Rational Solutions of First Order Linear Difference Systems

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Abstract

We propose an algorithm to compute rational function solutions for a first order system of linear difference equations with rational coefficients. This algorithm does not require preliminary uncoupling of the given system.

1 Introduction

Let K be a field of characteristic zero. A system of first order linear difference equations with rational coefficients over the field K is a system of the form :

$$y(x+1) = C(x)y(x) + r(x), \tag{1}$$

where $C(x)$ is a $n \times n$ matrix $r(x)$ and the unknown function $y(x)$ are a n-dimensional column-vectors. The entries of $C(x)$ and $r(x)$ are rational functions in x over the field K.

In this paper we consider and solve the problem of computing all the rational solutions of such a system. That is the functions $y \in K(x)^n$ that satisfy (1).
Algorithms for solving this problem in the scalar case (that is the case of a single scalar linear difference equation of arbitrary order) have been proposed in [1, 2, 3, 4, 5]. The algorithmic study of systems is, generally, less well developed. One possible approach for studying linear difference systems consists in reducing them, by means of cyclic vectors, to scalar linear difference equations. This idea goes back to Birkhoff [8]. This reduction can be very costly especially when n is "large". In this paper we propose an alternative approach to solve the above problem. It proceeds in two steps. In the first step we construct a *universal denominator*. We mean a polynomial $U(x) \in K[x]$ such that: for all $y \in K(x)^n$, if y is a solution of (1), then Uy is a polynomial vector. Then the substitution $y(x) = U(x)^{-1}z(x)$ into (1) reduces the problem to finding polynomial solutions of a system in $z(x)$ of the same type as (1). The second step of our method deals with this last problem.
The rest of this paper is organized as follows. In section 2 we give an algorithm for constructing polynomial solutions of a given difference system. The method followed is similar to the one used in [7] for differential systems. Section 3 gives an algorithm to find universal denominators. This algorithm generalizes and improves the one presented in [3] for scalar difference equations. We have implemented our algorithms under MAPLE V. Examples of computations are given in section 4.

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2 Polynomial Solutions

We begin by setting up some notation. If a rational function $a = u/v,\ u, v \in K[x]$ is not 0, we set $\operatorname{ord}(a) = -\deg a = -\deg u + \deg v$, and denote by $\ell c(a)$ the *leading coefficient* of a, that is the coefficient of $x^{-\operatorname{ord} a}$ in the expansion of a as a power series in x^{-1}. Thus

$$a = \ell c(a)x^{-\operatorname{ord} a} + \mathrm{O}(x^{-\operatorname{ord} a - 1}).$$

We set $\operatorname{ord}(0) = -\deg 0 = +\infty$ and $\ell c(0) = 0$.
If U is a matrix (or a vector) of rational functions then we define $\operatorname{ord} U$ to be the minimum of the orders of its entries. We define $\deg U = -\operatorname{ord} U$ and denote by $\ell c(U)$ the coefficient of $x^{-\operatorname{ord}(U)}$ in the expansion of U as a power series in x^{-1}.
By $\mathrm{Mat}_n(K(x))$ we denote the algebra of $n \times n$ matrices with entries in $K(x)$. We write $\mathrm{GL}(n, K(x))$ for the group of invertible matrices in $\mathrm{Mat}_n(K(x))$.
By I_n we denote the identity matrix of order n. By $\operatorname{diag}(a, b, \ldots, c)$ we denote the square diagonal matrix whose diagonal elements are $a, b, \ldots,$ c.

Let Δ denote the difference operator defined by

$$\Delta u(x) = x(u(x+1) - u(x)).$$

Then it is clear that any difference system of the form (1) can be rewritten as

$$\Delta y(x) - M(x)y(x) = f(x). \tag{2}$$

where $M \in \mathrm{Mat}_n(K(x))$ and $f \in K(x)^n$.

A difference system (2) corresponds to a difference operator

$$\mathcal{M} = \Delta - M$$

acting on y. Thus system (2) can be written

$$\mathcal{M}(y) = f.$$

Let $T \in \mathrm{GL}(n, K(x))$. The substitution $y = Tz$ transforms system (2) into a new system

$$\Delta z(x) - N(x)z(x) = g(x), \tag{3}$$

where

$$\begin{array}{rcl} N(x) & = & T[M](x) = T^{-1}(x+1)(M(x)T(x) - \Delta T(x)) \\ g(x) & = & T^{-1}(x+1)f(x) \end{array} \quad (4)$$

Two systems (2) and (3) (resp. the matrices M and N) are called *equivalent* if there exists $T \in \mathrm{GL}(n, K(x))$ such that (4) holds.

In this section, we shall develop an algorithm which allow to compute the set of the polynomial solutions (i.e. $y \in K[x]^n$) of a given difference system (2).
The class of difference systems of the form (2) has properties analogous to the class of differential systems

$$x\frac{dy}{dx} - M(x)y(x) = f(x). \quad (5)$$

It is in this spirit that we treat here difference systems. An algorithm for computing polynomial solutions of differential systems of the form (5) was presented in [7]. We will show that this algorithm can be adapted to compute polynomial solutions of difference systems of type (2). The method about to be presented may be summarized as follows:
Let a system of the form (2) be given. Our idea is to compute, one after another, the different monomials that occur in the (possible) general polynomial solution of (2). For this we put $y = cx^\mu + u$ where $c \in K^n$ and $\mu \in \mathbf{N} = \{0, 1, 2, \cdots\}$ are to be determined in such a way that $\mathcal{M}(y) = f$ and $\deg y = \mu > \deg u$. If such a couple (μ, c) is found then u satisfies the system :

$$\mathcal{M}(u) = f - \mathcal{M}(cx^\mu) \quad (6)$$

which is of the same type as system (2). We can then restart with u and system (6) and so on. This process can be repeated until we obtain a system of the form :

$$\mathcal{M}(w) = g = f - \mathcal{M}(c_1x^{\mu_1} + \cdots + c_\ell x^{\mu_\ell})$$

which has no polynomial solutions with degree $< \mu_\ell$. Hence, the original system (2) has a polynomial solution if and only if $g = 0$. This yields a system of linear algebraic equations for the components of the c_i's. Then, the general solution of this system gives the general polynomial solution of (2). We will show in section 2.2 how the couples (μ, c) can be determined (when it exist). For this we shall first introduce a useful notion namely *the indicial equation* for difference systems and explain how it can be computed.

2.1 The indicial equation

Consider a difference system of the form (2). For $1 \le i \le n$, let $\alpha_i = -\min(\mathrm{ord}(M_i), 0)$ where M_i denotes the ith row of the matrix M. Put

$$D(x) = \mathrm{diag}(x^{-\alpha_1}, \ldots, x^{-\alpha_n}).$$

Then multiplying on the left (2) by $D(x)$ gives

$$\mathcal{L}(y) = D(x)\Delta y(x) - A(x)y(x) = b(x), \quad (7)$$

where $A = DM$ and $b = Df$. Note that $\mathrm{ord}\,(A) \ge 0$ and $\mathrm{ord}\,(D) \ge 0$ (in fact the entries of the matrix $D(x)$ are polynomial in x^{-1}). It then makes sense to set

$$D_0 = D(\infty) \quad \text{and} \quad A_0 = A(\infty).$$

For $c \in K^n$ and $\lambda \in \mathbf{Z}$ we have

$$\mathcal{L}(cx^{-\lambda}) = \Delta(x^{-\lambda})(D_0 + \mathrm{O}(x^{-1}))c - x^{-\lambda}(A_0 + \mathrm{O}(x^{-1}))c.$$

Using the formal identity:

$$(x+1)^{-\lambda} = x^{-\lambda}(1 + x^{-1})^{-\lambda} = x^{-\lambda}(1 - \lambda x^{-1} + \cdots),$$

one sees that

$$\Delta(x^{-\lambda}) = x^{-\lambda}(-\lambda + \mathrm{O}(x^{-1})).$$

Thus

$$\mathcal{L}(cx^{-\lambda}) = -x^{-\lambda}((\lambda D_0 + A_0)c + \mathrm{O}(x^{-1})). \quad (8)$$

Now let $u \in K(x)^n$ (or more generally $u \in K[[x^{-1}]][x]^n$ a column vector of meromorphic formal power series in x^{-1}). By writing

$$u = \ell c(u) x^{-\mathrm{ord}\, u} + \mathrm{O}\left(x^{-\mathrm{ord}\, u - 1}\right),$$

using the linearity of $\mathcal{L}$ and (8) we see that:

$$\mathcal{L}(u) = -x^{-\mathrm{ord}\, u}\left((\mathrm{ord}\, u) D_0 + A_0\right) \ell c(u) + \mathrm{O}\left(x^{-\mathrm{ord}\, u - 1}\right)$$

It then follows that

$$\mathrm{ord}\, \mathcal{L}(u) \ge \mathrm{ord}\, u$$

and equality holds if and only if $(\mathrm{ord}\, u\ D_0 + A_0)\ell c(u) \ne 0$ or $u = 0$.
In particular if u is a nonzero solution of the homogeneous system $\mathcal{L}(u) = 0$ then

$$((\mathrm{ord}\, u)\ D_0 + A_0)\ell c(u) = 0$$

with $\ell c(u) \ne 0$, and therefore

$$\det\,(A_0 + (\mathrm{ord}\, u) D_0) = 0.$$

Thus we have proved the

Lemma 1 *Let a system of the form (7) be given. Then* $\mathrm{ord}\, \mathcal{L}(u) \ge \mathrm{ord}\, u$*, for all* $u \in K(x)^n$ *and equality holds if and only if* $u = 0$ *or* $\ell c(u) \notin \ker\,(A_0 + (\mathrm{ord}\, u) D_0)$*. In particular, if* $u \ne 0$ *and* $\mathcal{L}(u) = 0$ *then* $\det\,(A_0 + (\mathrm{ord}\, u) D_0) = 0$ *and* $\ell c(u) \in \ker\,(A_0 + (\mathrm{ord}\, u) D_0)$.

In view of the above result, it is natural to expect that the values of λ for which the determinant $\det\,(A_0 + \lambda D_0)$ is zero, will play an important and a particular role for the problem of searching polynomial solutions (or more generally formal power series solutions) of the given difference system. However, it may happen that this determinant vanishes identically in λ, in which case it is quite useless to us. This motivates the following definition

Definition 1 *System (2) (or the corresponding difference operator $\mathcal{M}$) is said to be* simple *if* $\det\,(A_0 + \lambda D_0) \not\equiv 0$ *(as a polynomial in λ). In this case the polynomial* $E(\lambda) = \det\,(A_0 + \lambda D_0)$ *will be called the indicial polynomial of (2) (or of $\mathcal{M}$).*

As an example of simple systems, take a system of the form (2) with $\mathrm{ord}(M) \geq 0$. In this case $D = I_n$ and $A = M$. So $D_0 = I_n$ and $A_0 = M(\infty)$. Hence $\det(A_0 + \lambda D_0) = \det(M(\infty) + \lambda I_n) \not\equiv 0$. Consequently, the system is simple and its indicial polynomial has degree n.
We will prove later (see the appendix) the following proposition

Proposition 1 *Every difference system of the form (2) can be reduced to an equivalent system (3) which is simple. Moreover the transformation T can be chosen so that its inverse T^{-1} be polynomial in x.*

The fact that the inverse transformation T^{-1} (in the above proposition) can be chosen polynomial is important when one is interested in the polynomial solutions of (2), since in this case, if y is a polynomial solution of (2) then $z = T^{-1}y$ is a polynomial solution of the equivalent system (3). Thus without any loss of generality we may suppose that the given system is *simple*.

2.2 Algorithm for finding polynomial solutions

Let a simple system of the form (7) be given. Remember that this means that the polynomial $E(\lambda) = det(A_0 + \lambda D_0) \not\equiv 0$.
In this section we shall develop an algorithm to compute the polynomial solutions of (7). In fact the method about to be presented solves the more general problem where the right-hand side b is assumed to linearly depend on some given parameters. More precisely, b is assumed to have the form :

$$b = b_0 + \sum_{i=1}^{m} p_i b_i,$$

where the b_i are column vectors with entries in $K(x)$ and the p_i's are given parameters. Our purpose is to determine the set of all parameters p_i for which (7) has polynomial solutions and to give these solutions.

If b is not zero, we set $\delta = \deg b$ and $\bar{b} = \ell c(b)$. If $b = 0$, we set $\delta = -\infty$ and $\bar{b} = 0$. Note that the components of $\bar{b}$ are polynomials of degree ≤ 1 in the parameters $p_1, \ldots, p_m$.

Now write $y = cx^\mu + u$, with $c \in K^n$ and $\mu \in \mathbf{N}$. Then $\mathcal{L}(y) = b$ gives $\mathcal{L}(u) = b - \mathcal{L}(cx^\mu)$ and hence, using (8), one finds

$$\mathcal{L}(u) = x^\delta \bar{b} + x^\mu(-\mu D_0 + A_0)c + \mathrm{O}\left(x^{\max(\mu,\delta)-1}\right) \qquad (9)$$

The question is : *can we find a vector $c \in K^n$ and a $\mu \in \mathbf{N}$ such that $\mathcal{L}(y) = b$ and $\deg u < \mu$?*
By Lemma 1 we know that $\deg \mathcal{L}(u) \leq \deg u$. It then follows that a necessary condition that μ and c exist is that the degree of the right-hand side of (9) must be $< \mu$.
Let

$$\mathcal{R} = \{\lambda \in \mathbf{N} |\ E(-\lambda) = 0\}.$$

Then several possibilities may occur :

1. If $\mathcal{R} = \emptyset$ and $\delta < 0$ then the degree of the right-hand side of (9) is equal to μ for all $\mu \in \mathbf{N}$. So, in this case there is no couple (μ, c) answering the above question.
2. If $\mathcal{R} \neq \emptyset$ and $\max \mathcal{R} > \delta$ then one may take $\mu = \max \mathcal{R}$ (or any element of $\mathcal{R}$ that is $> \delta$) and c an arbitrary nonzero element in $\ker(A_0 - \mu D_0)$.
3. If $\mathcal{R} \neq \emptyset$ and $\delta \geq \max \mathcal{R}$ or $\mathcal{R} = \emptyset$ and $\delta \geq 0$ then the only possible choice for μ is $\mu = \delta$. Indeed if one takes $\mu \neq \delta$ the degree of the right-hand side of (9) is $\geq \mu$ and hence $\deg u \geq \deg \mathcal{L}(u) \geq \mu$. So, we must choose $\mu = \delta$.
Now if one takes $\mu = \delta$ then (9) reduces to

$$\mathcal{L}(u) = x^\delta(\bar{b} - (\delta D_0 - A_0)c) + \text{ terms of degree } < \delta.$$

Consequently, in order that $\deg \mathcal{L}(u) < \mu = \delta$ holds c must be a solution of the linear system

$$\bar{b} - (\delta D_0 - A_0)c = 0. \qquad (10)$$

This last system has solutions if and only if $\bar{b}$ belongs to the *image* of $(A_0 - \delta D_0)$ (that is the space generated by the columns of $A_0 - \delta D_0$). Thus, one has to consider again two cases:

3.1 If one can choose the p_i's so that $\bar{b} \in \mathrm{Im}(A_0 - \delta D_0)$ then one can take $\mu = \delta$ and c any solution of the system (10).
For instance, if $E(-\delta) \neq 0$ then $\bar{b} \in \mathrm{Im}(A_0 - \delta D_0)$, for all values of the parameters, and in this case c is uniquely determined by $c = (\delta D_0 - A_0)^{-1}\bar{b}$.

3.2 If $\bar{b} \notin \mathrm{Im}(A_0 - \delta D_0)$ for all $p_1, \ldots, p_m$ then there is no couple (μ, c) answering our question.

Remark 1 It is clear from the discussion above that the degree of the any polynomial solution of a given simple system (7) is bounded by

$$\max(\mathcal{R} \cup \{\delta\}),$$

here $\mathcal{R}$ and δ are as defined above.

The above discussion leads to the following algorithm **next-term** which will be used later in the description of the main algorithm for polynomial solutions. It takes as input a rational function b, a list $\mathcal{R}$ of integers, a list $\mathcal{P}$ of parameters, a list $\mathcal{C}$ of linear relations on $\mathcal{P}$ (the constraints on the parameters) and a polynomial σ. The first call to this algorithm is done with b (the right hand-side of the given system), $\mathcal{R}$ and $\mathcal{P}$ as defined above, $\mathcal{C} = \emptyset$ and $\sigma = 0$. It produces a new set of parameters $\mathcal{P}$, a set $\mathcal{C}$ of linear constraints on these parameters and a polynomial σ, parameterized by the elements of $\mathcal{P}$, which represents the possible general solution of the given system.

Algorithm **next-term**$(b, \mathcal{R}, \mathcal{P}, \mathcal{C}, \sigma)$

0. $\delta := \deg b$; $\bar{b} := \ell c(b)$;

1. If $\mathcal{R} = \emptyset$ and $\delta < 0$ then return$(\sigma, \mathcal{P}, \mathcal{C})$.

2. If $\mathcal{R} \neq \emptyset$ and $\max \mathcal{R} > \delta$ then
set $\mu := \max \mathcal{R}$; compute a basis $e_1, \ldots, e_m$ of $\ker(A_0 - \mu D_0)$; put $c := c_1 e_1 + \cdots + c_m e_m$ where the c_i's designate arbitrary elements of K;
call **next-term** with $b := b - \mathcal{L}(cx^\mu)$, $\mathcal{R} := \mathcal{R} \setminus \{\mu\}$, $\mathcal{P} := \mathcal{P} \cup \{c_1, \ldots, c_m\}$, $\mathcal{C}$ *is not changed* and $\sigma := \sigma + cx^\mu$.
Note that in this case the number of elements of $\mathcal{R}$ decreases.

3. If ($\mathcal{R} \neq \emptyset$ and $\delta \geq \max \mathcal{R}$) or ($\mathcal{R} = \emptyset$ and $\delta \geq 0$) then

3.1 if $E(-\delta) \neq 0$ then set $c := (\delta D_0 - A_0)^{-1}\bar{b}$ and call **next-term** with : $b := b - \mathcal{L}(x^\delta c)$, $\sigma := \sigma + x^\delta c$, $\mathcal{R}, \mathcal{P}, \mathcal{C}$ *are not changed.*
Note that in this case the degree of b decreases.

3.2 if $E(-\delta) = 0$ then one has to know whether $\bar{b}$ belongs to $\mathrm{Im}(A_0 - \delta D_0)$ or not; this condition is equivalent to a system, say $\mathcal{G}$, of linear equations in the parameters.

- **(a)** If the relations $\mathcal{G}$ are compatible with the set $\mathcal{C}$ of constraints then solve $\bar{b} = (A_0 - \delta D_0)c$, let c be the general solution of this system, *(it depends on some arbitrary constants c_i)*, then call **next-term** with $b := b - \mathcal{L}(x^\delta c)$, $\mathcal{R} := \mathcal{R} \setminus \{\delta\}$, $\mathcal{P} := \mathcal{P} \cup \{c_i\}$, $\mathcal{C} := \mathcal{C} \cup \mathcal{G}$, and $\sigma := \sigma + x^\delta c$.
Note that in this case the degree of the right-hand side b and the number of elements of $\mathcal{R}$ decrease.
- **(b)** If the conditions $\mathcal{G}$ are not compatible with the constraints $\mathcal{C}$ then return $(\sigma, \mathcal{P}, \mathcal{C})$.

The above algorithm works for $\max(\mathbf{R} \cup \{\deg b\})$ decreases in every step. So, after a finite number of steps, one has $\mathcal{R} = \emptyset$ and $\delta < 0$, unless the situation in 3.2 (b) occurs in which case one stops.

Remark 2 The above algorithm computes, in fact, the *singular part* of the general meromorphic formal series solution at ∞ of the given system (that is solution y with entries in $K[[x^{-1}]][x]$). Note that only the monomials which really occur in this singular part are computed. Thus, the number of necessary steps for computing the candidate polynomial solution of a given system depends only on the number of the (non zero) monomials occuring in the singular part of its general meromorphic formal series solution at ∞. So, in case of sparse solutions our algorithm could be very fast.

We proceed now with a description of our algorithm for searching for polynomial solutions with a system $\mathcal{M}(y) = \Delta(y) - My = f$ as our starting point. Here $M \in \mathrm{Mat}_n(K(x))$ and $f = f_0 + \sum_{i=1}^m p_i f_i$ where the f_i's are in $K(x)^n$ and the p_i's are some parameters. The output is a triplet $(\mathcal{P}, \mathcal{C}, y)$ where $\mathcal{P}$ is a set of parameters, $\mathcal{C}$ is a set of linear relations on the elements of $\mathcal{P}$ and y is a polynomial parameterized by the entries of $\mathcal{P}$ which is solution of the given system when the constraints $\mathcal{C}$ hold.

Algorithm **PS**

1. Apply, if necessary, the algorithm of super-reduction (see the appendix) to reduce the given system to an equivalent simple system. Let $\mathcal{L}(z) = b$ denote the resulting system and T the matrix which achieves the reduction (one has $y = Tz$).
Note that the components of the new right-hand side b are (as the components of f) linear in the parameters p_i's.

2. Let $E(\lambda) := det(A_0 + \lambda D_0)$ be the indicial polynomial of $\mathcal{L}$. Set $\mathcal{R} := \{\lambda \in \mathbf{N}, E(-\lambda) = 0\}$, $\mathcal{P} := \{p_1, \ldots, p_m\}$ *($\mathcal{P}$ is the set of free parameters, it may be empty)*, $\mathcal{C} := \emptyset$ *(the set of constraints on these parameters)*, and $\sigma := 0$.

3. Call **next-term** with $b, \mathcal{R}, \mathcal{P}, \mathcal{C}$, and σ.
One then obtains a new set of parameters $\mathcal{P}$, a set $\mathcal{C}$ of linear constraints on these parameters and a polynomial $\sigma = \sum c_\mu x^\mu$ where the c_μ's are column vectors, the components of which are linear in the elements of $\mathcal{P}$.

4. Substituting $y = T\sigma$ in the system $\mathcal{M}(y) = f$ yields a system, say $\mathcal{F}$, of linear equations in the parameters $\mathcal{P}$.

5. If the system $\mathcal{F}$ is compatible with the constraints $\mathcal{C}$ then $y = T\sigma$ gives the general solution of our problem, otherwise, there is no polynomial solution.

In conclusion of this section we have to refine our suppositions on the field K. Indeed we must know how to find integer roots of an algebraic equation $E(\lambda) = 0$ over K. Our coefficient field is so-called *suitable* field in the sense of the following definition:
1) K is of characteristic zero;
2) there is an algorithm for finding integer roots of algebraic equations over K in one unknown.

The field $\mathbf{Q}$ is obviously suitable. It is easy to see that a simple extension (algebraic or transcendental) of a suitable field K is itself suitable.

3 Universal Denominators

Let a linear difference equation of the form

$$a_n(x)y(x+n) + \cdots + a_0(x)y(x) = b(x) \tag{11}$$

or a system of difference equations

$$u_1(x)y_1(x+1) + v_{11}(x)y_1(x) + \cdots + v_{1m}(x)y_m(x) = w_1(x)$$

$$\cdots \tag{12}$$

$$u_m(x)y_m(x+1) + v_{m1}(x)y_1(x) + \cdots + v_{mm}(x)y_m(x) = w_m(x)$$

with polynomial coefficients $a_i(x), b(x), u_i(x), v_{ij}(x), w_j(x)$ over the field K. One can start searching for rational function solutions of (11) or (12) with constructing a *universal denominator.* We mean a polynomial $U(x)$ which is a multiple of the denominator of any rational solution of (11) or (12). After constructing $U(x)$ one can substitute $z(x)/U(x)$ in (11) for $y(x)$ (resp. $z_1(x)/U(x), \ldots, z_m(x)/U(x)$ in (12) for $y_1(x), \ldots, y_m(x)$) , where $z(x), z_1(x), \ldots, z_m(x)$ are unknown polynomials. This results in an equation for $z(x)$ with polynomial coefficients and a polynomial right-hand side (or, resp. a system of such equations for $z_1(x), \ldots, z_m(x)$). The search for polynomial solutions has been considered in section 2.

The problem of constructing a universal denominator has been considered in the scalar case in [2, 3, 4]. The main aim of this section is the presentation of an algorithm to compute a universal denominator in the case of system (12). But first we recall briefly the situation in the scalar case (section 3.1). We will discuss some details of the algorithm described in [4] (the version of this algorithm described in [3] has a defect). We remind also the main notions and two theorems, which were proven in [4] and are needed to verify this algorithm. Then (section 3.2) we pass to the case of system (12) and show that the "scalar" algorithm lets one construct a universal denominator $U(x)$ for rational functions $y_1(x), \ldots, y_m(x)$. We demonstrate also how to modify one of mentioned theorem to verify this approach to construct $U(x)$.

3.1 Scalar equations

Recall the algorithm which has been proposed in [4] for (11). First of all we set

$$A(x) = a_n(x-n), \ B(x) = a_0(x)$$

and compute

$$\mathrm{dis}(A(x), B(x)) \tag{13}$$

meaning the greatest nonnegative integer N (if it exists) such that $A(x)$ and $B(x+N)$ have a nontrivial common divisor. If such N does not exist then we set $\mathrm{dis}(A(x), B(x)) = -1$. Observe that (13) can be compute as the largest nonnegative integer root of the polynomial $R(m)$ where $R(m) = Res_x(A(x), B(x+m))$. But it is not the only way: Y.K.Man and F.J.Write proposed a more effective algorithm in [10].

Set $N = \mathrm{dis}(A(x), B(x))$. Then the following algorithm $\mathbf{UD}(A, B, N)$ performance

1. $U(x) := 1$;
2. `for` $i = N, N-1, ..., 0$ `do`
 $d(x) := \gcd(A(x), B(x+i))$;
 $A(x) := A(x)/d(x)$;
 $B(x) := B(x)/d(x-i)$;
 $U(x) := U(x)d(x)d(x-1)\cdots d(x-i)$
 `od.`

lets one get a universal denominator (in [3] the loop
 `for` $i = 0, 1, ..., N$ `do`
was mistakenly used). To verify this algorithm some notions and theorems were proposed in [4].

First of all we call a polynomial *special* if its full factorization over K has the form

$$p^{\gamma_0}(x)p^{\gamma_1}(x+1)\dots p^{\gamma_h}(x+h) \tag{14}$$

where $p(x)$ is irreducible, $h, \gamma_0, \dots, \gamma_h$ are nonnegative integers. We will show the structure of a special polynomial by drawing in the plane (l, γ) for any $p(x+l)^\gamma, \gamma > 0$, the vertical segment with the ends $(l, 0), (l, \gamma)$. In Fig.1 is shown the diagram of the special polynomial

$$x(x+1)^2(x+2)^2(x+3)(x+4)^3(x+5)^5\cdots \tag{15}$$
$$(x+6)^3(x+7)^5(x+9)^3(x+11)^4(x+12)^2.$$

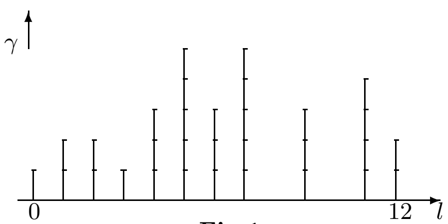

Fig.1

We call two special polynomials *related* if their product is special again.

Let $g(x)$ be a special polynomial of the form (14). We call a divisor

$$p^\sigma(x+l) \tag{16}$$

of $g(x)$ *critical of the first kind* if the relationship

$$p^{\sigma_1}(x+l_1)|g(x) \tag{17}$$

along with $l_1 > l$ implies $\sigma_1 < \sigma$ and along with $\sigma_1 > \sigma$ implies $l_1 < l$. We call a divisor of the form (16) of $g(x)$ *critical of the second kind* if the relationship (17) along with $l_1 < l$ implies $\sigma_1 < \sigma$ and along with $\sigma_1 > \sigma$ implies $l_1 > l$.

Let $p^{\alpha_1}(x+M_1), \dots, p^{\alpha_s}(x+M_s)$ be all critical divisors of the first kind, and $p^{\beta_1}(x+m_1), \dots, p^{\beta_t}(x+m_t)$ be all critical divisors of the second kind. Let $M_1 > \dots > M_s$ and $m_1 < \dots < m_t$. Then $\alpha_1 < \dots < \alpha_s$ and $\beta_1 < \dots < \beta_t$; $m_t \le M_s$. Let $\alpha_0 = \beta_0 = 0$ additionally.

In Fig.2 are marked all critical divisors of polynomial (15).

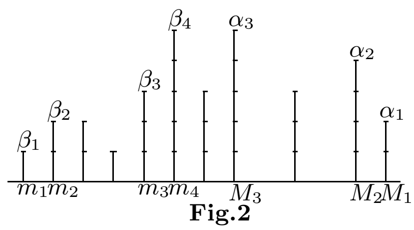

Fig.2

Let $A(x), B(x) \in K[x]$. We will call a special polynomial $g(x)$ of the form (14) *bounded* by the pair $(A(x), B(x))$ if

$$p^{\alpha_i - \alpha_{i-1}}(x+M_i)|A(x), i = 1, \dots, s, \tag{18}$$

$$p^{\beta_j - \beta_{j-1}}(x+m_j)|B(x), j = 1, \dots, t. \tag{19}$$

Theorem 1 ([4]) *Let the result of the substitution of a rational function $S(x)$ with special denominator in the left-hand side of (11) be a polynomial. Then the denominator of $S(x)$ is bounded by $(a_n(x-n), a_0(x))$.*

Theorem 2 ([4]) *Let a special polynomial $g(x)$ be bounded by $(A(x), B(x))$. Let (13) be positive and be equal to N. Let*

$$d(x) = \gcd(A(x), B(x+N)),$$
$$c(x) = d(x)d(x-1)\cdots d(x-N).$$

Let

$$\tilde{g}(x) = g(x)/\gcd(g(x), c(x)),$$
$$\tilde{A}(x) = A(x)/d(x), \tilde{B}(x) = B(x)/d(x-N).$$

Then $\tilde{g}(x)$ is bounded by $(\tilde{A}(x), \tilde{B}(x))$ and $g(x)|c(x)\tilde{g}(x)$.

Theorem 2 shows that the algorithm **UD** lets one compute a polynomial $U(x)$ divisible by any special polynomial bounded by $(A(x), B(x))$.

Any rational non–polynomial function $S(x)$ can be presented in the form $S_1(x) + \cdots + S_k(x)$ where $S_1(x), \dots, S_k(x)$ are rational functions with nonrelated special denominators. The product of the denominators is equal to the denominator of $S(x)$. Therefore if $S(x)$ is a solution of (11) then every $S_i(x), i = 1, \dots, k$, has the denominator which is bounded by $(a_n(x-n), a_0(x))$. And we obtain the desired universal denominator by algorithm **UD**.

3.2 Systems of equations

Let a system of the form (12) be given. We assume the determinant of the matrix

$$\begin{pmatrix} v_{11}(x) & \dots & v_{1m}(x) \\ \vdots & & \vdots \\ v_{m1}(x) & \dots & v_{mm}(x) \end{pmatrix} \tag{20}$$

to be a nonzero polynomial (otherwise either system (12) is incompatible or the number of unknowns in (12) can be reduced). Then there exist polynomials $\hat{u}_i(x), \hat{v}_{ij}(x), \hat{w}_j(x),\ i, j = 1, \ldots, m$, over K such that

$$\hat{u}_1(x)y_1(x)+\hat{v}_{11}(x)y_1(x+1)+\cdots+\hat{v}_{1m}(x)y_m(x+1) = \hat{w}_1(x)$$

$$\cdots$$

$$\hat{u}_m(x)y_m(x)+\hat{v}_{m1}(x)y_1(x+1)+\cdots+\hat{v}_{mm}(x)y_m(x+1) = \hat{w}_m(x).$$

Let's concentrate on the search for a universal denominator of rational solution of (12). Set

$$a_1 = \mathrm{lcm}(u_1(x), \ldots, u_m(x)), a_0 = \mathrm{lcm}(\hat{u}_1(x), \ldots, \hat{u}_m(x)) \quad (21)$$

and show that if

$$A(x) = a_1(x-1), B(x) = a_0(x), N = \mathrm{dis}(A(x), B(x)), \quad (22)$$

then performing algorithm **UD** gives a universal denominator $U(x)$. We start with the following analogue of Theorem 1.

Theorem 3 *Let $S_1(x), \ldots, S_m(x)$ be rational functions with related special denominators $g_1(x), \ldots, g_m(x)$. Let*

$$u_l(x)S_l(x+1) + v_{l1}(x)S_1(x) + \cdots + v_{lm}(x)S_m(x), \quad (23)$$

and

$$\hat{u}_l(x)S_l(x) + \hat{v}_{l1}(x)S_1(x+1) + \cdots + \hat{v}_{lm}(x)S_m(x+1)$$

be polynomials, for $l = 1, \ldots, m$. Let $g(x) = \mathrm{lcm}(g_1(x), \ldots, g_m(x))$ and polynomials $a_1(x), a_0(x)$ be defined by (21), $A(x) = a_1(x-1), B(x) = a_0(x)$. Then $g(x)$ is bounded by $(A(x), B(x))$.

Proof There exist an irreducible $p(x) \in K[x]$ and nonnegative integer $h, \gamma_{ij}, i = 0, \ldots, m, j = 1, \ldots, h$, such that

$$g_i(x) = p^{\gamma_{i0}}(x)p^{\gamma_{i1}}(x+1)\cdots p^{\gamma_{ih}}(x+h).$$

It is obvious that

$$g(x) = p^{\sigma_0}(x)p^{\sigma_1}(x+1)\cdots p^{\sigma_h}(x+h),$$

with $\sigma_i = \max\{\gamma_{1i}, \ldots, \gamma_{mi}\}, i = 0, \ldots, h$. Let $A(x) = a_1(x-1)$ and $p^{\alpha_1}(x+M_1), \ldots, p^{\alpha_s}(x+M_s)$ be all critical divisors of $g(x)$ of the first kind, $\alpha_0 = 0$. Let us prove (18). By definition of $\alpha_1, \ldots, \alpha_s$ we have

$$\alpha_{k-1} = \max\{\sigma_{M_k+1}, \sigma_{M_k+2}, \ldots, \sigma_h\} \quad (24)$$

for $k = 2, \ldots, s$ (see Fig.3).

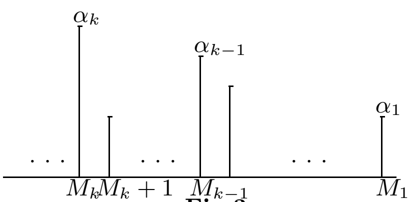

Fig.3

Let k be such that $1 \le k \le s$. Show that for any μ and j

$$(p^{\mu}(x+M_k+1)|g_j(x)) \Rightarrow (\mu \le \alpha_{k-1}). \quad (25)$$

If $k = 1$ then obviously $\mu = \alpha_0 = 0$. If $k \ge 2$ then

$$\mu \le \gamma_{j,M_k+1} \le \sigma_{M_k+1}$$

and $\mu \le \alpha_{k-1}$ due to (24). Thus (24) is proven. Let $\alpha_k = \gamma_{l,M_k}$. Consider the expression (23) which has to be equal to a polynomial. The denominator of $S_l(x+1)$ is divisible by $p^{\alpha_k}(x+M_k+1)$ while by (25) the denominator of any of rational functions $S_1(x), \ldots, S_m(x)$ is divisible at best by $p^{\alpha_{k-1}}(x+M_k+1)$. Thus

$$p^{\alpha_k-\alpha_{k-1}}(x+M_k+1)|u_l(x).$$

But $a_1(x) = \mathrm{lcm}(u_1(x), \ldots, u_m(x))$ and $A(x) = a_1(x-1)$, therefore

$$p^{\alpha_k-\alpha_{k-1}}(x+M_k)|A(x).$$

Similar reasoning lets one prove (19) for $B = a_0(x)$. □

Theorem 2 shows that applying algorithm **UD** to (22) allows to compute a polynomial $U(x)$ divisible by any special polynomial bounded by $(A(x), B(x))$. Let $y_1(x), \ldots, y_m(x)$ be rational functions which satisfy system (12) and suppose there is at least one non–polynomial among them. We can represent these functions in the form of sums

$$y_i(x) = S_{i1}(x) + \cdots + S_{ir}(x),$$

$i = 1, \ldots, m$; r is a natural number, in such a way that

a) in each of columns of the matrix

$$\begin{pmatrix} S_{11}(x) & \ldots & S_{1r}(x) \\ \vdots & & \vdots \\ S_{m1}(x) & \ldots & S_{mr}(x) \end{pmatrix}$$

placed rational functions with related special denominators and there is at least one non–polynomial function among them;

b) non–polynomial functions placed on different columns have nonrelated denominators. Then any rational function

$$u_i(x)S_{1j}(x+1) + v_{i1}(x)S_{1j}(x) + \cdots + v_{im}(x)S_{mj}(x),$$

$i = 1, \ldots, m,\ j = 1, \ldots, r$, is a polynomial.

By Theorem 3 the least common multiple $g_j(x)$ of the denominators of

$$S_{1j}(x), \ldots, S_{mj}(x)$$

$j = 1, \ldots, r$, is bounded by $(A(x), B(x))$. Therefore algorithm **UD** gives a polynomial $U(x)$ divisible by any $g_j(x),\ j = 1, \ldots, r$. It implies that $U(x)$ is divisible by the denominator of any rational function $S_{ij}(x)$ and it is evident that $U(x)$ is divisible by the denominator of any of rational functions $y_1(x), \ldots, y_m(x)$.

Remark that $\mathrm{lcm}(\hat{u}_1(x), \ldots, \hat{u}_m(x))$ is equal to the least common denominator of all the entries of the inverse of matrix (20) of the original system.

3.3 Preliminary transformations of equations and systems

We can add to **UD** preliminary steps which allow in some situations to decrease the degree of the universal denominator. We consider again two cases.

1. Equation (11). If the polynomials $a_0(x), a_1(x-1), \ldots, a_n(x-n)$ have a nontrivial common divisor $d(x)$ then the substitution $y(x) = z(x)/d(x)$ gives an equation with polynomial coefficients, whose degrees are decreased:

$$\widetilde{a}_n(x)y(x+n) + \cdots + \widetilde{a}_0(x)y(x) = b(x),$$

where $\widetilde{a}_j(x) = a_j(x)/d(x+j),\ j = 0, \ldots, n$. It makes sense to use the following version of **UD**:

Algorithm $\mathbf{UD}_{scal}$;
input: equation (11);
output: universal denominator $U(x)$;
1. $U(x) := \gcd(a_n(x-n), \ldots, a_1(x-1), a_0(x))$;
2. $A(x) := a_n(x-n)/U(x); B(x) := a_0(x)/U(x); N := \mathrm{dis}(A(x), B(x))$;
3. Step **2** of algorithm **UD**.

2. System (12). Analogous simplification is possible if for some $i, 1 \leq i \leq m$, the polynomials $v_{1i}(x), v_{2i}(x), \ldots, v_{mi}(x), u_i(x-1)$ have a nontrivial common divisor. It makes sense to use the following version of **UD**:

Algorithm $\mathbf{UD}_{sys}$;
input: system (12);
output: universal denominator $U(x)$;
1. $U(x) := 1$;
  `for` $i = 1, 2, \ldots, m$ `do`
    $d_i(x) := \gcd(u_i(x-1), v_{1i}(x), v_{2i}(x), \ldots, v_{mi}(x))$;
    $v_{1i}(x) := v_{1i}(x)/d_i(x); \ldots; v_{mi}(x) := v_{mi}(x)/d_i(x)$;
  `od.`
    $U(x) := \mathrm{lcm}(d_1(x), \ldots, d_m(x))$;
2. $A(x) := \mathrm{lcm}(u_1(x-1)/d_1(x), \ldots, u_m(x-1)/d_m(x))$;
3. Compute $B(x)$ which is equal to the lcm of the denominators of the elements of the matrix inverse of (20);
4. $N := \mathrm{dis}(A(x), B(x))$;
5. Step **2** of algorithm **UD**.

The following example shows that the described preliminary transformations can decrease the universal denominator. Consider the scalar equation

$$\begin{aligned} x(x+10)y(x) &- 2(x+1)(x+11)y(x+1) \\ &+ (x+2)(x+12)y(x+2) = 0 \end{aligned}$$

with solutions $\frac{1}{x}, \frac{1}{x+10}$. If we use **UD** then we get $U(x) = x(x+1)\cdots(x+10)$, but if we use $\mathbf{UD}_{scal}$ then we get $U(x) = x(x+10)$. In the same time $\mathbf{UD}_{scal}, \mathbf{UD}_{sys}$ can not increase the degree of $U(x)$ in comparison with **UD** because $A(x), B(x)$ which are used by $\mathbf{UD}_{scal}, \mathbf{UD}_{sys}$ divide those $A(x), B(x)$ which are used by **UD**.

4 Examples of computations

Our algorithms are implemented in MAPLE V. The two main functions are called **deltaRS** and **deltaPS**. They take as input a system $y(x+1) = A(x)y(x) + b(x)$ and return the general rational (resp. polynomial) solution of the given system if it exists and the empty set { } otherwise. We give here some examples solved by these two functions.

Consider the matrix $A(x)$ given by :

$$\begin{bmatrix} \frac{x-1}{x+5} & \frac{(7x+4+x^2)x}{x+5} & -x-1 & -\frac{(x^2+5x+5)(x-1)}{x+5} \\ \frac{x-1}{(x+1)(x+5)x} & \frac{x-1}{(x+1)(x+5)} & 0 & \frac{x-1}{(x+1)(x+5)} \\ \frac{x-1}{x+5} & \frac{x(x-1)}{x+5} & -x & -\frac{x^3+3x^2-5x-5}{x+5} \\ -\frac{x-1}{x(x+5)} & -\frac{x-1}{x+5} & 1 & \frac{(x+4)(x-1)}{x+5} \end{bmatrix}.$$

For the system $y(x+1) = A(x)y(x)$ the function **deltaPS** returns :

$$[xc_1, 0, (x-2)\,c_1, -c_1],$$

while the function **deltaRS** gives

$$\begin{aligned} [&\frac{x^7c_1 + 9\,x^6c_1 + x^5c_2 + 40\,x^4c_1 - x^4c_2 + 499\,x^3c_1 - 21\,x^3c_2}{x\,(x-1)\,(x+4)\,(x+3)\,(x+2)\,(x+1)} \\ &+ \frac{551\,x^2c_1 - 23\,x^2c_2 + 100\,xc_1 - 4\,xc_2 + 300\,c_1 - 12\,c_2}{x\,(x-1)\,(x+4)\,(x+3)\,(x+2)\,(x+1)}, \\ &-4\,\frac{-25\,c_1 + c_2}{x^2\,(x+2)\,(x+4)}, \\ &\frac{x^5c_1 + 8\,x^4c_1 + x^3c_2 - 10\,x^3c_1 - 70\,x^2c_1}{(x+1)\,(x+2)\,(x+3)\,(x+4)} \\ &+ \frac{2\,x^2c_2 - 7\,xc_2 + 99\,xc_1 - 11\,c_2 + 227\,c_1}{(x+1)\,(x+2)\,(x+3)\,(x+4)}, \\ &-\frac{x^2c_1 + 6\,xc_1 + c_2 - 17\,c_1}{(x+2)\,(x+4)}] \end{aligned}$$

Here c_1 and c_2 designate arbitrary constants. If one takes $c_2 = 25c_1$ then one gets the solution already found by **deltaPS**.

Now applying **deltaPS** on the non–homogeneous system $y(x+1) = A(x)y(x)+b(x)$ with $b = [x+1, 0, x+1, -1]$ yields the general solution:

$$[xc_1, 0, 1 + xc_1 - 2\,c_1, -c_1].$$

A Super–reduced Forms of linear difference systems

A.1 Definition and properties

The notion of super–irreducibilty has been introduced in a joint work of Hilali and Wazner [9], and is used there to study linear homogeneous differential systems near an irregular singularity. This notion has been generalized to difference systems (see chapter 7 of [6]). In this appendix, we will give the definition of super–reduced forms. Furthermore we will show the connexion between the super–irreducibility and the notion of simplicity introduced in section 2.1.

Consider a difference system of the form :

$$\Delta y(x) = M(x)y(x),\ \ M \in \mathrm{Mat}_n(K[[x^{-1}]][x]). \qquad (26)$$

Put $q = -\,\mathrm{ord}(M)$ and define the rational number $m(M)$ by

$$m(M) = \begin{cases} q + \frac{n_0}{n} + \frac{n_1}{n^2} + \cdots + \frac{n_{q-1}}{n^q} & \text{if} \quad q > 0 \\ 1 & \text{if} \quad q \leq 0 \end{cases}$$

where n_i denotes the number of rows of M of order $-q+i$ for $i = 0, \ldots, q-1$.
Finally define the rational number $\mu(M)$ by

$$\mu(M) = \min \{m(T[M]) \mid T \in \mathrm{GL}(n, K[[x^{-1}]][x])\}.$$

Definition 2 *System (26) (or the matrix M) is called* super–irreducible *iff $m(M) = \mu(M)$.*

In [6] a criterion to decide whether a system (26) is super–irreducible is given. We will repeat this criterion here since it will be used later.
Let us keep the notation above. Suppose $q > 0$ and define the integers $r_1, \ldots, r_q$ by

$$r_k = kn_0 + (k-1)n_1 + \cdots + n_{k-1}.$$

For $1 \le k \le q$ define

$$B_k(M, \lambda) = x^{-r_k} \det \left(\lambda I_n + x^{-q+k} M(x)\right)_{|x=\infty}.$$

Then $B_k(M, \lambda) \in K[\lambda]$ for all $1 \le k \le q$. In [6] the following result is proved.

Theorem 4 [6] *The system (26) is* super–irreducible *iff the polynomials $B_k(M, \lambda)$ do not vanish identically in λ, for $k = 1, \ldots, q$.*

We will now prove the

Proposition 2 *If system (26) is super–irreducible then it is simple.*

Proof Consider a system of the form (26) and put $q = -\mathrm{ord}(M)$. If $q \le 0$ then (as mentioned in section 2.1) the system is simple. Suppose that $q > 0$ and let $\alpha = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ with $\alpha_i = -\min(0, \mathrm{ord}(M_i))$, where M_i, $0 \le i \le n$, denotes the ith row of M. Then the matrix $D(x) = x^{-\alpha} \in \mathrm{Mat}_n(K[x^{-1}])$ and the matrix $A(x) = x^{-\alpha}M(x) \in \mathrm{Mat}_n(K[[x^{-1}]])$. Put $D_0 = D(\infty)$ and $A_0 = A(\infty)$, then one has

$$\det(A_0 + \lambda D_0) = B_q(M, \lambda).$$

Indeed, one easily verifies that $\det(x^{-\alpha}) = x^{-r_q}$ and then

$$\begin{aligned} x^{-r_q} \det(\lambda I_n + M(x)) &= \det(x^{-\alpha}) \det(\lambda I_n + M(x)) \\ &= \det(\lambda x^{-\alpha} + x^{-\alpha} M(x)). \end{aligned}$$

Hence

$$\begin{aligned} B_q(M, \lambda) &= x^{-r_q} \det\left(\lambda I_n + M(x)\right)_{|x=\infty} \\ &= \det\left(\lambda D(x) + A(x)\right)_{|x=\infty} \\ &= \det(\lambda D_0 + A_0). \end{aligned}$$

Now if (26) is super–irreducible then, by Theorem 4, the polynomial $B_q(M, \lambda)$ is not zero and (26) is simple. □

Remark 3 Note that a system may be simple without being super-irreducible. Indeed super-irreducibility requires that $B_k(M, \lambda) \not\equiv 0$ for all $1 \le k \le q$ while simplicity requires only that $B_q(M, \lambda) \not\equiv 0$ (as it was mentioned in the proof above).

An algorithm for computing a super–irreducible form of a difference system (26) is presented in [6]. It is similar to the super-reduction algorithm of Hilali-Wazner [1] for linear systems of differential equations. More precisely, given a matrix $M \in \mathrm{Mat}_n(K[[x^{-1}]][x])$ this algorithm produces a nonsingular matrix S which is polynomial in x^{-1} such that the equivalent matrix $N = S[M]$ is super–irreducible. Moreover, S satisfies $\det S = \gamma x^h$, for some integer h and $\gamma \in K \setminus \{0\}$. This last result implies that the matrix S^{-1} is of the form $S^{-1} = x^{-\nu}(S_0 + S_1 x + \cdots + S_d x^d)$ for some integers ν and d (here $S_0, \ldots, S_d$ are matrices with elements in K). If $\nu > 0$ then by setting $T = x^{-\nu} S$ the matrix T^{-1} is polynomial in x and the matrix

$$\begin{aligned} T[M] &= (1 + x^{-1})^\nu N(x) - x(1 - (1 + x^{-1})^\nu) I_n \\ &= (1 + \mathrm{O}(x^{-1})) N(x) + \mathrm{O}(1) I_n \end{aligned}$$

is still super–irreducible. Thus we have proved Proposition 1.

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