show in Theorem 2 that there are no restrictions on the validity of the discrete Newton-Leibniz formula for proper terms.

DEFINITION 3. *A hypergeometric term* $t(n)$ *defined on an interval* I *of integers is* proper *if there are*

- *a polynomial* $p \in K[n]$,
- *a constant* $z \in K$,
- *nonnegative integers* q, r,
- *constants* $\alpha_1, \ldots, \alpha_q$, $\beta_1, \ldots, \beta_r \in K$

such that

$$t(n) \;=\; p(n) z^n \frac{\prod_{i=1}^{q} (\alpha_i)_n}{\prod_{j=1}^{r} (\beta_j)_n} \tag{15}$$

for all $n \in I$.

THEOREM 2. *Let* $t(n)$ *be a proper hypergeometric term defined on an interval* I *of integers and given by (15). Denote* $a(n) = z\prod_{i=1}^{q}(n+\alpha_i)$ *and* $b(n) = \prod_{j=1}^{r}(n+\beta_j)$. *If a polynomial* $y \in K[n]$ *satisfies*

$$a(n)y(n+1) - b(n-1)y(n) \;=\; p(n) \tag{16}$$

and if

$$\bar{u}(n) \;=\; y(n) z^n \frac{\prod_{i=1}^{q} (\alpha_i)_n}{\prod_{j=1}^{r} (\beta_j)_{n-1}}$$

for all $n \in I^+$ *(see Section 2 for notation), then equation (13) holds for all* $n \in I$, *and the discrete Newton-Leibniz formula (14) is valid whenever* $[v,w] \cap \mathbb{Z} \subseteq I$.

Proof: By assumption, (16) holds for all $n \in I$. Multiplying it by $z^n \prod_{i=1}^{q}(\alpha_i)_n / \prod_{j=1}^{r}(\beta_j)_n$ yields

$$\begin{aligned} & z^n \frac{\prod_{i=1}^{q} (\alpha_i)_n}{\prod_{j=1}^{r} (\beta_j)_n}\, a(n)y(n+1) \\ - \;& z^n \frac{\prod_{i=1}^{q} (\alpha_i)_n}{\prod_{j=1}^{r} (\beta_j)_n}\, b(n-1)y(n) \\ = \;& t(n). \end{aligned} \tag{17}$$

Since $(\alpha_i)_n$ and $1/(\beta_j)_n$ are defined for all $n \in I$, $(\alpha_i)_{n+1}$ and $1/(\beta_j)_{n-1}$ are defined there too. By (6), $a(n)\prod_{i=1}^{q}(\alpha_i)_n = z\prod_{i=1}^{q}(\alpha_i)_{n+1}$ and $b(n-1)/\prod_{j=1}^{r}(\beta_j)_n = 1/\prod_{j=1}^{r}(\beta_j)_{n-1}$. Hence (17) is the same as (13), and (14) follows by summing it over $[v,w] \cap \mathbb{Z}$. □

EXAMPLE 5. *Even though the hypergeometric term (3) from Example 1 defined on* $I = \mathbb{Z}$ *can be written in terms of rising factorials as*

$$t(n) = \frac{(n-2)_n}{4^n (1)_n},$$

one can show that it is not *a proper term on* $\mathbb{Z}$. *However, it can also be written as*

$$t(n) = \begin{cases} 2\,t^*(n), & n < 2, \\ t^*(n), & n \geq 2, \end{cases}$$

where

$$t^*(n) = (2-n)\frac{(-1/2)_n}{4(1)_n}$$

is *a proper term (namely the one discussed in Example 4). So to evaluate* $\sum_{n=v}^{w} t(n)$ *one can first split the summation range at* $n=2$, *then use Theorem 2 on both subranges.* ■

5. WHEN CAN ACCURATE SUMMATION BE USED TO SUM P-RECURSIVE SEQUENCES?

By Accurate Summation algorithm (hereafter denoted by $\mathcal{AS}$) we mean a specific version of the general Accurate Integration algorithm given in [3] for integration/summation of solutions of Ore equations. This version, which is adapted for sequences that satisfy equations of the form (8) with $b(n)=0$, solves the following problem: Let a minimal annihilator L of the form (7) be known for a K-valued sequence $t(n)$. Determine if there exists a sequence $u(n)$ which satisfies (1), and has a minimal annihilator $\tilde{L}$ of order ρ.

It is shown in [3] that if such a u exists then it can be expressed as $R\,t$ where R is an operator of order $\rho-1$ with rational-function coefficients. $\mathcal{AS}$ constructs R if it exists. ($\mathcal{GA}$ solves this problem when $\rho=1$.) In order to analyze the validity of the discrete Newton-Leibniz formula (11) in this case, we need to express explicitly the quotient and the remainder of a linear difference operator when divided by the first-order operator $E-1$ from the left. The notion of the adjoint difference operator is useful here.

Definition 4. *Let*

$$L \;=\; \sum_{k=0}^{\rho} b_k(n)E^k$$

be an operator in $K(n)[E]$. *Its* adjoint $L^* \in K(n)[E^{-1}]$ *is defined as*

$$L^* \;=\; \sum_{k=0}^{\rho} E^{-k}\circ b_k(n) \;=\; \sum_{k=0}^{\rho} b_k(n-k)E^{-k}.$$

It is straightforward to verify that $(L_1\circ L_2)^* = L_2^*\circ L_1^*$.

Lemma 1. *Let* $P=\sum_{k=0}^{\rho} b_k(n)E^k$, $R=\sum_{k=0}^{\rho-1} c_k(n)E^k$ *be operators from* $K(n)[E]$, *and* $p\in K(n)$ *a rational function such that* $P=(E-1)\circ R+p$. *Then* $c_k(n)=\sum_{j=1}^{\rho-k} b_{k+j}(n-j)$ *and* $p=P^*(1)$.

Proof: Let

$$Q \;=\; \sum_{k=0}^{\rho-1}\sum_{j=1}^{\rho-k} b_{k+j}(n-j)E^k$$

and $q=P^*(1)$. Then

$$\begin{aligned}
&(E-1)\circ Q+q\\
&\quad= \sum_{k=0}^{\rho-1}\sum_{j=1}^{\rho-k} b_{k+j}(n-j+1)E^{k+1} - Q + P^*(1)\\
&\quad= \sum_{k=1}^{\rho}\sum_{j=1}^{\rho-k+1} b_{k+j-1}(n-j+1)E^k\\
&\quad- \sum_{k=0}^{\rho-1}\sum_{j=2}^{\rho-k+1} b_{k+j-1}(n-j+1)E^k + \sum_{j=0}^{\rho} b_j(n-j)\\
&\quad= \sum_{k=0}^{\rho}\sum_{j=1}^{\rho-k+1} b_{k+j-1}(n-j+1)E^k\\
&\quad- \sum_{k=0}^{\rho-1}\sum_{j=2}^{\rho-k+1} b_{k+j-1}(n-j+1)E^k\\
&\quad= b_\rho(n)E^\rho + \sum_{k=0}^{\rho-1} b_k(n)E^k \;=\; P.
\end{aligned}$$

As the quotient and remainder in operator division are unique, it follows that $R=Q$ and $p=q$. □

Note that just to find the remainder $p=P^*(1)$, it suffices to take adjoints on both sides of equation $P=(E-1)\circ R+p$ which results in $P^*=R^*\circ(E-1)^*+p=R^*\circ(E^{-1}-1)+p$, and apply this to 1.

Remark 2. *Let* $r\in K(n)$ *be a rational function, and* L *a difference operator as in (7). By Lemma 1, the remainder of* $1-rL$ *when divided by* $E-1$ *from the left is equal to*

$$(1-rL)^*(1) = (1-L^*\circ r)(1) = 1-(L^*\circ r)(1) = 1-L^*r.$$

Hence an operator R *such that* $1-rL=(E-1)\circ R$ *exists if and only if* $L^*r=1$. *This observation forms the basis of Accurate Summation.*

Algorithm $\mathcal{AS}$

Input: $L=\sum_{k=0}^{\rho} a_k(n)E^k \in K[n,E]$.
Output: $r\in K(n)$ and $R\in K(n)[E]$
such that $1-rL=(E-1)\circ R$, if they exist.

if there exists $r\in K(n)$ such that $L^*r=1$ **then**
  for $k := 0,1,\ldots,\rho-1$ **do**
    $c_k(n) := -\sum_{j=1}^{\rho-k} r(n-j)a_{k+j}(n-j)$;
  $R \;:=\; \sum_{k=0}^{\rho-1} c_k(n)E^k$;
  return $(r(n),R)$
else
  such $r(n)$ and R do not exist.

We can find a rational-function solution $r(n)$ of $L^*r=1$ using, e.g., the algorithm from [1] or the algorithm from [2].

A generalization of [3] was given in [5]; however, the approach taken in [3] has the advantage of simplicity, as it only uses the adjoint operator and algorithms for finding rational solutions. This simplifies the investigation of solutions that are obtained by $\mathcal{AS}$, and enables us to formulate *a priori* conditions for $\mathcal{AS}$, similar to Theorem 1 (see Theorems 3 and 5 below).

Assume that $\mathcal{AS}$ succeeds with L, returning r and R. It is shown in [3] that

- if L is a minimal annihilator for t, then a minimal annihilator for $u = R\,t$ is $\tilde{L} = 1 - R \circ (E - 1)$ (note that $\tilde{L}$ has the same order ρ as L);
- the sequence $u(n) = R\,t(n)$ satisfies (1), possibly with finitely many exceptions, for any sequence t such that $Lt = 0$ (not only for those t's whose minimal annihilator is L).

We now give two sufficient conditions for this $u(n)$ to satisfy equation (1) and for the discrete Newton-Leibniz formula (11) to be valid:

1. an *a posteriori* condition, depending on the poles of $r(n)$ and of the coefficients of R (Proposition 3),
2. an *a priori* condition, depending only on the integer singularities of L (Theorem 3).

In either case, we make the following assumptions:

- $L \in K[n, E]$ is an operator of type (7),
- $r \in K(n)$ is a rational function which satisfies $L^*r = 1$ as an equation in $K(n)$,
- $R \in K(n)[E]$ is an operator of order $\rho - 1$ which satisfies $1 - rL = (E - 1) \circ R$ in $K(n)[E]$,
- v, w are integers such that $v \le w - \rho$,
- $I_\rho := [v, w - \rho] \cap \mathbb{Z}$,
- $t(n)$ is a K-valued sequence which is defined for all $n \in [v, w] \cap \mathbb{Z}$ and satisfies $L\,t(n) = 0$ for all $n \in I_\rho$,
- $u(n)$ is a K-valued sequence such that $u(n) = R\,t(n)$ whenever $R\,t(n)$ is defined.

PROPOSITION 3. (*a posteriori* condition for $\mathcal{AS}$) *If $r(n)$ has no poles in I_ρ and the coefficients of R have no integer poles in $[v, w - \rho + 1]$, then equation (1) holds for all $n \in I_\rho$, and the discrete Newton-Leibniz formula*

$$\sum_{k=v}^{w} t(k) \;=\; u(w - \rho + 1) - u(v) + \sum_{k=1}^{\rho} t(w - \rho + k) \quad (18)$$

is valid.

Proof: By assumptions on t and R, $u(n)$ and $u(n + 1)$ are defined for all $n \in I_\rho$. As $r(n)$ has no poles in I_ρ,

$$\begin{aligned} u(n+1) - u(n) &= (E \circ R)\,t(n) - R\,t(n) \\ &= ((E-1) \circ R)\,t(n) \;=\; (1 - rL)\,t(n) \\ &= t(n) - r(n)\,L\,t(n) \;=\; t(n) \end{aligned}$$

for every $n \in I_\rho$. Thus (1) holds for all $n \in I_\rho$, and summing it over I_ρ yields (18). □

LEMMA 2. *Let $\alpha, \beta \in \mathbb{Z} \cup \{-\infty, \infty\}$. If $r(n)$ has no integer poles in $[\alpha, \beta]$ then the coefficients of R have*

(i) no integer poles in $[\alpha + \rho, \beta + 1]$, and also

(ii) no integer poles in $[\alpha, \beta - \rho + 1]$.

Proof: Write $R = \sum_{k=0}^{\rho-1} c_k(n)E^k$. By Lemma 1,

$$c_k(n) = -\sum_{j=1}^{\rho-k} r(n-j)a_{k+j}(n-j)$$

for $0 \le k \le \rho - 1$.

(i) By assumption, $r(n - j)$ has no integer poles in $[\alpha + j, \beta + j]$, hence $c_k(n)$ has no integer poles in $\bigcap_{1\le j\le\rho-k}[\alpha + j, \beta + j] = [\max_{1\le j\le\rho-k}(\alpha + j), \min_{1\le j\le\rho-k}(\beta+j)] = [\alpha+\rho-k, \beta+1]$, and the coefficients of R have no integer poles in $\bigcap_{0\le k\le\rho-1}[\alpha+\rho-k, \beta+1] = [\max_{0\le k\le\rho-1}(\alpha+\rho-k), \beta+1] = [\alpha+\rho, \beta+1]$.

(ii) To prove the second assertion, we need to express the coefficients $c_k(n)$ in a different way. Since $1 - rL = (E - 1) \circ R$, it follows from Remark 2 that $L^*r = \sum_{j=0}^{\rho} r(n - j)a_j(n - j) = 1$. Shifting this k times we find that $\sum_{j=0}^{\rho} r(n+k-j)a_j(n+k-j) = \sum_{j=-k}^{\rho-k} r(n-j)a_{k+j}(n - j) = 1$. Therefore

$$\begin{aligned} c_k(n) &= -\sum_{j=1}^{\rho-k} r(n-j)a_{k+j}(n-j) \\ &\quad + \sum_{j=-k}^{\rho-k} r(n-j)a_{k+j}(n-j) - 1 \\ &= \sum_{j=0}^{k} r(n+j)a_{k-j}(n+j) - 1 \end{aligned}$$

for $0 \le k \le \rho - 1$. By assumption, $r(n + j)$ has no integer poles in $[\alpha - j, \beta - j]$, hence $c_k(n)$ has no integer poles in $\bigcap_{0\le j\le k}[\alpha - j, \beta - j] = [\max_{0\le j\le k}(\alpha - j), \min_{0\le j\le k}(\beta - j)] = [\alpha, \beta - k]$, and the coefficients of R have no integer poles in $\bigcap_{0\le k\le\rho-1}[\alpha, \beta - k] = [\alpha, \min_{0\le k\le\rho-1}(\beta - k)] = [\alpha, \beta - \rho + 1]$. □

THEOREM 3. (*a priori* condition for $\mathcal{AS}$) *If $[v, w - \rho] \cap [m_{Lt}, M_{Ll} - \rho] = \emptyset$, then equation (1) holds for all $n \in I_\rho$, and the discrete Newton-Leibniz formula (18) is valid.*

Proof: Rewrite $L^*r = 1$ in the equivalent form $L'r = 1$ where $L' = E^\rho \circ L^* = \sum_{k=0}^{\rho} a_{\rho-k}(n + k)E^k \in K[n, E]$. By Lemma 1, $r(n)$ has no integer poles outside $[m_{L'l}, M_{L't}]$. But $S_{L'l} = S_{Lt}$ and $S_{L't} = S_{Ll} - \rho$, therefore $m_{L'l} = m_{Lt}$ and $M_{L't} = M_{Ll} - \rho$, hence $r(n)$ has no integer poles outside $[m_{Lt}, M_{Ll} - \rho]$.

If $m_{Lt} \le M_{Ll} - \rho$, then both intervals $[v, w - \rho]$ and $[m_{Lt}, M_{Ll} - \rho]$ are nonempty, hence either $w - \rho < m_{Lt}$ or $M_{Ll} - \rho < v$. In the former case, $r(n)$ has no integer poles in $(-\infty, w - \rho]$, so by Lemma 2(i), the coefficients of R have no integer poles in $(-\infty, w-\rho+1]$. In the latter case, $r(n)$ has no integer poles in $[v, \infty)$, so by Lemma 2(ii), the coefficients of R have no integer poles in $[v, \infty)$. In either case, the result follows from Proposition 3.

If $m_{Lt} > M_{Ll} - \rho$ then $r(n)$ has no integer poles at all. By Lemma 2, the coefficients of R also have no integer poles, and the result again follows from Proposition 3. □

A similar remark to the one stated immediately after the proof of Theorem 1 about the use of the *a priori* and *a posteriori* conditions in practice applies here as well.

EXAMPLE 6. *Let* $L = (n-3)(n-2)(n+1)E^2 - (n-3)(n^2-2n-1)E - (n-2)^2$. *Define* $t(n)$ *by the initial values* $t(2) = a$, $t(3) = 0$, $t(4) = b$, $t(5) = c$ *where* a, b, c *are arbitrary fixed complex numbers, and by the recurrence* $L\,t(n) = 0$ *when* $n \le 1$ *or* $n \ge 6$. *Then it can be checked that* $L\,t(n) = 0$ *for all* $n \in \mathbb{Z}$. *Algorithm* $\mathcal{AS}$ *succeeds with input* L *and returns* $r(n) = -1/((n-2)(n-3))$, $R = nE + 1/(n-3)$. *In this case* $S_{Lt} = \{2\}$, $m_{Lt} = 2$, $S_{Ll} = \{1, 4, 5\}$, $M_{Ll} - \rho = 5 - 2 = 3$. *So both the* a posteriori *and the* a priori *conditions reduce to* $3 \notin [v, w-1]$. *This is the best possible, as the sequence* $u(n) = R\,t(n)$ *is undefined at* $n = 3$, *and equation (1) does not hold for* $n = 2, 3$. *It can be verified that except in the special case* $b + 4c = 0$, *there is no way to define* $u(3)$ *so that (1) would hold for all* $n \in \mathbb{Z}$. ∎

REMARK 3. *When* $\rho = 1$ *and* $v \le w - 1$, *Theorem 3 implies Theorem 1 in the following way. If* $r(n)$ *satisfies (10) then it is easy to verify that* $\bar{r}(n) := -r(n+1)/a_1(n)$ *satisfies* $L^*\bar{r} = 1$, *and* $1 - \bar{r}\,L = (E-1)\circ R$ *where* $R = r(n)$ *is an operator of order* 0. *Thus* $u(n) = r(n)t(n)$ *of Theorem 1 agrees with* $u(n) = R\,t(n)$ *of Theorem 3. By the assumption of Theorem 1,* $[v, w] \cap [m_{Lt} + 1, M_{Ll} - 1] = \emptyset$.

If $m_{Lt} + 1 \le M_{Ll} - 1$ *then either* $w \le m_{Lt}$ *or* $M_{Ll} - 1 \le v - 1$, *hence* $[v, w-1] \cap [m_{Lt}, M_{Ll} - 1] = \emptyset$, *and the conclusion follows by Theorem 3. If* $m_{Lt} + 1 > M_{Ll} - 1$ *then* $m_{Lt} \ge M_{Ll} - 1$. *Again we distinguish two cases: If* $m_{Lt} > M_{Ll} - 1$, *the conclusion follows by Theorem 3. If* $m_{Lt} = M_{Ll} - 1$, *then* $a_0(n)$ *and* $a_1(n)$ *have a common zero since* $a_0(m_{Lt}) = a_1(M_{Ll} - 1) = 0$. *But this contradicts the assumption of relative primality of the coefficients of* L.

Note that when $\rho \ge 2$, *polynomials* $a_0(n)$ *and* $a_\rho(n)$ *need not be relatively prime.*

6. EXPLOITING POLYNOMIAL FACTORS

In this section we show how polynomial factors of hypergeometric terms (even non-proper ones, such as the one in Example 3) and of P-recursive sequences can be used to strengthen the statements of Theorems 1 and 3 (i.e., to weaken the *a priori* conditions for validity of the discrete Newton-Leibniz formula).

THEOREM 4. *Assume that*

- $L = a_1(n)E + a_0(n)$ *and* $\bar{L} = \bar{a}_1(n)E + \bar{a}_0(n)$ *are operators of type (7) with* $\rho = 1$,
- $t(n)$, $\bar{t}(n)$ *are* K*-valued sequences with infinitely many nonzero values, defined on an infinite interval of integers* I *and satisfying* $L\,t(n) = \bar{L}\,\bar{t}(n) = 0$ *on* I^- *(see Section 2 for notation),*
- $p \in K[n]$ *is a polynomial such that* $t(n) = p(n)\bar{t}(n)$ *for all* $n \in I$,
- $r \in K(n)$ *is a rational function which satisfies (10) as an equation in* $K(n)$,
- $\bar{r} = p\,r \in K(n)$,
- $\bar{u}(n)$ *is a* K*-sequence such that* $\bar{u}(n) = \bar{r}(n)\bar{t}(n)$ *whenever both* $\bar{r}(n)$ *and* $\bar{t}(n)$ *are defined,*
- v, w *are integers such that* $v \le w$ *and* $[v, w] \cap \mathbb{Z} \subseteq I$.

If $I \cap [m_{\bar{L}t} + 1, M_{\bar{L}l} - 1] = \emptyset$ *then equation (13) holds for all* $n \in I^-$, *and the discrete Newton-Leibniz formula*

$$\sum_{k=v}^{w} t(k) \;=\; \bar{u}(w) - \bar{u}(v) + t(w) \tag{19}$$

is valid.

Proof: By assumption, we have for all $n \in I^-$,

$$\bar{a}_1(n)\bar{t}(n+1) + \bar{a}_0(n)\bar{t}(n) \;=\; 0, \tag{20}$$

$$a_1(n)p(n+1)\bar{t}(n+1) + a_0(n)p(n)\bar{t}(n) \;=\; 0. \tag{21}$$

Multiplying (20) by $a_1(n)p(n+1)$, (21) by $\bar{a}_1(n)$ and subtracting, we find $\bar{a}_0(n)\bar{t}(n)a_1(n)p(n+1) = a_0(n)p(n)\bar{t}(n)\bar{a}_1(n)$. Since $\bar{t}(n)$ has infinitely many nonzero values on I^-, this implies that

$$a_0(n)p(n)\bar{a}_1(n) \;=\; \bar{a}_0(n)a_1(n)p(n+1) \tag{22}$$

holds infinitely often, hence also as an equation in $K[n]$. Multiplying (10) by $\bar{a}_1(n)p(n)$, (22) by $r(n+1)$, subtracting, and cancelling $a_1(n)$ in $K(n)$, we obtain

$$\bar{a}_0(n)\bar{r}(n+1) + \bar{a}_1(n)\bar{r}(n) \;=\; -\bar{a}_1(n)p(n) \tag{23}$$

as an equation in $K(n)$. It follows from Proposition 1 that $\bar{r} \in K(n)$ has no integer poles outside the interval $[m_{\bar{L}t} + 1, M_{\bar{L}l} - 1]$. Therefore $\bar{r}(n)$ and $\bar{u}(n)$ are defined on I, $\bar{r}(n+1)$ and $\bar{u}(n+1)$ are defined on I^-, and (23) is valid for all $n \in I^-$.

Pick an $n \in I^-$. Multiplying (20) by $\bar{r}(n+1)$, (23) by $\bar{t}(n)$ and subtracting, we obtain

$$\bar{a}_1(n)\bar{r}(n+1)\bar{t}(n+1) - \bar{a}_1(n)\bar{r}(n)\bar{t}(n) \;=\; \bar{a}_1(n)p(n)\bar{t}(n).$$

If $\bar{a}_1(n) \ne 0$ this reduces to (13). If $\bar{a}_1(n) = 0$ then, by assumption, $\bar{a}_0(n) \ne 0$, so (20) implies $\bar{t}(n) = t(n) = \bar{u}(n) = 0$ and (23) implies $\bar{r}(n+1) = \bar{u}(n+1) = 0$, hence (13) holds in this case as well. So (13) holds for all $n \in I^-$, and the second assertion follows by summing (13) on n from v to $w - 1$. □

EXAMPLE 7. *Let* $t(n)$ *be the hypergeometric term from Example 3 which satisfies* $L\,t(n) = 0$ *for all* $n \in \mathbb{Z}$ *where* $L = (n-5)(n-3)\,E - (n-4)(n-1)n$. *Define* $p(n) = (n-5)(n-3)(n-2)$ *and*

$$\bar{t}(n) \;=\; \begin{cases} (n-1)!, & n \ge 2, \\ \frac{(-1)^n}{(-n)!}, & n \le 1. \end{cases}$$

Then $\bar{L}\,\bar{t}(n) = 0$ *for all* $n \ne 1$ *where* $\bar{L} = E - n$. *So, in the notation of Theorem 4, the maximal possible interval* I *is either* $(-\infty, 1] \cap \mathbb{Z}$ *or* $[2, \infty) \cap \mathbb{Z}$. *As* $\bar{L}$ *has no leading singularities,* $M_{\bar{L}l} = -\infty$ *and* $[m_{\bar{L}t} + 1, M_{\bar{L}l} - 1] = \emptyset$. *With* $r(n) = (n-6)/((n-2)(n-3))$, $\bar{r}(n) = (n-5)(n-6)$ *and* $\bar{u}(n) = \bar{r}(n)\bar{t}(n)$, *all the assumptions of Theorem 4 are satisfied, and it follows that formula (19) is valid provided that* $w \le 1$ *or* $v \ge 2$. ∎

Now we consider the general case with $\rho \ge 1$.

PROPOSITION 4. *Let* $(r(n), R)$ *be the result of applying* $\mathcal{AS}$ *to an operator* $L \in K[n, E]$ *of type (7), and let* $r = s/q$ *where* $s, q \in K[n]$. *Then there exist* $p \in K[n]$ *and* $\bar{L} \in K[n, E]$ *such that*

$$L \circ p = q\bar{L}, \tag{24}$$

$(E-1)\circ R \circ p = p - s\,\bar{L}$, *and* $R \circ p \in K[n, E]$.

Proof: Let $d \in K[n]$ be a polynomial and $B \in K[n, E]$ an operator such that

$$E^\rho \circ L^* \circ \frac{1}{q} = \frac{1}{d} B.$$

Then

$$\frac{1}{q} L \circ E^{-\rho} = B^* \circ \frac{1}{d}.$$

Therefore $L \circ E^{-\rho} \circ d = qB^*$. Multiplying this by E^ρ on the right gives $L \circ E^{-\rho} \circ d \circ E^\rho = L \circ d(n-\rho) = qB^* \circ E^\rho$. Take $p(n) = d(n-\rho)$ and $\bar{L} = B^* \circ E^\rho$. Then (24) is satisfied and $p - s\,\bar{L} = p - rq\bar{L} = p - rL \circ p = (1 - rL) \circ p = (E-1) \circ R \circ p$. Hence the operator $R \circ p$ is the left quotient of $p - s\,\bar{L}$ by $E - 1$ and, consequently, has polynomial coefficients. □

THEOREM 5. *Let*

- $L, \bar{L}, R, r, p, q$ *be such as in Proposition 4,*
- $v, w \in \mathbb{Z}$ *be such that* $v \le w - \rho$*,*
- $I_\rho = [v, w-\rho] \cap \mathbb{Z}$*,*
- $\bar{t}(n)$ *be a* K*-valued sequence defined for all* $n \in [v, w] \cap \mathbb{Z}$ *such that* $\bar{L}\bar{t}(n) = 0$ *for all* $n \in I_\rho$*.*

Then the K*-valued sequence* $t(n) = p(n)\bar{t}(n)$ *satisfies* $Lt(n) = 0$ *for all* $n \in I_\rho$*, and the discrete Newton-Leibniz formula (18) can be applied to* $t(n)$ *with* $u(n) = (R \circ p)\bar{t}(n)$*.*

Proof: By (24),

$$Lt(n) = (L \circ p)\bar{t}(n) = q\bar{L}\bar{t}(n) = 0$$

for all $n \in I_\rho$. Also, $u(n) = (R \circ p)\bar{t}(n)$ and $u(n+1) = (E \circ R \circ p)\bar{t}(n)$ are defined for all $n \in I_\rho$. Therefore, by Proposition 4,

$$\begin{aligned} u(n+1) - u(n) &= ((E-1) \circ R \circ p)\bar{t}(n) = (p - s\bar{L})\bar{t}(n) \\ &= p(n)\bar{t}(n) - s(n)\bar{L}\bar{t}(n) = t(n) \end{aligned}$$

for all $n \in I_\rho$, and (18) follows by summing this over I_ρ. □

EXAMPLE 8. *Consider again the operator*

$$L = 2(n+1)(n-2)E - (2n-1)(n-1)$$

from Example 2. Here $m_{Lt} = 1$ *and* $M_{Ll} = 3$*, so, following Theorem 3, we can apply formula (18) if* $[v, w-1] \cap [1, 2] = \emptyset$*. Using the algorithm from [1] or the algorithm from [2] we compute the solution* $r(n) = -(n+2)/((n-1)(n-2))$ *of* $L^* r = 1$*, and set* $q(n) = (n-1)(n-2)$*. Then*

$$E \circ L^* \circ \frac{1}{q(n)} = \frac{1}{n-1}(-(2n+1)E + 2(n+1)),$$

therefore we have

$$\begin{aligned} d(n) &= n - 1, \\ B &= -(2n+1)E + 2(n+1), \\ \bar{L} &= B^* \circ E = 2(n+1)E - (2n-1), \qquad (25) \\ p(n) &= n - 2, \\ R &= 2n(n+1)/(n-2), \\ u(n) &= 2n(n+1)\bar{t}(n). \end{aligned}$$

Let $\bar{t}(n)$ *be a sequence defined for all* $n \in [v, w] \cap \mathbb{Z}$ *and satisfying* $\bar{L}\bar{t}(n) = 0$ *for all* $n \in [v, w-1] \cap \mathbb{Z}$ *where* $\bar{L}$ *is given in (25). Then by Theorem 5, the sequence* $t(n) = (n-2)\bar{t}(n)$ *satisfies* $Lt(n) = 0$ *for all* $n \in [v, w-1] \cap \mathbb{Z}$*, and the formula*

$$\sum_{n=v}^{w-1} t(n) = 2w(w+1)\bar{t}(w) - 2v(v+1)\bar{t}(v) \qquad (26)$$

is valid whenever $v \le w - 1$*. The general solution of* $\bar{L}y = 0$ *is* $\bar{t}(n) = c\frac{(-1/2)_n}{(1)_n}$ *where* c *is an arbitrary constant. Thus, by taking* $c = -1/4$*, we see that (26) can be used to sum the term* $t(n) = (n-2)\bar{t}(n) = (2-n)\frac{(-1/2)_n}{4(1)_n}$ *considered in Example 4.* ■

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