On Truncated Series Involved in Exponential-logarithmic Solutions of Truncated LODEs

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Abstract. Previously, the authors proposed algorithms for finding exponential-logarithmic solutions of linear ordinary differential equations with coefficients in the form of series, for which only a finite number of initial terms is known. Each solution involves a finite set of power series, for which the maximum possible number of terms is calculated. Below, these algorithms are supplemented with the option to confirm the impossibility of obtaining a larger number of terms in the series without using additional information about the given equation. Such a confirmation has the form of a counterexample to the assumption that it is possible to obtain additional terms of the series involved in the solution that are invariant under all prolongations of the given equation.

Keywords: Differential equations · Truncated power series · Computer algebra systems

1 Introduction

The representation of solutions of linear ordinary differential equations requires the use of power and Laurent series. This is the subject of many theoretical studies (see, e.g., [19–23, 26, 27]) and found numerous application in computer algebra (see, e.g., [1–5, 8, 11, 17, 28]).

The proposed paper is a continuation of the series of works by the authors on LODE with coefficients, having the form of such power series, with respect to which only their first terms are known. Thus, about the considered equations, there is only some incomplete information. In our previous papers, we proposed algorithms for finding solutions of such equations in the form of Laurent series, as well as the search for regular and exponential-logarithmic solutions. It has been proven that these algorithms allow one to find the maximum possible number of terms of those series that are included in the solutions. The algorithms are implemented by the authors as a package of procedures. The user of these procedures may find it is desirable to obtain some visual arguments in favor of the maximum number of found terms of the series. Below, the authors proposed such visual arguments: for an arbitrary equation with truncated coefficients, a new algorithm presents two prolonged versions of the original equation whose solutions differ from each other in subsequent (not included in the number of previously found) terms of the series included in the solutions.

2 Truncated Equations

Suppose that K is an algebraically closed field of characteristics 0. The standard notation $K[x]$ is used below for a ring of polynomials in x over K. A ring of formal power series in x over K is denoted by $K[[x]]$, a field of formal Laurent series is denoted by $K((x))$. It is clear that $K[x] \subset K[[x]] \subset K((x))$. For any nonzero element $a(x) = \sum a_i x^i$ in $K((x))$, its *valuation* $\operatorname{val} a(x)$ is defined by the equality $\operatorname{val} a(x) = \min \{i \mid a_i \neq 0\}$, while $\operatorname{val} 0 = \infty$.

The differential equations in the paper are represented with $\theta = x\frac{d}{dx}$ instead of $\frac{d}{dx}$. It is convenient for the algorithms to solve linear ordinary differential equations with coefficients in the form of truncated series (see [6, 7, 13, 14, 16]). We consider such equations in the form

$$a_r(x)\theta^r y(x) + a_{r-1}(x)\theta^{r-1} y(x) + \cdots + a_0(x)y(x) = 0, \tag{1}$$

where $y(x)$ is an unknown function of x. *The equation coefficients* $a_0(x), a_1(x), \ldots, a_r(x)$ are *truncated series*, i.e., for each $i = 0, 1, \ldots, r$ we have

$$a_i(x) = \sum_{j=0}^{t_i} a_{ij} x^j + O(x^{t_i+1}) \tag{2}$$

where $a_{ij} \in K$; t_i is an integer such that $t_i \geqslant -1$ (if $t_i = -1$ then the sum in (2) is 0). Hereinafter, the symbol $O(x^t)$ involved in the formal expressions denotes some series, whose valuation is not less than t. For a series

$$\sum_{k=l}^{t} a_k x^k + O(x^{t+1}),$$

$a_k \in K$, l, t are integer, $t \geqslant l$, we call t *the truncation degree*. Note that a coefficient in (1) can be in the form $O(x^m)$, $m \geqslant 0$.

We refer as *a prolongation* of equation (1) to any equation

$$\tilde{a}_r(x)\theta^r y(x) + \tilde{a}_{r-1}(x)\theta^{r-1} y(x) + \cdots + \tilde{a}_0(x)y(x) = 0,$$

such that $\tilde{a}_i(x) - a_i(x) = O(x^{t_i+1})$, i.e. $\operatorname{val}(\tilde{a}_i(x) - a_i(x)) > t_i$, $i = 0, 1, \ldots, r$. We consider as prolongations both equations with truncated coefficients, and equations with completely specified series coefficients, i.e., equations

$$\left(\sum_{j=0}^{\infty} \tilde{a}_{rj} x^j\right) \theta^r y(x) + \left(\sum_{j=0}^{\infty} \tilde{a}_{r-1,j} x^j\right) \theta^{r-1} y(x) + \cdots$$
$$\cdots + \left(\sum_{j=0}^{\infty} \tilde{a}_{0j} x^j\right) y(x) = 0. \quad (3)$$

3 Truncated Solutions

Formal *exponential-logarithmic* solutions of equation (3) are solutions in the form

$$e^{Q(x^{-1/q})}\, x^{\lambda} w(x^{1/q}), \quad (4)$$

where Q is a polynomial with coefficients in K, $q \in \mathbb{Z}_{>0}$, $\lambda \in K$,

$$w(x) = \sum_{s=0}^{m} w_s(x) \ln^s x,$$

$m \in \mathbb{Z}_{\geq 0}$, $w_s(x) \in K((x))$, $s = 0, \ldots, m$, and $w_m(x) \neq 0$. In (4), the factor $x^{\lambda} w(x^{1/q})$ is *the regular part*, $Q(x^{-1/q})$ is *the exponent of irregular part*, and q is *the ramification index*.

When $q = 1$ and $Q \in K$, solution (4) is called *formal regular* solution, otherwise it is called *irregular*. When $q = 1$, $Q \in K$, $\lambda \in \mathbb{Z}$ and $w(x) \in K((x))$, formal regular solution (4) is called *Laurent* one. In the further references of solutions in the paper we skip the word "formal", but it is assumed.

Suppose that the leading coefficient $\tilde{a}_r(x)$ is nonzero in equation (3) with completely specified coefficients. It is known (see e.g. [20, Ch. V], [26, 29, 17]) that for equation (3), there exist r solutions in form (4), which are linearly independent over K. Algorithms are proposed in [26, 29, 17, 18] for finding the ramification index q and the exponent of irregular part $Q(x^{-1/q})$ for r linearly independent solutions of the form (4). Suppose that the valuation of at least one of the coefficients in (3) is equal to 0. Then, to construct the ramification index q and the exponent of irregular part $Q(x^{-1/q})$ for all solutions, it is sufficient to know $r \operatorname{val} \tilde{a}_r(x)$ initial coefficients of all $\tilde{a}_i(x)$, $i = 0, 1, \ldots, r$ (see e.g. [25]). To construct the regular part of the solution with any given truncation degree of the series in $w(x)$, the algorithms proposed in [20, ch. IV], [21], [22, ch. II, VIII] may be used. For this construction, it is also sufficient to know some finite number of initial coefficients of all $\tilde{a}_i(x)$ ([3, Prop. 1]).

Let $Q(x^{-1/q}) \in K[x^{-1/q}]$, $q \in \mathbb{Z}_{>0}$, $\lambda \in K$ and

$$w_s^{\langle k_s \rangle}(x) = \sum_{j=j_s}^{k_s} w_{s,j} x^j + O(x^{k_s+1}),$$

$j_s, k_s \in \mathbb{Z}$, $k_s \geq j_s$, $s = 0, \ldots, m$, and $w_{m,j_m} \neq 0$. For equation (1) with truncated coefficients, the expression

$$e^{Q(x^{-1/q})} x^{\lambda} \sum_{s=0}^{m} w_s^{\langle k_s \rangle}(x^{1/q}) \ln^s x, \tag{5}$$

is referred to as *a solution with a truncated regular part* if any equation that is a prolongation of (1) has the solution $e^{Q(x^{-1/q})} x^{\lambda} \tilde{w}(x^{1/q})$ that is *a prolongation of solution* (5), i.e., $\tilde{w}(x)$ has a form

$$\tilde{w}(x) = \sum_{s=0}^{m} \tilde{w}_s(x) \ln^s x$$

and it is satisfied that $\tilde{w}_s(x) - w_s^{\langle k_s \rangle}(x) = O(x^{k_s+1})$, i.e., $\mathrm{val}\,(\tilde{w}_s(x) - w_s^{\langle k_s \rangle}(x)) > k_s$, $s = 0, 1, \ldots, m$. Such truncated solution is described as *invariant* to the prolongations of equation (1).

In [6, 7, 13, 14, 16] it is shown that for an equation of the form (1), it is possible to construct all truncated solutions with the maximum possible truncation degree of the series involved in the solution. The maximum possible truncation degree in the invariant solution s_{max} means that there is no invariant solution s that is a prolongation of s_{max} such that the truncation degree of at least one series in s is greater than the truncation degree of the corresponding series in s_{max}. We describe this case as the exhaustive use of information on a given equation in constructing truncated solutions. The above articles present algorithms for solving this problem and their implementation in Maple.

In [24, 12], we have considered the question of automatic confirmation of such an exhaustive use of information about a given equation for the construction of Laurent and regular truncated solutions. Confirmation is presented as a counterexample with two different prolongations of the given equation, which lead to the appearance of different additional terms in the solutions.

Algorithms for constructing both the truncated solutions themselves and counterexamples of the described type are based on finding solutions with *literals*, i.e., symbols used to represent unspecified coefficients of a series involved in the equation (see [7]). Literals denote the coefficients of the terms of the series, the degrees of which are greater than the truncation degree of the series. Finding solutions using literals means representing subsequent (non-invariant for all possible prolongations of the equation) terms of the series by formulas containing literals, i.e., unspecified coefficients. This allows us to clarify the influence of unspecified coefficients on the subsequent terms of the series in the solution.

Remark 1 Thus, literals are something close to undetermined coefficients. But for literals, it is not supposed to find specific values that allow one to find out all the solutions to the original differential equation. Here the goal is to find out whether the unknown coefficients of the series included in the equation have an effect on the initial terms of those series that are included in the solutions.

In this article, we extend the results obtained in [24, 12] to the case of exponential-logarithmic solutions with a truncated regular part. The problem of presenting two different prolongations of the original equation, which form a counterexample to the assumption about the possibility of adding invariant terms to the series involved in the truncated exponential-logarithmic solutions of the given truncated equation, is solved.

4 The Case of Exponential-logarithmic Solutions

Prolongations of equation (1) which contain literals $U_{[i,j]}$ look like the following:

$$\left(\sum_{j=0}^{t_r} a_{rj}x^j + \sum_{j=t_r+1}^{\infty} U_{[r,j]}x^j\right)\theta^r y(x) + \\ + \left(\sum_{j=0}^{t_{r-1}} a_{r-1,j}x^j + \sum_{j=t_{r-1}+1}^{\infty} U_{[r-1,j]}x^j\right)\theta^{r-1} y(x) + \cdots \\ \cdots + \left(\sum_{j=0}^{t_0} a_{0j}x^j + \sum_{j=t_0+1}^{\infty} U_{[0,j]}x^j\right) y(x) = 0 \quad (6)$$

(we use the notation $U_{[i,j]}$ rather than, say $U_{i,j}$ to emphasize the special status of these unknowns).

The algorithms from [17, 18] allow computing exponential parts $e^{Q(x^{-1/q})}$ of all solutions in form (4) for equation (6). We are only interested in the exponential parts that have ramification indices q and coefficients of polynomials Q that do not depend on literals. For each of such pairs q, Q the substitution

$$x = t^q, \;\; y(x) = e^{Q(1/t)}z(t) \quad (7)$$

is made in equation (6), where t is a new independent variable, and $z(t)$ is a new unknown function. As a result of the substitution with further multiplication of the equation by $e^{-Q(1/t)}$, we obtain a new equation, whose coefficients are Laurent series in t. The coefficients of the series are polynomials in literals over K. The regular solutions $t^\lambda w(t)$ of the new equation are then constructed using the version of the algorithm ([14, Sect.4.2]). For each series involved in the regular solutions, the version of the algorithm computes the maximum number of terms which are invariant under the prolongations of the equation, and one more term which depends on literals. Such a coefficient will be a polynomial over K in a finite number of literals.

In such a way we get a finite set of polynomials in literals for the exponential-logarithmic solution with regular part (5). The set may be used to construct a counterexample.

In [12], we proved the following theorem for the case of truncated Laurent and regular solutions.

Theorem 1. *([12], Theorem 1) Suppose that solutions of equation (6) involve m truncated power series*

$$c_{i0} + c_{i1}x + \cdots + c_{ik_i}x^{k_i} + p_i(u_1, \ldots, u_l)x^{k_i+1} + O(x^{k_i+2}), \qquad (8)$$

where $u_1, \ldots, u_l$ are literals, the coefficients c_{ij} are independent from the literals, while the coefficient $p_i(u_1, \ldots, u_l)$ is a non-constant polynomial in the literals, $i = 1, \ldots, m$. Then, there are $\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l \in K$ such that two prolongations of the equation that correspond to $u_j = \alpha_j$, $u_j = \beta_j$, $j = 1, \ldots, l$, lead to the occurrence of different very first additional terms in the truncated series involved in the solutions.

Now we show that a similar statement is valid for exponential-logarithmic solutions with a truncated regular part.

Theorem 2. *Let $\mathcal{E}$ be an equation of the form (1) and s be its truncated solution of the form (5), computed using the algorithm from [16]. Then there exist $\mathcal{E}_1$ and $\mathcal{E}_2$, which are two different prolongations of the equation $\mathcal{E}$ such that $\mathcal{E}_1$ has a truncated solution s_1, $\mathcal{E}_2$ has a truncated solution s_2, both solutions s_1 and s_2 are prolongations of s, and any truncated series involved in s has a prolongation both in s_1, and in s_2, while the very first additional terms of those prolongations are different.*

Proof. The algorithm from [16] is based on the construction of the truncated solutions in form (5), each series in the solutions being constructed up to the first term that contains literals and that is not included in the resulting truncated solutions. Before dropping the terms with literals each series in the truncated solutions is in form (8). Theorem 1 can be applied to all these truncated series together. Thus, there are two different sets of values $\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l \in K$ for the literals $u_1, \ldots, u_l$, which are used to construct the prolongations $\mathcal{E}_1$ and $\mathcal{E}_2$ that have truncated solutions s_1 and s_2 with different additional terms $p_i(\alpha_1, \ldots, \alpha_l)x^{k_i+1}$ and $p_i(\beta_1, \ldots, \beta_l)x^{k_i+1}$ not containing literals. □

An algorithm to compute two different sets $\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l \in K$ may be based on the approach used in [12] to prove Theorem 1.

5 Automatic Confirmation of the Solutions Truncation Degree Maximality

The counterexample computation is implemented by us as an extension of *FormalSolution* procedure from *TruncatedSeries* package. The package contains our implementation of the algorithms presented in [6, 7, 13, 14, 16, 24, 12] in Maple. The Maple library with the *TruncatedSeries* package and Maple worksheets with examples of using its commands are available from [30].

The first argument of *FormalSolution* procedure is a differential equation in the form (1). The application of θ^k to the unknown function $y(x)$ is written as $\theta(y(x), x, k)$. The truncated coefficients $a_i(x)$ of the equation, i.e., the coefficients in the form (2) are written as $b_i(x) + \mathrm{O}(x^{t_i+1})$, where $b_i(x)$ is a polynomial of the degree not higher than t_i over the field of algebraic numbers.

An unknown function of the equation is specified as the second argument of the procedure.

A row of optional arguments are also supported in the procedure (see [7, 9, 15] for details). We introduce a new optional argument '*counterexample*' = '*Eqs*', which allows obtaining the automatically constructed counterexample assigned to the variable *Eqs* in addition to the computed solution itself. The use of some optional parameters are demonstrated below.

In order to use the package download TruncatedSeries2021.zip from [30]. This archive includes two files: maple.ind and maple.lib. Put these files to some directory, for example to "/usr/userlib". Assign

$> libname := \text{"/usr/userlib"}, libname:$

in the Maple session. Make the short form name of *FormalSolution* procedure available:

$> with(TruncatedSeries):$

Consider the third-order equation with coefficients truncated to different degrees:

$$> eq := (x^4 + \mathrm{O}(x^7))\theta(y(x), x, 3) + (3x + \mathrm{O}(x^5))\theta(y(x), x, 2) + (1 + 3x^3 + 2x^2 + x + \mathrm{O}(x^4))\theta(y(x), x, 1) + \mathrm{O}(x^5)y(x) = 0:$$

Using the *FormalSolution* command we obtain exponential-logarithmic solutions whose regular parts are calculated to the maximum possible degrees:

$> FormalSolution(eq, y(x))$

$$\left[\_c_1 + \mathrm{O}\left(x^5\right) + \mathrm{e}^{\frac{1}{3x}} x^{\frac{2}{3}} \left(\_c_2 + \frac{35\_c_2 x}{27} + \frac{8947\_c_2 x^2}{1458} + \mathrm{O}\left(x^3\right)\right) + \mathrm{e}^{\frac{1}{x^3} - \frac{1}{3x}} y_{reg}(x)\right] \tag{9}$$

The first two terms of the result, i.e., $\_c_1+\mathrm{O}\left(x^5\right)$, mean that all prolongations of *eq* have Laurent solutions with valuation 0, and their initial segment till the degree 4 is equal to $\_c_1$ where $\_c_1$ is an arbitrary constant $\_c_1$.

The third term means that all prolongations of the equation *eq* have irregular solutions with the exponential part $\mathrm{e}^{\frac{1}{3x}}$ and the regular part, which is the same up to an arbitrary constant $\_c_2$ for all prolongations of the original equation.

The fourth term means that all prolongations of the equation *eq* have irregular solutions with the exponential part $\mathrm{e}^{\frac{1}{x^3} - \frac{1}{3x}}$. Moreover, there are such prolongations that their regular parts differ by λ.

If, when calling the *FormalSolution* command, the optional argument '*output*' = '*literal*' is used, then the regular parts of the solution are calculated to the maximum degree and, furthermore, terms are added with coefficients depending on literals. In some cases, it is possible to obtain the expression for λ which also depends on literals.

> *FormalSolution*(*eq*, *y*(*x*), '*output*' = '*literal*')

$$\begin{aligned}
&_c_1 - \frac{U_{[0,5]}_c_1 x^5}{5} + \mathrm{O}\left(x^6\right) + \mathrm{e}^{\frac{1}{3x}} x^{\frac{2}{3}} \left(_c_2 + \frac{35_c_2 x}{27} + \frac{8947_c_2 x^2}{1458}\right.\\
&\left.+ \left(\frac{5832431}{118098}_c_2 - \frac{1}{9}_c_2 U_{[1,4]} + \frac{1}{27}_c_2 U_{[2,5]}\right) x^3 + \mathrm{O}\left(x^4\right)\right) \qquad (10)\\
&+\mathrm{e}^{\frac{1}{x^3} - \frac{1}{3x}} x^{\frac{19}{3} + 3U_{[3,7]}} \left(_c_3 + \mathrm{O}(x)\right)
\end{aligned}$$

Here the literal $U_{[i,k]}$ denotes the coefficient of $x^k\theta^i$. There are two sets of values from $\bar{\mathbb{Q}}$ for these literals such that the expressions

$$\frac{U_{[0,5]}\_c_1}{5},$$

$$\frac{5832431}{118098}\_c_2 - \frac{1}{9}\_c_2 U_{[1,4]} + \frac{1}{27}\_c_2 U_{[2,5]}$$

and

$$\frac{19}{3} + 3U_{[3,7]}$$

take different values. These two sets correspond to two prolongations of the equation *eq*. Their solutions are different prolongations of solution (9) and all regular parts of the solution are prolonged. We call such prolongations a counterexample. Obviously, there are an infinite number of counterexamples. As a result of running the *FormalSolution* command with the new optional argument '*counterexample*' = '*Eqs*', the variable *Eqs* will be assigned a pair of the equations which forms one of the possible counterexamples:

> *FormalSolution*(*eq*, *y*(*x*), '*counterexample*' = '*Eqs*') :

For the first counterexample equation

> *Eqs*[1]

$$\begin{aligned}
&\left(x^5 + \mathrm{O}\left(x^6\right)\right) y(x) + \left(3x^3 + 2x^2 + x + 1 + 4x^4 + \mathrm{O}\left(x^5\right)\right) \theta(y(x), x, 1)\\
&+ \left(3x + \mathrm{O}\left(x^6\right)\right) \theta(y(x), x, 2) + \left(x^4 - 4x^7 + \mathrm{O}\left(x^8\right)\right) \theta(y(x), x, 3) = 0
\end{aligned} \qquad (11)$$

using *FormalSolution* we obtain a truncated solution

> *FormalSolution*$(Eqs[1], y(x))$

$$\left[\_c_1 - \frac{\_c_1 x^5}{5} + \mathrm{O}\left(x^6\right) + \mathrm{e}^{\frac{1}{3x}} x^{\frac{2}{3}} \left(\_c_2 + \frac{35\_c_2 x}{27} + \frac{8947\_c_2 x^2}{1458} + \frac{5779943\_c_2 x^3}{118098} + \mathrm{O}\left(x^4\right)\right) + \frac{\mathrm{e}^{\frac{1}{x^3} - \frac{1}{3x}} \left(\_c_3 + \mathrm{O}(x)\right)}{x^{\frac{17}{3}}}\right] \tag{12}$$

For the second counterexample equation

> *Eqs*[2]

$$\begin{aligned}&\left(5x^5 + \mathrm{O}\left(x^6\right)\right) y(x) + \left(3x^3 + 2x^2 + x + 1 - 2x^4 + \mathrm{O}\left(x^5\right)\right) \theta(y(x), x, 1) \\ &+ \left(3x + \mathrm{O}\left(x^6\right)\right) \theta(y(x), x, 2) + \left(x^4 - x^7 + \mathrm{O}\left(x^8\right)\right) \theta(y(x), x, 3)\end{aligned} \tag{13}$$

we obtain

> *FormalSolution*$(Eqs[2], y(x))$

$$\left[-\_c_1 x^5 + \_c_1 + \mathrm{O}\left(x^6\right) + \mathrm{e}^{\frac{1}{3x}} x^{\frac{2}{3}} \left(\_c_2 + \frac{35\_c_2 x}{27} + \frac{8947\_c_2 x^2}{1458} + \frac{5858675\_c_2 x^3}{118098} + \mathrm{O}\left(x^4\right)\right) + \mathrm{e}^{\frac{1}{x^3} - \frac{1}{3x}} x^{\frac{10}{3}} \left(\_c_3 + \mathrm{O}(x)\right)\right] \tag{14}$$

It can be seen that (12) and (14) are prolongations of (9), they differ in all regular parts. The exponents λ of the third regular part are also different: $\lambda = -\frac{17}{3}$ for (12) and $\lambda = \frac{10}{3}$ for (14).

6 Conclusion

In this paper, we have described an algorithm which confirms the exhaustive use of the information contained in a truncated LODE in the process of finding truncated exponential-logarithmic solutions by our algorithms which were published earlier.

The mathematical techniques we employ in this paper use the algebras of differential operators and polynomials, and we give the explicit counterexample for the supposition that additional terms of solutions of a given LODE can de obtained.

From our work, new questions arise. For example, can our results be extended to systems of LODEs? We will continue to investigate this line of enquiry.

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