ISSN 0361-7688, Programming and Computer Software, 2008, Vol. 34, No. 4, pp. 187–190. © Pleiades Publishing, Ltd., 2008.

On the Bottom Summation

S. A. Abramov[a] and M. Petkovšek[b 1]

[a] *Russian Academy of Sciences, Dorodnicyn Computing Centre, ul. Vavilova 40, Moscow GSP-1, 119991 Russia*

[b] *Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia*

e-mail: sabramov@ccas.ru, marko.petkovsek@fmf.uni-lj.si

Received September 10, 2007

Abstract—We consider summation of consecutive values $\varphi(v)$, $\varphi(v+1)$, …, $\varphi(w)$ of a meromorphic function $\varphi(z)$, where $v, w \in \mathbb{Z}$. We assume that $\varphi(z)$ satisfies a linear difference equation $L(y) = 0$ with polynomial coefficients, and that a summing operator for L exists (such an operator can be found—if it exists—by the Accurate Summation algorithm, or, alternatively, by Gosper's algorithm when $\operatorname{ord} L = 1$). The notion of *bottom summation* which covers the case where $\varphi(z)$ has poles in $\mathbb{Z}$ is introduced.

DOI: 10.1134/S0361768808040014

1. INTRODUCTION

The object of this note is to present the results of [1] related to the so-called "bottom summation" in a simpler form. The object of our investigation is correctness of the discrete Newton–Leibniz formula for definite summation in the case where a summing operator has been successfully constructed by the Accurate Summation algorithm [2] or by Gosper's algorithm [3]. In the detailed proofs given in [1], many abstract notions were used. In addition, it was necessary to prove a number of auxiliary statements. As a result, the paper [1] is quite difficult to read. However, the main results of [1] are of some practical interest for computer algebra and their short presentation without complicated proofs can be useful.

Below, we present the main results of [1] in a simpler form and give some illustrations. Full proofs can be found in [1].

2. SUMMING OPERATORS

Let E be the shift operator such that $E(f(k)) = f(k+1)$ for sequences $f(k)$, where $k \in \mathbb{Z}$, and $E(\varphi(z)) = \varphi(z+1)$ for analytic functions, $z \in \mathbb{C}$. Let

$$L = a_d(k)E^d + \ldots + a_1(k)E + a_0(k) \in \mathbb{C}(k)[E].$$

We say that an operator $R \in \mathbb{C}(k)[E]$ is a *summing* operator for L if

$$(E-1)\circ R = 1 + M \circ L \qquad (1)$$

for some $M \in \mathbb{C}(k)[E]$. We can assume without loss of generality that $\operatorname{ord} R = \operatorname{ord} L - 1 = d - 1$:

$$R = r_{d-1}(k)E^{d-1} + \ldots + r_1(k)E + r_0(k) \in \mathbb{C}(k)[E].$$

[1] The text was submitted by the authors in English.

3. THE DISCRETE NEWTON–LEIBNIZ FORMULA

If a summing operator exists, then it can be constructed by the Accurate Summation algorithm [4] or, when $d = 1$, by Gosper's algorithm [3]. At first glance, in those cases where $R \in \mathbb{C}(k)[E]$ exists, equality (1) gives us an opportunity to use the discrete Newton–Leibniz formula (DNLF)

$$\sum_{k=v}^{w-1} f(k) = g(w) - g(v)$$

for all integers $v < w$ and for any sequence f such that $L(f) = 0$ taking $g = R(f)$. Indeed, we can apply both sides of $(E-1)\circ R = 1 + M \circ L$ to f. This gives

$$(E-1)(R(f)) = f + M(L(f)).$$

Set $g = R(f)$. Taking into account that $L(f) = 0$, we get

$$(E-1)g = f,$$

or, equivalently,

$$g(k+1) - g(k) = f(k).$$

As a consequence, the DNLF is applicable:

$$\sum_{k=v}^{w-1} f(k) = \sum_{k=v}^{w-1} (g(k+1) - g(k))$$
$$= g(w) - g(w-1) + g(w-1)$$
$$- g(w-2) + \ldots + g(v+1) - g(v)$$
$$= g(w) - g(v)$$

(the telescoping effect).

However, it was shown that, if R has rational-function coefficients that have poles in $\mathbb{Z}$, then this formula may give incorrect results (an example will be demonstrated below). This gives rise to defects in many implementations of summation algorithms.

Example 1. Consider the sequence

$$f(k) = \frac{\binom{2k-3}{k}}{4^k},$$

which satisfies the first-order recurrence relation $2(k+1)(k-2)f(k+1) - (2k-1)(k-1)f(k) = 0$.

Although Gosper's algorithm succeeds on this sequence, producing $R(k) = \frac{2k(k+1)}{k-2}$, and $f(k)$ is defined for all $k \in \mathbb{Z}$, the discrete Newton–Leibniz formula

$$\sum_{k=0}^{w-1} f(k) = R(w)f(w) - R(0)f(0)$$

$$= \frac{2w(w+1)\binom{2w-3}{w}}{(w-2)4^w}$$

is not correct: if we assume that the value of $\binom{2k-3}{k}$ is 1 when $k = 0$ and -1 when $k = 1$ (as is common practice in combinatorics), then the expression on the right gives the true value of the sum only at $w = 1$.

4. THE BOTTOM SUMMATION

Suppose that L acts on analytic functions:

$$L = a_d(z)E^d + \ldots + a_1(z)E + a_0(z) \in \mathbb{C}(z)[E]. \quad (2)$$

We consider the summing operator (if it exists) in the form

$$R = r_{d-1}(z)E^{d-1} + \ldots + r_1(z)E + r_0(z) \in \mathbb{C}(z)[E].$$

Let $\varphi(z)$ be a meromorphic solution of $L(y) = 0$.

It turns out that, if $\varphi(z)$ has no pole in $\mathbb{Z}$, then neither does $R(\varphi)(z)$, and we can use the DNLF to sum values $\varphi(k)$ for $k = v, v + 1, \ldots, w$. So, such undesirable phenomena as demonstrated in Example 1 cannot occur if the elements of the sequence under summation are the values $\varphi(k)$, $k \in \mathbb{Z}$, of an analytic function $\varphi(z)$, which satisfies (in the complex plane $\mathbb{C}$) the same difference equation with polynomial coefficients as does the original sequence (at integer points).

This follows from a stronger statement. The fact is that, even if $\varphi(z)$ has some poles in $\mathbb{Z}$, the summation task can nevertheless be performed correctly.

For any $k \in \mathbb{Z}$, the function $\varphi(z)$ can be represented by Laurent's series

$$\varphi(z) = c_{k,\rho_k}(z-k)^{\rho_k} + c_{k,\rho_k+1}(z-k)^{\rho_k+1} + \ldots$$

with $\rho_k \in \mathbb{Z}$ and $c_{k,\rho_k} \neq 0$. If $L(\varphi) = 0$, then there exists the minimal element ρ in the set of all ρ_k, $k \in \mathbb{Z}$. This ρ we call the *depth* of $\varphi(z)$ and denote it by depth (φ).

We associate with $\varphi(z)$ the sequence $f(k)$ such that $f(k) = c_{k,\rho_k}$ if $\rho_k = \rho$, and $f(k) = 0$ otherwise. This $f(k)$ we call the *bottom* of $\varphi(z)$ and denote it as bott(φ).

We illustrate these notions by the following simple example.

It is well known that $\Gamma(z)$ has finite values when $z = 1, 2, \ldots$ and has simple poles when $z = 0, -1, -2, \ldots$.

We have

$$\mathrm{depth}(\Gamma) = -1$$

and

$$\mathrm{bott}(\Gamma)(k) = \begin{cases} 0, & \text{if } k > 0 \\ \dfrac{(-1)^{k+1}}{(-k-1)!}, & \text{if } k \le 0. \end{cases}$$

If we consider $\Gamma(z)$ only in the half-plane $\mathrm{Re}\,z > 0$, then its depth is 0 and the bottom is the sequence

$$f(k) = (k-1)!, \quad k = 1, 2, \ldots.$$

Proposition 1. *Let* $L(\varphi) = 0$. *Then*, $L(\mathrm{bott}(\varphi)) = 0$.

Proposition 2. *Let* $L(\varphi) = 0$, *and let R be a summing operator for L. Then*, $\mathrm{depth}(\varphi) = \mathrm{depth}(R(\varphi))$.

Theorem 1 (on the bottom summation). *Let* $L(\varphi(z)) = 0$, *and let R be a summing operator for L. Denote* $\psi(z) = R(\varphi(z))$. *Then the bottom summation formula*

$$\sum_{k=v}^{w-1} \mathrm{bott}(\varphi)(k) = \mathrm{bott}(\psi)(w) - \mathrm{bott}(\psi)(v)$$

is valid for any $v < w$. *In particular*, *if* φ *has no pole in* $\mathbb{Z}$ *(i.e., depth*$(\varphi) \ge 0$*), then the function* $\psi(z)$ *has no pole in* $\mathbb{Z}$, *and the discrete Newton–Leibniz formula*

$$\sum_{k=v}^{w-1} \varphi(k) = \psi(w) - \psi(v)$$

is valid for any $v < w$.

Example 1 (continued). Assume that the value of $\binom{2k-3}{k}$ is defined as

$$\lim_{z \to k} \frac{\Gamma(2z-2)}{\Gamma(z+1)\Gamma(z-2)}; \quad (3)$$

this is a natural extension of the formula

$$\binom{2k-3}{k} = \frac{(2k-3)!}{k!(k-3)!}$$

for all $k \in \mathbb{Z}$.

Set

$$\varphi(z) = \frac{\Gamma(2z-2)}{\Gamma(z+1)\Gamma(z-2)4^z},$$

and

$$\psi(z) = \frac{2z(z+1)}{z-2}\varphi(z).$$

The limit in (3) exists for all $k \in \mathbb{Z}$, and depth(φ) = 0. Now, the DNLF gives the correct result

$$\sum_{k=0}^{w-1} \frac{\Gamma(2k-2)}{\Gamma(k+1)\Gamma(k-2)4^k} = \frac{2w(w+1)\Gamma(2w-2)}{(w-2)\Gamma(w+1)\Gamma(w-2)4^w}$$

for $w = 1, 2, \ldots$.

Earlier, we assumed that the value of $\binom{2k-3}{k}$ is 1 when $k = 0$ and –1 when $k = 1$ (as is common in combinatorics). However,

$$\lim_{z\to 0} \frac{\Gamma(2z-2)}{\Gamma(z+1)\Gamma(z-2)} = \frac{1}{2} \neq 1$$

and

$$\lim_{z\to 1} \frac{\Gamma(2z-2)}{\Gamma(z+1)\Gamma(z-2)} = -\frac{1}{2} \neq -1.$$

This example demonstrates a conflict between the combinatorial and analytic definitions of the symbol $\binom{p}{q}$.

Example 2. The function $\varphi(z) = z\Gamma(z+1)$ satisfies the equation $L(y) = 0$, where $L = zE - (z+1)^2$. We have $R = \frac{1}{z}$, ord$R$ = 0, and $\psi(z) = R(\varphi)(z) = \Gamma(z+1)$. Evidently, $\varphi(z)$ has finite values when $z = 0, 1, \ldots$ and has simple poles when $z = -1, -2, \ldots$. We have depth(φ) = depth(ψ) = –1 and

$$\operatorname{bott}(\varphi)(k) = \begin{cases} \frac{(-1)^{k+1}k}{(-k-1)!}, & \text{if} \quad k < 0 \\ 0, & \text{if} \quad k \geq 0, \end{cases}$$

$$\operatorname{bott}(\psi)(k) = \begin{cases} \frac{(-1)^{k+1}}{(-k-1)!}, & \text{if} \quad k < 0 \\ 0, & \text{if} \quad k \geq 0. \end{cases}$$

The bottom summation gives us

$$\sum_{k=v}^{w-1} \frac{(-1)^k k}{(-k-1)!} = \frac{(-1)^w}{(-w-1)!} - \frac{(-1)^v}{(-v-1)!}$$

for any $v < w \leq 0$, or, equivalently,

$$\sum_{k=v}^{w-1} \frac{(-1)^k k}{(k-1)!} = \frac{(-1)^{w+1}}{(w-2)!} - \frac{(-1)^{v+1}}{(v-2)!}$$

for any $1 \leq v < w$.

If we consider $\varphi(z)$ in the half-plane Re$z \geq 0$, then depth(φ) = depth(ψ) = 0, and we have

$$\sum_{k=v}^{w-1} k\Gamma(k+1) = \Gamma(w+1) - \Gamma(v+1)$$

for any $0 \leq v < w$ or, equivalently,

$$\sum_{k=v}^{w-1} k \cdot k! = w! - v!.$$

The equation $L(y) = 0$ with L of the form (2) always has a non-zero solution, which is meromorphic in $\mathbb{C}$. This is a consequence of the following result of M. Barkatou and J.-P. Ramis:

Theorem 2 ([5]). *Let L be of the form* (2), *where* $a_0(z)$ *is a non-zero polynomial. Let* $c \in \mathbb{R}$ *be such that the real part of each of the roots of* $a_0(z)$ *is not larger than c. Then*, *the equation* $L(y) = 0$ *has a solution which is holomorphic* (*i.e.*, *analytic and having no singularity*) *in the half-plane* Re$z > c$.

5. ADDITIONAL EXAMPLES

Example 3. The rational function $\varphi(z) = \frac{1}{z(z+1)}$ satisfies the equation $L(y) = 0$, where $L = (z+2)E - z$. We have $R = -z - 1$ and $\psi(z) = R(\varphi)(z) = -\frac{1}{z}$. It is easy to see that depth(φ) = depth(ψ) = –1 and

$$\operatorname{bott}(\varphi)(k) = \begin{cases} 1, & \text{if} \quad k = 0 \\ -1, & \text{if} \quad k = -1 \\ 0, & \text{otherwise}, \end{cases}$$

$$\operatorname{bott}(\psi)(k) = \begin{cases} -1, & \text{if} \quad k = 0 \\ 0, & \text{otherwise}. \end{cases}$$

A simple direct check shows that, for any $v < w$,

$$\sum_{k=v}^{w-1} \text{bott}(\varphi)(k) = \text{bott}(\psi)(w) - \text{bott}(\psi)(v).$$

If we consider $\varphi(z)$ only in the half-plane $\text{Re}\,z > 0$, then $\text{depth}(\varphi) = \text{depth}(\psi) = 0$, and we have

$$\sum_{k=v}^{w-1} \frac{1}{k(k+1)} = -\frac{1}{w} + \frac{1}{v}$$

for any $1 \le v < w$.

Example 4. For the second-order operator

$$L = (z-3)(z-2)(z+1)E^2$$
$$-(z-3)(z^2-2z-1)E-(z-2)^2,$$

there exists the first-order summing operator

$$R = zE + \frac{1}{z-3}.$$

It follows from Theorem 2 that the equation $L(y) = 0$ has solutions holomorphic in the half-plane $\text{Re}\,z > 2$. Denote by $\varphi(z)$ an arbitrary solution of this kind. By our theorem, the DNLF must be correct for $3 \le v < w$ in spite of the fact that one of the coefficients of R has a pole at $z = 3$.

We can find the values of $\varphi(z)$ and of $\psi(z) = R(\varphi)(z)$ when $z = 3$. An algorithm from [2] yields

$$\varphi(z) = (4\varphi(5) - 2\varphi(4))(z-3) + O((z-3)^2),$$
$$z \longrightarrow 3,$$

which gives $\varphi(3) = 0$, $\psi(3) = \varphi(4) + 4\varphi(5)$. This value of $\psi(3)$ can be used in the DNLF when $3 = v < w$.

$$\sum_{k=3}^{w-1} \varphi(k) = w\varphi(w+1) + \frac{\varphi(w)}{w-3} - 4\varphi(5) - \varphi(4).$$

CONCLUSIONS

Indiscriminate application of the discrete Newton–Leibniz formula to the output of Gosper's algorithm or of the Accurate Summation algorithm in order to compute a definite sum can lead to incorrect results. This can be observed in many implementations of these algorithms in computer algebra systems.

In this paper it is shown, in particular, that such undesirable phenomena cannot occur if the elements of the sequence under summation are the values $\varphi(k)$, $k \in \mathbb{Z}$, of an analytic function $\varphi(z)$, which satisfies (in the complex plane $\mathbb{C}$) the same difference equation with polynomial coefficients as does the original sequence (at integer points).

A practical consequence of this is as follows. If these conditions are satisfied, then a computer-algebra-system user can be sure that the obtained sum was computed correctly.

On the more theoretical side, if $\varphi(z)$ mentioned above has some poles at integer points, then, nevertheless, one can find the sum of a sequence, which, however, is not the sequence of values of $\varphi(k)$, $k \in \mathbb{Z}$, but is associated with $\varphi(z)$ in a natural way. This can yield an interesting (and, probably, unexpected) identity. We call this sequence associated with $\varphi(z)$, the *bottom* of $\varphi(z)$. If $\varphi(z)$ is defined for all $z \in \mathbb{Z}$, then its bottom coincides with the sequence $\varphi(k)$, $k \in \mathbb{Z}$.

ACKNOWLEDGMENTS

This work was supported in part by the Russian Foundation for Basic Research, project no. 07-01-00482-a, and by MVZT RS project no. P1-0294.

REFERENCES

1. Abramov, S.A. and Petkovšek, M., Analytic Solutions of Linear Difference Equations, Formal Series, and Bottom Summation, *Proc. Computer Algebra in Scientific Computing, 10th Int. Workshop, CASC'07* (Bonn, Germany, 2007), *Lecture Notes in Computer Science*, 2007, vol. 4770, pp. 1–10.
2. Abramov, S.A. and van Hoeij, M., Set of Poles of Solutions of Linear Difference Equations with Polynomial Coefficients, *Zh. Vychisl. Mat. Mat. Fiz.*, 2003, vol. 43, no. 1, pp. 60–65 [*J. Comput. Math. Math. Phys.*, 2003, vol. 43, no. 1, pp. 57–62].
3. Gosper, R.W., Jr., Decision Procedure for Indefinite Hypergeometric Summation, *Proc. Natl. Acad. Sci. USA*, 1978, vol. 75, pp. 40–42.
4. Abramov, S.A. and van Hoeij, M., Integration of Solutions of Linear Functional Equations, *Integral Transforms Special Functions*, 1999, vol. 8, pp. 3–12.
5. Barkatou, M.A., Contribution à l'étude des équations différentielles et aux différences dans le champ complexe, *PhD Dissertation*, Grenoble, France: INPG, 1989.