On Polynomial Solutions of Linear Operator Equations

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1 Introduction

Let K be a field of characteristic 0 and $L : K[x] \to K[x]$ an endomorphism of the K-linear space of univariate polynomials over K. We consider the following computational tasks concerning L:

T1. **Homogeneous equation** $Ly = 0$**:** Compute a basis of $\operatorname{Ker} L$ in $K[x]$.

T2. **Inhomogeneous equation** $Ly = f$**:** Given $f \in K[x]$, compute a basis of the affine space $L^{-1}(f)$ in $K[x]$.

T3. **Parametric inhomogeneous equation** $Ly = \sum_{i=1}^{m} \lambda_i f_i$**:** Given $f_1, f_2, \ldots, f_m \in K[x]$, compute a basis of $\operatorname{Ker} L'$ where $L' : (K[x] \oplus K^m) \to K[x]$ and $L' : (y, \lambda) \mapsto Ly - \sum_{i=1}^{m} \lambda_i f_i$, for $y \in K[x]$, $\lambda \in K^m$.

Many problems and algorithms in differential and difference algebra contain these tasks as subproblems which, however conceptually simple, often account for a fair share of the overall computing time. For instance, the algorithms for

- finding all the rational solutions of differential and (q-)difference equations [Abr89b, Abr95],
- finding Liouvillian solutions of differential equations [Sin91],
- finding (q-)hypergeometric solutions of (q-)difference equations [Pet92, Abr&Pet95],
- factoring linear differential and difference operators with rational coefficients [Bro&Pet94],
- indefinite hypergeometric summation (Gosper's algorithm) [Gos78],
- definite hypergeometric summation (Zeilberger's "creative telescoping" algorithm) [Zei91],

all need to find polynomial solutions of operator equations at some stage. For instance, task T3 is crucial for the algorithms presented in [Sin91] and [Zei91]. Clearly, it is important to use as efficient algorithms as possible for such time-critical subproblems.

We consider three types of linear operators: differential, difference, and q-difference operators, all with polynomial coefficients. Throughout the paper, L is the operator, r its order and d the maximum degree of its coefficients. To find polynomial solutions of an equation of the form $Ly = f$, it is natural to proceed in two steps:

1. Compute a degree bound N for polynomial solutions of the equation.
2. Given N, compute polynomial solution(s) of degree(s) $\leq N$.

Degree bounds required in step 1 are given in [Abr89b] for the differential and difference cases. Here we improve slightly upon these bounds and also give them for the q-difference case. Our main contribution, however, is to step 2 of the above procedure. Usually, it is carried out by means of *undetermined coefficients*: expand $y(x)$ with respect to the usual power basis $(x^n)_{n=0}^{\infty}$ and substitute this expression into the equation, obtaining a triangular system of linear algebraic equations for c_i's. Then there is a one-to-one correspondence between solutions of this system and those of the original equation.

The problem with the method of undetermined coefficients is that the number of unknown coefficients, $N + 1$ (respectively $N + m + 1$ in the parametric case), is often much larger than the order of the equation, r. We show that by using appropriate polynomial bases, the resulting linear system has a band-diagonal matrix which represents a recurrence for the unknown coefficients. After taking full advantage of this recurrence, the remaining linear system that needs to be solved is of size corresponding to the bandwidth of that matrix.

In particular, for differential, difference and q-difference equations, we reduce step 2 of the above procedure to a system of $r + d$ linear algebraic equations with only r unknowns (respectively $r + m$ in the parametric case). Thus, in order

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to minimize the number of unknowns in the final linear system, one should use the method proposed here when $N \geq r$, and the method of undetermined coefficients when $N < r$.

In Section 2 we define formal series expansions with respect to distinct-degree bases of $K[x]$, and use them to find polynomial solutions of operator equations. Section 3 gives algorithms to solve problems T1, T2 and T3. Given a recurrence we show how to find all sequences satisfying it on a specified interval, on which its leading coefficient may vanish at some points. Section 4 applies these results to the cases when L is a linear differential, difference, or q-difference operator. In Section 5 we present some empirical evidence suggesting that our method can be much faster than the commonly used method of undetermined coefficients. Finally we show in section 6 how our method can also be used to compute formal series solutions of the equations we are considering.

Throughout the paper, the set of nonnegative integers is denoted by $\mathbb{N}$. We take the degree of the zero polynomial to be $-\infty$.

2 Formal series expansions in $K[x]$

Let $(P_n(x))_{n=0}^{\infty}$ be a sequence of polynomials from $K[x]$ such that

P1. $\deg P_n = n$ for $n \geq 0$,

P2. $P_n \mid P_m$ for $0 \leq n < m$.

Then $\{P_0, P_1, \ldots\}$ is a basis of the K-linear space $K[x]$ and every polynomial from $K[x]$ has a unique expansion in terms of polynomials $P_n(x)$. Let $l_n : K[x] \to K$ be linear functionals such that $l_n(P_m) = \delta_{mn}$. Then

$$p(x) = \sum_{n=0}^{\deg p} l_n(p)\, P_n(x) \tag{1}$$

for all $p \in K[x]$.

Example 1 Take $P_n(x) = (x-a)^n/n!$ where $a \in K$ is arbitrary. Then $l_n(p) = p^{(n)}(a)$, and (1) is Taylor's formula for $p(x)$.

Example 2 Take $P_n(x) = \binom{x-a}{n}$ where $a \in K$ is arbitrary. Then $l_n(p) = \Delta^n p(a)$, and (1) is Newton's interpolation formula for $p(x)$.

Example 3 Take $P_n(x) = x^n/(n_q)!$ where $(n_q)! = 1_q 2_q \ldots n_q$ and $m_q = 1+q+\ldots+q^{m-1}$. Then $l_n(p) = D_q^n p(0)$, where $D_q p(x) = (p(qx) - p(x))/(qx - x)$, and (1) is a q-analogue of Taylor's formula for $p(x)$.

Let $K[[(P_n)_{n=0}^{\infty}]]$ denote the algebra of formal series of the form

$$s(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

where $c_n \in K$, and let $\tilde{l}_n : K[[(P_n)_{n=0}^{\infty}]] \to K$ be linear functionals such that $\tilde{l}_n(s(x)) = c_n$. Operations in $K[[(P_n)_{n=0}^{\infty}]]$ are defined by setting

$$\tilde{l}_n(s+t) = \tilde{l}_n(s) + \tilde{l}_n(t), \tag{2}$$

$$\tilde{l}_n(s\,t) = \sum_{k,m:\, k+m \geq n} \tilde{l}_k(s)\tilde{l}_m(t)\, l_n(P_k P_m), \tag{3}$$

for all $s, t \in K[[(P_n)_{n=0}^{\infty}]]$. From property P2 it follows that $l_n(P_k P_m) = 0$ when $n < \max\{k, m\}$, hence the sum in (3) is finite. By virtue of (1), (2) and (3), $K[x]$ is a subalgebra of $K[[(P_n)_{n=0}^{\infty}]]$, and $\tilde{l}_n$ agrees with l_n on $K[x]$.

Example 4 If $P_n(x) = x^n$ then $K[[(P_n)_{n=0}^{\infty}]] = K[[x]]$, the usual algebra of formal power series over K.

In order to extend L to an endomorphism $\tilde{L}$ of the K-linear space $K[[(P_n)_{n=0}^{\infty}]]$, we assume additionally that

P3. there are $A, B \in \mathbb{N}$, and elements $\alpha_i(n) \in K$ for $n = 0, 1, 2, \ldots$ and $i = -A, 1-A, \ldots, B$, such that

$$LP_n = \sum_{i=-A}^{B} \alpha_i(n)\, P_{n+i} \tag{4}$$

where $\alpha_{-A}(n)$ is not identically zero, and P_k is taken to be 0 when $k < 0$.

Define $\tilde{L}$ on $s \in K[[(P_n)_{n=0}^{\infty}]]$ by

$$\tilde{L}s = \sum_{n=0}^{\infty} \left(\sum_{i=-A}^{B} \alpha_i(n-i)\, \tilde{l}_{n-i}(s) \right) P_n \tag{5}$$

where $\tilde{l}_k$ is taken to be 0 when $k < 0$.

The operator $\tilde{L}$ has been defined in (5) so as to agree with L on $K[x]$. This enables us to replace the equation $Ly = f$ in $K[x]$ with $\tilde{L}y = f$ in $K[[(P_n)_{n=0}^{\infty}]]$. By (5), the latter equation is equivalent to

$$\sum_{i=-B}^{A} \alpha_{-i}(n+i)\, \tilde{l}_{n+i}(y) = l_n(f), \tag{6}$$

for all $n \geq 0$. This is a recurrence for the unknown sequence $\langle \tilde{l}_n(y) \rangle_{n=0}^{\infty} \in K$, of effective order at most $A+B$ (in fact $A+b$ where b is given below in (8)), with leading coefficient $\alpha_{-A}(n+A)$, and homogeneous for $n > \deg f$.

Theorem 1 *Let $Ly = f$ where y is a polynomial. Then $\deg y \leq N$ where*

$$N = \max\{-b-1, \deg f - b\} \cup \{n \in \mathbb{N};\ \alpha_b(n) = 0)\}, \tag{7}$$

$$b = \max\{k \in \mathbb{Z};\ \alpha_k(n) \not\equiv 0\} \tag{8}$$

and α_k is as in (4).

Proof Let y be a polynomial of degree N satisfying $Ly = f$. Clearly $b \leq B$; it can even happen that $b < 0$. We distinguish two cases: either $N + b < 0$ and hence $N < -b$, or $N + b \geq 0$. In the latter case, equation (6) holds for $n \geq N + b$. When $n > N + b$, the left-hand side of (6) is zero. For $n = N + b$ it equals $\alpha_b(N) l_N(y)$, as $l_{N+b-i}(y) = 0$ when $i < b$. Now either $\alpha_b(N) \neq 0$ and then $N + b = \deg f$, or else N is a root of $\alpha_b(x)$. In summary: either $N < -b$, or $N = \deg f - b$, or $\alpha_b(N) = 0$. □

Our algorithm is based on the following simple observation.

Theorem 2 *Let N and b be as in (7) and (8), respectively, with $N \geq 0$. Then for every $y \in K[[(P_n)_{n=0}^{\infty}]]$ the following are equivalent:*

(i) y is a polynomial which satisfies $Ly = f$,

(ii) the sequence $\langle \tilde{l}_n(y)\rangle_{n=0}^{\infty}$ *satisfies (6) for* $n \le N+b$, *and* $\tilde{l}_n(y) = 0$ *for* $n > N$.

Proof By (5) and since N is an upper bound for degree of polynomial solutions of $Ly = f$, it is clear that (i) implies (ii).

Conversely, to see that (ii) implies (i), let $n > N+b$. Then $n-b > N$, hence all terms appearing on the left of (6) are zero. Also, $n > N+b \ge \deg f - b + b = \deg f$, thus the right side of (6) is zero as well. It follows that (6) is satisfied for all $n \ge 0$. Then by (5), y is a polynomial which satisfies $Ly = f$. □

Thus to find all polynomial solutions $y \in K[[(P_n)_{n=0}^{\infty}]]$ of $Ly = f$, it suffices to compute all vectors $(\tilde{l}_{-B}(y), \tilde{l}_{1-B}(y), \ldots, \tilde{l}_{N+A+B}(y))$ which satisfy (6) for $0 \le n \le N+b$, then select those with $\tilde{l}_n(y) = 0$ for $n < 0$ and for $n > N$ and set $y = \sum_{n=0}^{N} \tilde{l}_n(y) P_n(x)$. This can be done by using recurrence (6) in either the forward or backward direction, taking conditions at one end as initial conditions and those at the other end as constraints on the free parameters. The backward direction approach in the case of differential and difference equations and the power basis $P_n(x) = x^n$ has been essentially used already in [Abr89a, Abr&Kva91]. Here we describe the forward direction approach, which can be implemented more efficiently, specially in the nonsingular case where computing with indeterminates can be completely avoided. In the differential and difference case, it is always possible to choose a basis for which the forward direction will be nonsingular (see section 4).

3 The algorithm

3.1 Forward solutions of recurrences

Here we show how to find a generating set for the affine space of vectors $v = (v_0, v_1, \ldots, v_{N+A+b}) \in K^{N+A+b+1}$ which satisfy

$$\sum_{i=-b}^{A} \alpha_{-i}(n+i)\, v_{n+i} = l_n(f), \qquad (9)$$

for $0 \le n \le N+b$ (take $v_n = 0$ for $n < 0$). Let

$$\mathcal{S} = \{n \in \mathbb{N};\ \alpha_{-A}(n) = 0\}$$

be the set of *singularities* of (9), and $\mathcal{N} = \{0, 1, \ldots, A-1\} \cup \mathcal{S}$. Denote by σ the cardinality of the set $\{n \in \mathcal{S};\ A \le n \le N+A+b\}$.

As we compute from (9), we maintain a list of vectors $\mathcal{V}$, a list of indeterminates $\mathcal{I}$, a list of equations $\mathcal{E}$, and an additional vector g. Throughout the course of the algorithm, $\mathcal{V}$ and $\mathcal{I}$ will have equal length whose current value will be denoted by t. The only operations performed on these lists are appending a new element, and extending all vectors in $\mathcal{V}$ by one component. We denote the current elements of $\mathcal{V}$ and $\mathcal{I}$ by $\mathcal{V} = (v^{(1)}, v^{(2)}, \ldots, v^{(t)})$ and $\mathcal{I} = (c_{i_1}, c_{i_2}, \ldots, c_{i_t})$, respectively. Note that we start indexing elements of lists with 1 and components of vectors with 0.

Initially, the lists $\mathcal{V}, \mathcal{I}, \mathcal{E}$ are empty, and g is the empty vector. Next, for $n = 0, 1, \ldots, N+A+b$, repeat the following: If $n \notin \mathcal{N}$ then set

$$v_n = -\left(\sum_{i=1}^{A+b} \alpha_{i-A}(n-i)\, v_{n-i}\right) / \alpha_{-A}(n) \qquad (10)$$

for all $v \in \mathcal{V}$, and

$$g_n = \left(l_{n-A}(f) - \sum_{i=1}^{A+b} \alpha_{i-A}(n-i)\, g_{n-i}\right) / \alpha_{-A}(n) \qquad (11)$$

while leaving $\mathcal{E}$ and $\mathcal{I}$ unchanged.
Otherwise ($n \in \mathcal{N}$) set $v_n = 0$ for all $v \in \mathcal{V}$, $g_n = 0$, and append a new element to each of the lists $\mathcal{V}$, $\mathcal{I}$, $\mathcal{E}$:

$$\begin{aligned} \mathcal{V} &:= \mathcal{V} + (e^{(n+1)}), \\ \mathcal{I} &:= \mathcal{I} + (c_n), \\ \mathcal{E} &:= \mathcal{E} + (E_{n-A}), \quad \text{provided that } n \ge A. \end{aligned}$$

Here $e^{(k)}$ denotes the vector of length k with a 1 in the last position and zeros everywhere else, + denotes concatenation of lists, and E_n is Equation (9). Before appending E_{n-A} to $\mathcal{E}$, replace each indeterminate c_k with $\sum_{j=1}^{t-1} c_{i_j} v_k^{(j)}$. This ensures that only indeterminates from $\mathcal{I}$ appear in $\mathcal{E}$. Should the equation be $0 = 0$ after this replacement, we do not need to add it to $\mathcal{E}$.

After termination of this extension loop, $t = A + \sigma$, $\mathcal{E}$ contains σ equations, and the vectors $v^{(j)}$ and g have length $N+A+b+1$. Denote by $s_j(\tilde{L})$ and $h(\tilde{L}, f)$ the corresponding polynomials

$$s_j(\tilde{L}) = \sum_{n=0}^{N+A+b} v_n^{(j)} P_n \qquad \text{for } j = 1, 2, \ldots, t,$$

and

$$h(\tilde{L}, f) = \sum_{n=0}^{N+A+b} g_n P_n\,.$$

Denote by $\mathcal{E}(\tilde{L}, f)$ the final list of equations. Our algorithm proceeds from here on in slightly different ways for tasks T1, T2, and T3. Those are covered in the next 3 subsections.

In the nonsingular case ($\sigma = 0$), this algorithm reduces to starting with

$$\begin{aligned} \mathcal{V} &:= (e^{(1)}, e^{(2)}, \ldots, e^{(A)}), \\ \mathcal{I} &:= (c_0, c_1, \ldots, c_{A-1}), \\ \mathcal{E} &:= (), \\ g &:= (0, 0, \ldots, 0) \in K^A \end{aligned}$$

and simply using (10) and (11) to extend g and the vectors in $\mathcal{V}$ for $n = A, A+1, \ldots, N+A+b$. This makes $\mathcal{I}$ and $\mathcal{E}$ unnecessary in that case.

Note that in the course of repeatedly applying formulas (10) and (11), we have to evaluate the polynomials α_i at successive integer values of n. Through either the method of finite differences or chains of recurrences [Zima84, Bach&al94], this can be done efficiently, using only $\deg(\alpha_i)$ additions in K for each evaluation.

A detailed example of the singular case of this algorithm is given in Section 4.3.

3.2 Homogeneous equation

Compute $s_j = s_j(\tilde{L})$ for $j = 1, 2, \ldots, t$ and find the general solution $(c_{i_1}, c_{i_2}, \ldots, c_{i_t})$ of the homogeneous linear system composed of $\mathcal{E}(\tilde{L}, 0)$ and of

$$\sum_{j=1}^{t} c_{i_j} l_n(s_j) = 0, \quad \text{for } N < n \le N+A+b.$$

Then $y = \sum_{j=1}^{t} c_{i_j} s_j$ is the general polynomial solution of $Ly = 0$ over K.

3.3 Inhomogeneous equation

Compute $s_j = s_j(\tilde{L})$ for $j = 1, 2, \ldots, t$ and $h = h(\tilde{L}, f)$. Then find the general solution $(c_{i_1}, c_{i_2}, \ldots, c_{i_t})$ of the inhomogeneous linear system composed of $\mathcal{E}(\tilde{L}, f)$ and of

$$\sum_{j=1}^{t} c_{i_j} l_n(s_j) = -l_n(h), \quad \text{for } N < n \le N + A + b.$$

Then $y = \sum_{j=1}^{t} c_{i_j} s_j + h$ is the general polynomial solution of $Ly = f$ over K. Clearly, any particular solution of the above linear system yields a particular polynomial solution of $Ly = f$.

3.4 Parametric inhomogeneous equation

Let N_i denote the degree bound corresponding to f_i, and $N = \max_{1 \le i \le m} N_i$. Compute $s_j = s_j(\tilde{L})$ for $j = 1, 2, \ldots, t$ and $h_i = h(\tilde{L}, f_i)$, for $i = 1, 2, \ldots, m$. Let $\mathcal{E}(\tilde{L}, f_i) = (u^{(1)} \cdot c = r_{i1}, u^{(2)} \cdot c = r_{i2}, \ldots, u^{(\sigma)} \cdot c = r_{i\sigma})$, where $c = (c_{i_1}, c_{i_2}, \ldots, c_{i_t})$ is the vector of indeterminates, $u^{(k)}$ are constant vectors of length t, and $\cdot$ denotes inner product. Now find the general solution $(c_{i_1}, c_{i_2}, \ldots, c_{i_t}, \lambda_1, \lambda_2, \ldots, \lambda_m)$ of the homogeneous linear system composed of

$$u^{(k)} \cdot c = \sum_{i=1}^{m} \lambda_i r_{ik}, \quad \text{for } 1 \le k \le \sigma,$$

and

$$\sum_{j=1}^{t} c_{i_j} l_n(s_j) + \sum_{i=1}^{m} \lambda_i l_n(h_i) = 0, \quad \text{for } N < n \le N + A + b.$$

Then $y = \sum_{j=1}^{t} c_{i_j} s_j + \sum_{i=1}^{m} \lambda_i h_i$ is the general polynomial solution of $Ly = \sum_{i=1}^{m} \lambda_i f_i$ over K.

3.5 The size of the linear system

The final linear system has at most $\sigma + A + b$ equations and $\sigma + A$ unknowns (resp. $\sigma + A + m$ unknowns, in the parametric case).

4 Choosing the polynomial basis

4.1 Differential equations

Let $L = \sum_{k=0}^{r} p_k(x) D^k$ where $p_k(x) \in K[x]$, $p_r \ne 0$, and $Dp(x) = dp(x)/dx$. Choose $a \in K$ such that $p_r(a) \ne 0$. The polynomials $P_n(x) = (x-a)^n/n!$ clearly satisfy properties P1 and P2 from Section 2. Let $d = \max \cup_{k=0}^{r} \{j;\ l_j(p_k) \ne 0\}$. Since $P_m P_n = \binom{m+n}{m} P_{m+n}$ and $D^k P_n = P_{n-k}$, using (1) on p_k we obtain

$$LP_n = \sum_{k=0}^{r} p_k P_{n-k} = \sum_{i=-r}^{d} P_{n+i} \sum_{j=0}^{d} \binom{n+i}{j} l_j(p_{j-i})$$

where $p_k = 0$ when $k < 0$ or $k > r$. Comparing this with (4) we read off

$$A = r,$$
$$B = d,$$
$$\alpha_i(n) = \sum_{j=0}^{d} \binom{n+i}{j} l_j(p_{j-i}),$$

thus P3 is satisfied. Obviously, $\alpha_i(n)$ is a polynomial in n. Since $\alpha_{-A}(n+A) = \sum_{j=0}^{d} \binom{n}{j} l_j(p_{j+r}) = l_0(p_r) = p_r(a) \ne 0$ by the choice of a, recurrence (9) has no singularities and $\sigma = 0$. Hence the final linear system has at most $r + d$ equations and r unknowns.

The value of b required for the degree bound is the maximum i such that $l_j(p_{j-i}) \ne 0$, for some j. This is the same as the maximum i such that $\deg p_{j-i} = j$. Writing $k = j - i$, we obtain $b = \max_{0 \le k \le r} (\deg p_k - k)$.

Alternatively, we can work with the power basis $P_n(x) = x^n$ which also satisfies P1, P2 and P3, but here recurrence (9) can have singularities.

4.2 Difference equations

Let $L = \sum_{k=0}^{r} p_k(x) \Delta^k$ where $p_k(x) \in K[x]$, $p_r \ne 0$, and $\Delta p(x) = p(x+1) - p(x)$. Choose $a \in K$ such that $p_r(n+a) \ne 0$ for all nonnegative integer n. For example, take $a = 0$ if $p_r(x)$ has no nonnegative integer zero, else take $a = \max\{x \in \mathbb{N};\ p_r(x) = 0\} + 1$. The polynomials $P_n(x) = \binom{x-a}{n}$ clearly satisfy properties P1 and P2 from Section 2. Let $d = \max \cup_{k=0}^{r} \{i;\ l_i(p_k) \ne 0\}$ as before. Multiplying a variant of the well-known Chu-Vandermonde identity

$$\binom{x}{n} = \sum_{i} \binom{x-m}{i-m} \binom{m}{m+n-i}$$

by $\binom{x}{m}$, revising binomials and replacing x by $x - a$, we get

$$P_m P_n = \sum_{i} \binom{i}{m} \binom{m}{i-n} P_i. \tag{12}$$

Since $\Delta^k P_n = P_{n-k}$, using (1) on p_k and then (12) we find

$$\begin{aligned} LP_n &= \sum_{k=0}^{r} p_k P_{n-k} \\ &= \sum_{i=-r}^{d} P_{n+i} \sum_{k=0}^{r} \sum_{j=0}^{d} \binom{n+i}{j} \binom{j}{i+k} l_j(p_k) \end{aligned}$$

where $p_k = 0$ when $k < 0$ or $k > r$. Comparing this with (4) we read off

$$A = r,$$
$$B = d,$$
$$\alpha_i(n) = \sum_{k=0}^{r} \sum_{j=0}^{d} \binom{n+i}{j} \binom{j}{i+k} l_j(p_k),$$

thus P3 is satisfied. Obviously, $\alpha_i(n)$ is a polynomial in n. Since $\alpha_{-A}(n+A) = \sum_{k=0}^{r} \sum_{j=0}^{d} \binom{n}{j} \binom{j}{k-r} l_j(p_k) = \sum_{j=0}^{d} \binom{n}{j} l_j(p_r) = p_r(n+a) \ne 0$ for all nonnegative integer n, by the choice of a, recurrence (9) has no singularities and $\sigma = 0$. Hence the final linear system has at most $r + d$ equations and r unknowns.

The value of b required for the degree bound can be shown to be $b = \max_{0\le k\le r}(\deg p_k - k)$ as in the differential case.

If we work with the basis $P_n(x) = \binom{x}{n}$ conditions P1, P2, P3 are satisfied but recurrence (9) may have singularities. The power basis $P_n(x) = x^n$ does not satisfy P3.

4.3 q-Difference equations

Here we assume that $q \in K$ is not zero and not a root of unity. Let $L = \sum_{k=0}^{r} p_k(x)Q^k$ where $Qp(x) = p(qx)$, $p_k(x) \in K[x]$, $p_r \neq 0$, and not all $l_0(p_k)$ are zero. The polynomials $P_n(x) = x^n$ clearly satisfy properties P1 and P2 from Section 2. Let $d = \max \cup_{k=0}^{r}\{j;\ l_j(p_k) \neq 0\}$. Since $P_m P_n = P_{m+n}$ and $Q^k P_n = q^{nk} P_n$, using (1) on p_k we obtain

$$LP_n = \sum_{k=0}^{r} p_k q^{nk} P_n = \sum_{i=0}^{d} P_{n+i} \sum_{k=0}^{r} q^{nk} l_i(p_k)$$

where $p_k = 0$ when $k < 0$ or $k > r$. Comparing this with (4) we read off

$$\begin{aligned} A &= 0, \\ B &= d, \\ \alpha_i(n) &= \sum_{k=0}^{r} q^{nk} l_i(p_k), \end{aligned} \tag{13}$$

thus P3 is satisfied. Obviously, $\alpha_i(n)$ is a polynomial in q^n.

The value of b required for the degree bound equals d.

However, as $\alpha_{-A}(n + A) = \alpha_0(n) = \sum_{k=0}^{r} q^{nk} l_0(p_k)$ can have nonnegative integer roots, recurrence (9) may be singular with $\sigma \le r$. Hence the final linear system has at most $r + d$ equations and r unknowns.

Unlike the cases of difference and differential equations, there seems to be no obvious choice of a polynomial basis P_n which would guarantee nonsingularity of (9) in the q-difference case.

Example 5 Consider the equation

$$\begin{aligned} &(1 - q^{10} - (q - q^{10})\,x)\; y(q^2 x) - \\ &(1 - q^{20} - (q^2 - q^{20})\,x)\; y(qx) + \\ &q^{10}(1 - q^{10} - (q^2 - q^{11})\,x)\; y(x) = \\ &\qquad (q^{21} - q^{20} - q^{12} + q^{10} + q^2 - q)\,x \end{aligned} \tag{14}$$

Here $r = 2$, $d = 1$ and $\deg f = 1$. From (13) we find

$$\begin{aligned} \alpha_0(n) &= (1 - q^{10})(q^n - 1)(q^n - q^{10}), \\ \alpha_1(n) &= (q^{10} - q)(q^n - q)(q^n - q^{10}). \end{aligned} \tag{15}$$

Since $b = d = 1$, degrees of polynomial solutions either equal $\deg f - b = 0$ or are roots of $\alpha_d(n) = \alpha_1(n)$, therefore they belong to the set $\{0, 1, 10\}$ and we take $N = 10$. From (15) we find $\mathcal{N} = \mathcal{S} = \{n \in \mathbb{N};\ \alpha_0(n) = 0\} = \{0, 10\}$. Recurrence (9) in this case is

$$\alpha_0(n)v_n + \alpha_1(n-1)v_{n-1} = l_n(f), \qquad \text{for } 0 \le n \le 11.$$

We start with $\mathcal{V} = \mathcal{I} = \mathcal{E} = ()$ and $g = ()$. At $n = 0$ we set $g = (0)$ and put the vector (1) into $\mathcal{V}$, c_0 into $\mathcal{I}$ and equation $\alpha_0(0)c_0 = l_0(f)$ into $\mathcal{E}$. This equation turns out to be the identity $0 = 0$, so $\mathcal{E}$ remains empty. We perform next the extension step for $n = 1, 2, \ldots, 11$. Except at $n = 10$ we use the formulas $v_n = -\alpha_1(n-1)v_{n-1}/\alpha_0(n)$ and $g_n = (l_n(f) - \alpha_1(n-1)g_{n-1})/\alpha_0(n)$, obtained from (10) and (11). At $n = 9$ we have $\mathcal{V} = ((1, -1, 0, 0, 0, 0, 0, 0, 0, 0))$ and $g = (0, 1, 0, 0, 0, 0, 0, 0, 0, 0)$. At $n = 10$ we extend both vectors with 0 and append $(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$ to $\mathcal{V}$, c_{10} to $\mathcal{I}$ and equation $\alpha_1(9)c_9 = 0$ to $\mathcal{E}$. However, since c_9 is not in $\mathcal{I}$ we replace it by $v_9^{(1)} c_0$. But $v_9^{(1)} = 0$, hence the new equation is $0 = 0$ again, and $\mathcal{E}$ remains empty. After one more step (which extends all three vectors by 0 thanks to $\alpha_1(10) = 0$) the extension loop terminates with $\mathcal{V} = ((1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0))$, $g = (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$, $\mathcal{I} = (c_0, c_{10})$ and $\mathcal{E} = ()$. The corresponding polynomials are $s_1 = 1 - x$, $s_2 = x^{10}$ and $h = x$. The final linear system consists of a single equation $c_0 l_{11}(s_1) + c_{10} l_{11}(s_2) = -l_{11}(h)$ which holds identically. Hence the general polynomial solution of (14) is $y(x) = x + C_1(1 - x) + C_2 x^{10}$. As it happens, there exist solutions for all three possible degrees (0, 1, and 10).

5 Some time comparisons

Table 1 shows the times that were needed to find the rational solutions of the Legendre equation

$$L_m(y) = (1 - x^2)\,y'' - 2x\,y' + m(m+1)\,y = 0$$

where n is a fixed positive integer. This equation has a one-dimensional space of rational solutions generated by the m^{th} Legendre polynomial which has degree m. For that equation and the basis $P_n(x) = x^n/n!$ we get $r = 2$, $d = 2$, $A = 2$, $B = 2$, $p_0 = m(m+1)$, $p_1 = -2x$, $p_2 = 1 - x^2$, $\alpha_{-2} = 1$, $\alpha_{-1} = \alpha_1 = \alpha_2 = 0$, $\alpha_0(n) = m(m+1) - n(n+1)$, $b = 0$ and $N = \max\{0, m\} = m$, so recurrence (9) is:

$$v_{n+2} + (m(m+1) - n(n+1))\,v_n = 0$$

for $n \ge 0$.

m	*Ratlode*	*Series*	*Orthopoly*
20	0.25	0.18	0.06
30	0.35	0.20	0.13
40	0.60	0.23	0.23
50	0.83	0.28	0.43
100	1.98	0.35	3.62
200	6.35	0.67	56.17
500	40.05	1.52	2736.72
1000	161.67	5.70	•
10000	•	398.92	•
15000	•	935.70	•
19000	•	1612.50	•

Table 1: *Maple* CPU seconds to solve $L_m(y) = 0$ (SPARC 10/41).

Ratlode is the `ratlode` package of the *Maple* share library, which uses a straightforward implementation of the method of undetermined coefficients; *Series* is a straightforward implementation of the method presented here, and *Orthopoly* is the built-in `orthopoly[P]` function of *Maple* which computes Legendre polynomials. A dot indicates that the computation did not terminate after 24 hours.

Table 2 shows the times that were needed to find polynomial solutions of the equation

$$\tilde{B}_m(y) = y(x+1) - y(x) = mx^{m-1}$$

where m is a fixed positive integer. This equation has general solution of the form $y(x) = B_m(x) + C$ where $B_m(x)$ is the Bernoulli polynomial of degree m, and C is an arbitrary constant. For that equation and the basis $P_n(x) = \binom{x}{n}$ we get $r = 1$, $d = 0$, $A = 1$, $B = 0$, $p_0 = 0$, $p_1 = 1$, $\alpha_{-1} = 1$, $\alpha_0 = 0$, $b = -1$ and $N = \max\{0, m\} = m$, so recurrence (9) is simply the recurrence of order 0

$$v_{n+1} = \Delta^n f(0)$$

for $n \geq 0$ where $f(x) = mx^{m-1}$.

n	*Poly*	*Series*	*BernoulliB*
20	2.48	0.43	0.25
30	7.90	0.72	1.82
40	20.40	1.12	7.30
50	45.12	1.48	20.90
100	635.83	6.07	554.43
200	9471.98	36.70	14768.7
300	49663.60	114.57	•
400	•	282.83	•
700	•	2084.08	•

Table 2: *Mathematica* CPU seconds to solve $\tilde{B}_n(y) = nx^{n-1}$ (SPARC 10/41).

Poly is a straightforward implementation of the method of undetermined coefficients; *Series* is a straightforward implementation of the method presented here, and *BernoulliB* is the built-in `BernoulliB[n, x]` function of *Mathematica* which computes Bernoulli polynomials. As the first two implementations return solutions satisfying $y(0) = 0$, the n-th Bernoulli number (`BernoulliB[n]` of *Mathematica*) has been added to their respective outputs in order to obtain identical results in all three cases. A dot indicates that the computation did not terminate after 24 hours.

6 Formal series solutions

In conclusion, we remark that our algorithm can also be used to compute formal series solutions in $K[[(P_n)_{n=0}^{\infty}]]$ of equations of the form $Ly = f$: let $M = \max(\mathcal{S})$. Using our extension loop for $n = 0, 1, \ldots, M$, we can construct the set $\mathcal{P}$ of all the polynomials $p \in K[x]$ satisfying $\deg(p) \leq M$ and

$$Lp(x) \equiv f \pmod{P_{M+1}(x)}. \qquad (16)$$

The set $\mathcal{P}$ together with recurrence (9) describes all the solutions in $K[[(P_n)_{n=0}^{\infty}]]$ of $Ly = f$, in the sense that any such solution must be an element of $\mathcal{P}$ prolongated by (9).

Example 6 Consider the equation

$$(q^6x^3 + 1)\, y(q^2x) - q^{14}y(x) = 0$$

with the usual power basis $P_n(x) = x^n$. The recurrence (9) in this case is

$$\left((q^n)^2 - q^{14}\right) v_n + (q^n)^2 v_{n-3} = 0 \quad \text{for } n \geq 0 \qquad (17)$$

so $\mathcal{N} = \mathcal{S} = \{7\}$ and $M = 7$. After performing the extension step for $n = 0, 1, \ldots, 7$, solving (16) yields $\mathcal{P} = \{Cx^7\}$ where C is an arbitrary constant. $\mathcal{P}$ together with (17) describes all the power series solutions of the original equation, namely

$$\begin{aligned} y(x) &= C\left(x^7 + \frac{q^6}{1-q^6}x^{10} + \frac{q^6}{1-q^6}\frac{q^{12}}{1-q^{12}}x^{13} + \cdots\right) \\ &= C\sum_{k=0}^{\infty} \frac{q^{3k(k+1)}}{(q^6;q^6)_k}x^{3k+7} \end{aligned}$$

where $(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$.

In the differential and (q-)difference cases, we can compute [Abr89b, Abr95] a universal denominator $d(x) \in K[x]$ such that any solution of $Ly = f$ of the form

$$\frac{z(x)}{p(x)} \qquad (18)$$

where $z(x) \in K[[(P_n)_{n=0}^{\infty}]]$ and $p \in K[x]$ can be written as $\tilde{z}(x)/d(x)$ where $\tilde{z}(x) \in K[[(P_n)_{n=0}^{\infty}]]$. So we can apply our algorithm after doing the change of variable $z(x) = d(x)y(x)$, obtaining all solutions of the form (18). This leads to formal Laurent series in the differential case.

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