On Strongly Non-Singular Polynomial Matrices

Sergei A. Abramov and Moulay A. Barkatou

Abstract We consider matrices with infinite power series as entries and suppose that those matrices are represented in an "approximate" form, namely, in a truncated form. Thus, it is supposed that a polynomial matrix P which is the l-truncation (l is a non-negative integer, $\deg P = l$) of a power series matrix M is given, and P is non-singular, i.e., $\det P \neq 0$. We prove that the strong non-singularity testing, i.e., the testing whether P is not a truncation of a singular matrix having power series entries, is algorithmically decidable. Supposing that a non-singular power series matrix M (which is not known to us) is represented by a strongly non-singular polynomial matrix P, we give a tight lower bound for the number of initial terms of M^{-1} which can be determined from P^{-1}. In addition, we report on possibility of applying the proposed approach to "approximate" linear differential systems.

Keywords Polynomial matrices, Strong non-singularity, Linear differential systems, Truncated series

1 Introduction

We discuss an "approximate" representation of infinite power series which appear as inputs for computer algebra algorithms. A well-known example is given in [10], it is related to the number of terms in M that can influence some components of formal

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exponential-logarithmic solutions of a differential system $x^{s+1}y' = My$, where s is a non-negative integer, M is a matrix whose entries are power series; see also its generalization in [11] and our previous paper [4].

In the present paper, we consider matrices with infinite power series (over a field K of characteristic 0) as entries and suppose that those series are represented in a truncated form. Thus, it is assumed that a polynomial matrix P which is the l-truncation $M^{\langle l \rangle}$ (l is a non-negative integer, $\deg P = l$) of a power series matrix M is given, and P is non-singular, i.e., $\det P \neq 0$. We prove that the the question of strong non-singularity, i.e., the question whether P is not the l-truncation of a singular matrix having power series entries, is algorithmically decidable.

Assuming that a non-singular power series matrix M (which is not known to us) is represented by a strongly non-singular polynomial matrix P, we give a tight lower bound for the number of initial terms of M^{-1} which can be determined from P^{-1}. As it turns out, for the answer to these questions, the number $h = \deg P + \operatorname{val} P^{-1}$ plays the key role, and $h \geqslant 0$ is a criterion of the impossibility of a prolongation of polynomials to series so that the determinant of the matrix vanishes. If this inequality holds then first, for any prolongation, the valuations of the determinant and the inverse of the approximate matrix and, resp., of the prolonged matrix coincide. Second, in the expansions of the determinants of the approximate and prolonged matrices the coefficients coincide for $x^{\operatorname{val}\det P}$, as well as h subsequent coefficients (for larger degrees of x). The similar statement holds for the inverse matrices.

In addition, we prove that if M is an $n \times n$-matrix having power series entries, $\det M \neq 0$ then there exists a non-negative integer l such that $M^{\langle l \rangle}$ is a strongly non-singular polynomial matrix. If the entries of M are represented algorithmically (for each power series that is an entry of M, an algorithm is specified that, given an integer i, finds the coefficient of x^i) then an upper bound for such l can be computed.

In Section 7, we discuss the possibility of applying the proposed approach to approximate linear differential systems of arbitrary order with power series matrix coefficients: if a system S is given in the approximate truncated form $\tilde{S}$, $\operatorname{ord}\tilde{S} = \operatorname{ord} S$, and the leading matrix of $\tilde{S}$ is strongly non-singular then one can guarantee, under some extra specific conditions, that Laurent series solutions of the truncated system $\tilde{S}$ coincide with Laurent series solutions of the system S up to some degree of x that can be estimated by the algorithm we proposed in [4].

In our paper we are considering a situation where a truncated system is initially given and we do not know the original system. We are trying to establish, whether it is possible to get from the solutions of this system an information on solutions of any system obtained from this system by a prolongation of the polynomial coefficients to series. In comparison with, e.g., [8, 10], this is a different task.

2 Preliminaries

Let K be a field of characteristic 0. We denote by $K[[x]]$ the ring of *formal power series* and $K((x)) = K[[x]][x^{-1}]$ its quotient field (the field of *formal Laurent series*) with coefficients in K. For a nonzero element $a = \sum a_i x^i$ of $K((x))$ we denote by $\mathrm{val}\, a$ the *valuation* of a defined by $\mathrm{val}\, a = \min\{i \text{ such that } a_i \neq 0\}$; by convention, $\mathrm{val}\, 0 = \infty$.

If $l \in \mathbb{Z}, a(x) \in K((x))$ then we define the *l-truncation* $a^{\langle l \rangle} \in K[x, x^{-1}]$ as the Laurent polynomial obtained by omitting all the terms of valuation larger than l in a.

The ring of $n \times n$ matrices with entries belonging to a ring (a field) R is denoted by $\mathrm{Mat}_n(R)$. The *identity* $n \times n$-matrix is denoted by I_n. The notation M^T is used for the transpose of a matrix (vector) M.

For $M \in \mathrm{Mat}_n(K((x)))$ we define $\mathrm{val}\, M$ as the minimum of the valuations of the entries of M. We define the *leading coefficient* of a nonzero matrix $M \in \mathrm{Mat}_n(K((x)))$ as $\mathrm{lc}\, M = (x^{-\mathrm{val}\, M} M)|_{x=0}$. For $M \in \mathrm{Mat}_n(K[x])$ we define $\deg M$ as the maximum of the degrees of the entries of M.

A matrix $M \in \mathrm{Mat}_n(K((x)))$ is *non-singular* if $\det M \neq 0$, otherwise M is *singular*.

For $M \in \mathrm{Mat}_n(K((x)))$ we denote by M^* the adjugate matrix of M, i.e. the transpose of the cofactor matrix of M. One has

$$MM^* = M^* M = (\det M) I_n,$$

and, when M is non-singular

$$M^{-1} = (\det M)^{-1} M^*, \tag{1}$$

Given $M \in \mathrm{Mat}_n(K((x)))$, we define $M^{\langle l \rangle} \in \mathrm{Mat}_n(K[x, x^{-1}])$ obtained by replacing the entries of M by their l-truncations (if $M \in \mathrm{Mat}_n(K[[x]])$ then $M^{\langle l \rangle} \in \mathrm{Mat}_n(K[x])$).

3 Strongly Non-Singular Polynomial Matrices

Definition 1 Let $P \in \mathrm{Mat}_n(K[x])$ be a non-singular polynomial matrix and denote by d its degree. We say that P is *strongly non-singular* if there exists no singular matrix $M \in \mathrm{Mat}_n(K[[x]])$ such that $M^{\langle d \rangle} = P$.

Remark 1 Clearly, a non-singular matrix $P \in \mathrm{Mat}_n(K[x])$ of degree d is strongly non-singular if and only if there exists no $Q \in \mathrm{Mat}_n(K[[x]])$ such that $P + x^{d+1} Q$ is singular.

Now we prove a simple criterion for a polynomial matrix to be strongly non-singular.

Proposition 1 *Let $P \in Mat_n(K[x])$, $\det P \neq 0$. Then P is strongly non-singular if and only if*

$$\deg P + \mathrm{val}\, P^* \geqslant \mathrm{val}\, \det P. \tag{2}$$

Proof. Let $d = \deg P$, $v = \mathrm{val}\det P$ and

$$\tilde{P} = (P^*)^T$$

be the cofactor matrix of P.
Necessity: Suppose that the condition (2) is not satisfied. Let $\tilde{P} = (\tilde{p}_{i,j})_{i,j=1,\dots,n}$, and $\tilde{p}_{i_0,j_0}$ an entry of $\tilde{P}$ such that

$$d + \mathrm{val}\,\tilde{p}_{i_0,j_0} < v,$$

then $v - \mathrm{val}\,\tilde{p}_{i_0,j_0} \geqslant d+1$. Divide $\det P$ by $\tilde{p}_{i_0,j_0}$, considering them as power series. The quotient is a power series q, $\mathrm{val}\,q \geqslant d+1$. For the matrix $Q = (q_{i,j})_{i,j=1,\dots,n}$ such that

$$q_{i,j} = \begin{cases} -x^{-d-1}q, & \text{if } i = i_0 \text{ and } j = j_0, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

we get $\det(P + x^{d+1}Q) = 0$ by the Laplace expansion along the i_0-th row. According to Remark 1 the matrix P is not strongly non-singular.

Sufficiency: Suppose that the condition (2) is satisfied and let $Q \in \mathrm{Mat}_n(K[[x]])$. Since $\mathrm{val}\,x^{d+1}Q \geqslant d+1$ we have

$$\mathrm{val}\det(P + x^{d+1}Q) = v,$$

and $\det(P + x^{d+1}Q) \neq 0$.

Remark 2 Obviously, the inequality (2) can be rewritten in the equivalent form

$$\mathrm{val}\,(P^{-1}) + \deg P \geqslant 0, \tag{4}$$

since $\mathrm{val}\,P^* - \mathrm{val}\det P = \mathrm{val}\,(P^{-1})$ due to (1). Note also that

$$\deg P \geqslant \mathrm{val}\det P \tag{5}$$

is a sufficient condition for a matrix P to be strongly non-singular, since (5) implies (2).

Example 1 Every non-singular constant matrix is strongly non-singular. More generally, every polynomial matrix P such that $\mathrm{val}\det P = 0$ is strongly non-singular. □

It follows from the given proof of Proposition 1 that if P is not strongly non-singular, then one can *construct explicitly* a matrix $Q \in \mathrm{Mat}_n(K[[x]])$ such that $\det(P + x^{\deg P+1}Q) = 0$, and Q has only one nonzero entry, which is factually a rational function of x that can be expanded into a power series.

Example 2 Consider the following matrix

$$P = \begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix}. \tag{6}$$

One has $\deg P = 1, \operatorname{val} P^* = 0, \operatorname{val}\det P = 2$, so inequalities (2), (4) are not satisfied. Hence P is not strongly non-singular. Its cofactor matrix $\tilde{P}$ is given by

$$\tilde{P} = \begin{pmatrix} x & -1 \\ 0 & x \end{pmatrix},$$

In accordance with (3), the corresponding matrix Q is

$$Q = \begin{pmatrix} 0 & -x^{-2}\,\frac{x^2}{-1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The matrix

$$P + x^2 Q = \begin{pmatrix} x & x^2 \\ 1 & x \end{pmatrix}$$

is singular as expected. □

Proposition 2 *Let P be a strongly non-singular polynomial matrix of degree d. Let $v = \operatorname{val}\det P$ and $h = \operatorname{val} P^{-1} + \deg P$. Then for any $Q \in Mat_n(K[[x]])$ one has*

$$\det(P + x^{d+1}Q) - \det P = O(x^{v+h+1}). \tag{7}$$

Proof. Put $\bar{P} = x^{-\operatorname{val} P^{-1}} P^{-1}$ so that $\operatorname{val}\bar{P} \geqslant 0$. For any $Q \in \mathrm{Mat}_n(K[[x]])$ one has

$$P + x^{d+1}Q = P(I_n + x^{d+1}P^{-1}Q) = P(I_n + x^{d+1+\operatorname{val} P^{-1}}\bar{P}Q)$$

Hence

$$\det(P + x^{d+1}Q) = \det P \det(I_n + x^{h+1}\bar{P}Q).$$

The matrix P is strongly non-singular hence $h \geqslant 0$, and since $\operatorname{val}(\bar{P}Q) \geqslant 0$ it follows that

$$\det(I_n + x^{h+1}\bar{P}Q) = 1 + O(x^{h+1}),$$

and therefore $\det(P + x^{d+1}Q) = \det P + O(x^{v+h+1})$.

As a consequence, Proposition 2 states that $\det(P + x^{d+1}Q)$ and $\det P$ have the same valuation v for any $Q \in \mathrm{Mat}_n(K[[x]])$. Moreover, the $h+1$ first terms in the power series expansion of $\det(P + x^{d+1}Q)$ coincide with the corresponding terms of $\det P$.

Example 3 Let

$$P = \begin{pmatrix} 1+x & 0 \\ 1 & 1-x \end{pmatrix}.$$

Here $\det P = 1 - x^2$, $v = \operatorname{val}\det P = 0$ hence the matrix P is strongly non-singular. Here $\deg P = 1$ and $h = \operatorname{val} P^{-1} + \deg P = 1$ Let

$$Q = \begin{pmatrix} 1 + x + x^2 + \cdots & 0 \\ 0 & 0 \end{pmatrix}$$

then

$$P + x^2 Q = \begin{pmatrix} 1+x+x^2+\cdots & 0 \\ 1 & 1-x \end{pmatrix}.$$

We have

$$\det P = 1 - x^2,\ \det(P + x^2 Q) = 1$$

and (7) holds (here $x^{v+h+1} = x^2$). □

4 Inverse Matrix

The following proposition states that if P is strongly non-singular then for any $Q \in \mathrm{Mat}_n(K[[x]])$, the Laurent series expansions of the matrices P^{-1} and $(P + x^{d+1}Q)^{-1}$ have the same valuation and their first $h+1$ terms coincide where $h = \deg P + \mathrm{val}\, P^{-1}$.

Proposition 3 *Let P be a strongly non-singular polynomial $n \times n$-matrix of degree d and let $h = \deg P + \mathrm{val}\, P^{-1}$. Then for any $Q \in Mat_n(K[[x]])$ the Laurent series expansions of $(P + x^{d+1}Q)^{-1}$ and P^{-1} coincide up to order $\mathrm{val}\, P^{-1} + h$, i.e.,*

$$(P + x^{d+1}Q)^{-1} - P^{-1} = O(x^{\mathrm{val}\, P^{-1} + h + 1}). \tag{8}$$

In particular, one has

$$\mathrm{val}\,(P + x^{d+1}Q)^{-1} = \mathrm{val}\, P^{-1} \quad \textit{and} \quad \mathrm{lc}(P + x^{d+1}Q)^{-1} = \mathrm{lc}\, P^{-1} \tag{9}$$

for any $Q \in Mat_n(K[[x]])$.

Proof. Let $\bar{P} = x^{-\mathrm{val}\, P^{-1}} P^{-1}$ so that $\mathrm{val}\, \bar{P} \geqslant 0$. For any $Q \in \mathrm{Mat}_n(K[[x]])$ one has

$$(P + x^{d+1}Q)^{-1} = (I_n + x^{d + \mathrm{val}\, P^{-1} + 1} \bar{P} Q)^{-1} P^{-1}$$

It follows from (4) that $h \geqslant 0$, hence

$$(I_n + x^{h+1} \bar{P} Q)^{-1} = I_n + x^{h+1} C_1 + x^{h+2} C_2 + \cdots$$

where the C_i are constant matrices and the dots denote terms of higher valuation. It follows that

$$(P + x^{d+1}Q)^{-1} = P^{-1} + O(x^{h+1}) \cdot P^{-1} = x^{\mathrm{val}\, P^{-1}} (\bar{P} + O(x^{h+1}) \cdot \bar{P})$$

Hence

$$(P + x^{d+1}Q)^{-1} = x^{\mathrm{val}\, P^{-1}} (\bar{P}^{<h+1>} + O(x^{h+1}))$$

and the claim follows.

Example 4 Going back to the matrices P, Q from Example 3, we see that

$$P + x^2 Q = \begin{pmatrix} \frac{1}{1-x} & 0 \\ 1 & 1-x \end{pmatrix}$$

and we can compute

$$(P + x^2 Q)^{-1} = \begin{pmatrix} 1-x & 0 \\ -1 & \frac{1}{1-x} \end{pmatrix} = \begin{pmatrix} 1-x & 0 \\ -1 & 1+x+x^2+\cdots \end{pmatrix}$$

while

$$P^{-1} = \begin{pmatrix} \frac{1}{1+x} & 0 \\ \frac{-1}{1-x^2} & \frac{1}{1-x} \end{pmatrix} = \begin{pmatrix} 1-x+x^2+\cdots & 0 \\ -1-x^2-\cdots & 1+x+x^2+\cdots \end{pmatrix}.$$

Taking into account that here $d = 1$, $h = 1$, we see that (8) and (9) hold. □

Remark 3 Examples 3, 4 show that estimates (7), (8) are tight: $O(x^{v+h+1})$ and $O(x^{\operatorname{val} P^{-1}+h+1})$ cannot be replaced by $O(x^{v+h+2})$ and, resp., $O(x^{\operatorname{val} P^{-1}+h+2})$.

5 Product of Strongly Non-Singular Matrices

The product of two strongly non-singular matrices is not in general a strongly non-singular matrix.

Example 5 By Proposition 1, the matrices

$$P_1 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & x \\ 0 & -x^2 \end{pmatrix}$$

are both strongly non-singular, but their product

$$P_1 P_2 = \begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix},$$

is not, as it has been shown in Example 2. □

However, the following proposition holds:

Proposition 4 *Let $P_1, P_2 \in Mat_n(K[x])$ be strongly non-singular, and such that*

$$\deg P_1 P_2 = \deg P_1 + \deg P_2. \tag{10}$$

Then $P_1 P_2$ is a strongly non-singular matrix.

Proof. The inequality

$$\operatorname{val}(P_1 P_2)^{-1} \geqslant \operatorname{val} P_1^{-1} + \operatorname{val} P_2^{-1} \tag{11}$$

takes place (it holds for any matrices). Thus, it follows that if (10) is satisfied and (4) holds for both P_1 and P_2 then it holds for $P_1 P_2$ as well.

6 Width and s-Width of Non-Singular Matrices with Power Series Entries

In [3], the *width* of a non-singular (full rank) matrix $M \in \mathrm{Mat}_n(K[[x]])$ was defined as the minimal non-negative integer w such that any truncation $M^{\langle l \rangle}$ of M, $l \geqslant w$, is non-singular. Besides the notion of the width we will consider a similar notion related to the strong non-singularity.

Definition 2 The *s-width* (the *strong width*) of a non-singular (full rank) matrix $M \in \mathrm{Mat}_n(K[[x]])$ is the minimal non-negative integer w_s such that any $\hat{M} \in \mathrm{Mat}_n(K[[x]])$ which satisfies $\hat{M}^{\langle w_s \rangle} = M^{\langle w_s \rangle}$ is a non-singular matrix.

We will use the notations $w(M), w_s(M)$ when it is convenient.

It was shown in [3, Rmk 3] that the width $w(M)$ is well defined for any non-singular matrix $M \in \mathrm{Mat}_n(K[[x]])$. The following Proposition states that the s-width $w_s(M)$ is also well defined for any non-singular $M \in \mathrm{Mat}_n(K[[x]])$ and it is bounded by $-\mathrm{val}\,(M^{-1})$.

Proposition 5 *Let $M \in Mat_n(K[[x]])$ with $\det M \neq 0$ and set $l_0 = -\mathrm{val}\,(M^{-1})$. Then the matrix $(M^{\langle l \rangle} + x^{l+1}Q)$ is non-singular for any $Q \in Mat_n(K[[x]])$ and any $l \geqslant l_0$.*

Proof. For any $Q \in \mathrm{Mat}_n(K[[x]])$ and for any non-negative integer l one has

$$M^{\langle l \rangle} + x^{l+1}Q = M + O(x^{l+1}) = M(I_n + x^{l+1+\mathrm{val}\,M^{-1}}O(1)).$$

Hence

$$\det\,(M^{\langle l \rangle} + x^{l+1}Q) = (\det M)\det\,(I_n + x^{l+1+\mathrm{val}\,M^{-1}}O(1))$$

If we take $l \geqslant -\mathrm{val}\,(M^{-1})$ then

$$\mathrm{val}\,(\det\,(M^{\langle l \rangle} + x^{l+1}Q)) = \mathrm{val}\,(\det M)$$

for all $Q \in \mathrm{Mat}_n(K[[x]])$. Thus the claim follows.

Evidently,

$$w_s(M) \geqslant w(M)$$

for any non-singular matrix $M \in \mathrm{Mat}_n(K[[x]])$. However, as it is shown by the following example, it may happen that $w_s(M) > w(M)$; in other words, $w_s(M) \neq w(M)$ in general.

Example 6 Consider the matrix

$$M = \begin{pmatrix} x & x^3 \\ 1 & x \end{pmatrix}. \tag{12}$$

One has $\det M = x^2 - x^3 \neq 0$, $\det M^{\langle 0 \rangle} = 0$,

$$M^{\langle 1 \rangle} = M^{\langle 2 \rangle} = \begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix}, \;\det \begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix} \neq 0,$$

and $M^{\langle l\rangle} = M$ for $l \geqslant 3$. Thus $w(M) = 1$. However, $w_s(M) > 1$, due to the fact that

$$\det \begin{pmatrix} x & x^2 \\ 1 & x \end{pmatrix} = 0.$$

It is easy to check that $w_s(M) = 2$. □

Remark 4 The above example shows that the matrix $M^{\langle w_s\rangle}$ is not necessarily a strongly non-singular matrix. In fact, the matrix $M^{\langle w_s\rangle}$ is strongly non-singular if, and only if, $\deg M^{\langle w_s\rangle} = w_s$.

Proposition 6 *Let $M \in Mat_n(K[[x]])$ with $\det M \neq 0$. Then the set*

$$L(M) = \{l \in \mathbb{N} \mid M^{\langle l\rangle} \text{ is strongly non-singular}\} \tag{13}$$

is non-empty if, and only if, M is either an infinite power series matrix or a polynomial matrix which is strongly non-singular.

Proof. Let $d = \deg M$, with $d = +\infty$ when $M \in \mathrm{Mat}_n(K[[x]] \setminus K[x])$, and $l_0 = -\mathrm{val}\,(M^{-1})$. If $L(M) \neq \varnothing$ and $d < +\infty$ then $M^{\langle l\rangle}$ is strongly non-singular for some integer $l \geqslant 0$, and hence M is strongly non-singular as well. Reciprocally, suppose that $d = +\infty$. Then there exists an $l \geqslant l_0$ such that $\deg M^{\langle l\rangle} = l$. Now, according to Proposition 5, the matrix $M^{\langle l\rangle}$ is strongly non-singular. Hence $L(M) \neq \varnothing$.

Proposition 7 *Let $M \in Mat_n(K[[x]])$ with $\det M \neq 0$. Suppose that $L(M) \neq \varnothing$ (see (13)), and denote the smallest element of $L(M)$ by $\widetilde{w}_s(M)$. Then*

$$w_s(M) \leqslant -\mathrm{val}\, M^{-1} \leqslant \widetilde{w}_s(M).$$

In particular, the three quantities coincide if, and only if, the matrix $M^{\langle w_s\rangle}$ is strongly non-singular.

Proof. We know that the first inequality $w_s(M) \leqslant -\mathrm{val}\,(M^{-1})$ always holds. It remains to prove the second inequality. Let $l \in L(M)$ and set $P = M^{\langle l\rangle}$. One has $l \geqslant \deg P$ and $\deg P + \mathrm{val}\, P^{-1} \geqslant 0$, since P is strongly non-singular. On the other hand, by Proposition 3, one has $\mathrm{val}\, P^{-1} = \mathrm{val}\, M^{-1}$. It follows that

$$l + \mathrm{val}\, M^{-1} \geqslant \deg P + \mathrm{val}\, P^{-1} \geqslant 0.$$

The last part of the proposition follows from the fact that $M^{\langle w_s\rangle}$ is strongly non-singular if, and only if, $w_s \in L(M)$.

The matrix M in Example 6 satisfies the inequalities

$$w_s(M) = -\mathrm{val}\, M^{-1} = \mathrm{val}\det M = 2 < \widetilde{w}_s(M) = 3.$$

This shows in particular that, in general, val $\det M$ is not an upper-bound of $\widetilde{w}_s$, while we always have

$$w_s(M) \leqslant -\mathrm{val}\,(M^{-1}) \leqslant \mathrm{val}\,\det M.$$

The following example shows that $w_s(M)$ is not always equal to val $\det M$.

Example 7 Let

$$M = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

It is easy to check that $w_s(M) = 1 = \widetilde{w}_s(M)$. Indeed, $\det M^{\langle 0 \rangle} = 0$ and

$$\det M^{\langle 1 \rangle} = x^2, \;\; \det(M^{\langle 1 \rangle} + x^2 Q) = \det \begin{pmatrix} x + O(x^2) & O(x^2) \\ O(x^2) & x + O(x^2) \end{pmatrix} = x^2 + O(x^3).$$

In the same time, val $\det M = 2$. □

Proposition 8 *There exists an algorithm which, given a non-singular matrix $M \in Mat_n(K[[x]] \setminus K[x])$ that is represented algorithmically*[1] *computes $\widetilde{w}_s(M)$.*

Proof. For $l = 0, 1, \ldots$, we set step-by-step $P = M^{\langle l \rangle}$ and test wether condition (4) holds. Proposition 6 guarantees that this process terminates.

Note that the existence of an algorithm for computing $w_s(M)$ for a non-singular matrix M represented algorithmically is still an open problem, although we can compute upper bounds $\widetilde{w}_s(M)$, val $\det M$ for it.

7 Linear Differential Systems with Truncated Coefficients

We write ϑ for $x\frac{d}{dx}$ and consider linear differential systems with power series coefficients of the form

$$A_r(x)\vartheta^r y + A_{r-1}(x)\vartheta^{r-1} y + \cdots + A_0(x) y = 0 \tag{14}$$

where $y = (y_1, y_2, \ldots, y_m)^T$ is a column vector of unknown functions of x and where the coefficient matrices

$$A_0(x), A_1(x), \ldots, A_r(x) \tag{15}$$

belong to $\mathrm{Mat}_m(K[[x]])$. We suppose that $A_0(x), A_r(x)$ are non-zero and $\min_i\{\mathrm{val} A_i\} = 0$. For a system S of the form (14) we define the l-truncation $S^{\langle l \rangle}$ as the differential system with polynomial matrix coefficients obtained from S by omitting all the terms of valuation larger than l in the coefficients of S (the l-truncation is with respect to x only, not with respect to ϑ).

[1] For each power series that is an entry of M, an algorithm is specified that, given an integer i, finds the coefficient of x^i — see [2].

7.1 Width and s-Width of Differential Systems of Full Rank

Definition 3 Let S be a system of full rank over $K[[x]][\vartheta]$. The minimal integer w such that $S^{\langle l\rangle}$ is of full rank for all $l \geqslant w$ is called the *width* of S; this notion was first introduced in [3]. The minimal integer w_s such that any system S_1 satisfying the condition $S_1^{\langle w_s\rangle} = S^{\langle w_s\rangle}$, is of full rank, is called the *s-width* (the *strong width*) of S.

We will use the notations $w(S), w_s(S)$ when it is convenient.

Any linear algebraic system can be considered as a linear differential system of zero order. This let us state using Example 6 that for an arbitrary differential system S we have $w_s(M) \neq w(M)$ in general. However the inequality

$$w_s(S) \geqslant w(S)$$

holds.

It was proven in [3, Thm 2] that if a system S of the form (14) is of full rank then the width w of S is well defined, and the value w may be computed if the entries of S are represented algorithmically.

Concerning the s-width, we get the following proposition:

Proposition 9 *Let S be a full rank system of the form (14). Then the s-width $w_s(S)$ is defined. If the power series coefficients of S are represented algorithmically then we can compute algorithmically a non-negative integer N such that $w_s(S) \leqslant N$.*

Proof. The idea that was used to prove the mentioned Theorem 2 from [3] can be used here as well. For this, the induced recurrent system R is considered (such R is a specific recurrent system for the coefficients of Laurent series solutions of S). This system has polynomial coefficients of degree less than or equal to $r = \operatorname{ord} S$. The original system S is of full rank if and only if R is of full rank as a recurrent system. A recurrent system of this kind can be transformed by a special version of EG-eliminations ([3, Sect.3]) into a recurrent system $\tilde{R}$ whose leading matrix is non-singular. It is important that only a finite number of the coefficients of R are involved in the obtained leading matrix of $\tilde{R}$ (due to some characteristic properties of the used version of EG-eliminations). Each of polynomial coefficients of R is determined from a finite number (bounded by a non-negative integer N) of the coefficients of the power series involved in S. This proves the existence of the width and of the s-width as well. The mentioned number N can be computed algorithmically when all power series are represented algorithmically.

In conclusion of the proof, note that we can compute the width of S since we can test ([1, 5, 7]) whether a finite order differential system with polynomial coefficients is of full rank or not. From this point we can consider step-by-step $S^{\langle N-1\rangle}, S^{\langle N-2\rangle}, \ldots, S^{\langle 1\rangle}, S^{\langle 0\rangle}$ until the first one of them is not of full rank. If all the truncated systems are of full rank then $w = 0$. However, it is not exactly clear how to find $w_s(S)$, using the upper bound N. Is this problem algorithmically solvable? The question is still open.

Remark 5 If A_r, the leading matrix of S, is non-singular then $w_s(S) \leqslant w_s(A_r)$, since a system with non-singular leading matrix is necessarily of full rank.

7.2 When Only a Truncated System is Known

In this section we are interested in the following question (this is the main issue of the whole Section 7): suppose that for a system S of the form (14) only a finite number of terms of the entries of $A_0(x), A_1(x), \dots, A_r(x)$ is known, i.e., we know not the system S itself but the system $S^{\langle l \rangle}$ for some non-negative integer l. Suppose that we also know that

(a) $\operatorname{ord} S^{\langle l \rangle} = \operatorname{ord} S$, and
(b) $A_r(x)$ is invertible.

Is it possible to check the existence of nonzero Laurent series of S from the given approximate system $S^{\langle l \rangle}$ and if yes how many terms of these solutions of S can be computed from the solutions of $S^{\langle l \rangle}$? We will show that under the condition that the leading (polynomial) matrix of $S^{\langle l \rangle}$ is strongly non-singular we can apply our approach from [4] to get a non-trivial answer to this question.

We first recall the following result that we proved in [4]:

Proposition 10 *([4, Prop. 6]) Let S be a system of the form (14) having a non-singular $A_r(x)$ and*

$$\gamma = \min_i \operatorname{val} \left(A_r^{-1}(x) A_i(x)\right), \;\; q = \max\{-\gamma, 0\}.$$

There exists an algorithm, that uses only the terms of valuation less than

$$rmq + \gamma + \operatorname{val} \det A_r(x) + 1 \tag{16}$$

of the entries of the matrices $A_0(x), A_1(x), \dots, A_r(x)$, and computes a nonzero polynomial $I(\lambda)$ (the so-called indicial *polynomial* [9, Ch. 4, §8], [6, Def. 2.1], [4, Sect. 3.2]*) such that:*

- *if $I(\lambda)$ has no integer root then (14) has no solution in $K((x))^m \setminus \{0\}$,*
- *otherwise, there exist Laurent series solutions of S. Let e_*, e^* be the minimal and maximal integer roots of $I(\lambda)$; then the sequence*

$$a_k = rmq + \gamma + \operatorname{val} \det A_r(x) + \max\{e^* - e_* + 1,\, k + (rm-1)q\}, \tag{17}$$

$k = 1, 2, \dots$, is such that the system S possesses a solution $y(x) \in K((x))^m$ if and only if, the system $S^{\langle a_k \rangle}$ possesses a solution $\tilde{y}(x) \in K((x))^m$ such that $\tilde{y}(x) - y(x) = O(x^{e+k})$.

Let us now assume that we are given a truncated system $S^{\langle l \rangle}$ and denote by $\tilde{A}_i$ its coefficients so that $\tilde{A}_i = A_i^{\langle l \rangle}$ for $i = 0, \dots, r$. Suppose that its leading coefficient

$\tilde{A}_r$ is strongly non-singular and let $d = \deg \tilde{A}_r$, $p = -\mathrm{val}\,\tilde{A}_r^{-1}$ and $h = d - p$. Since $h \geqslant 0$, we have that $p \leqslant d \leqslant l$. Moreover, using (7) and (8) we have that

$$\mathrm{val}\,(\det A_r) = \mathrm{val}\,(\det \tilde{A}_r), \;\; \mathrm{val}\,(\det A_r^{-1}) = \mathrm{val}\,(\det \tilde{A}_r^{-1}),$$

and

$$A_r^{-1} = \tilde{A}_r^{-1} + O(x^{-p+h+1}).$$

Hence, for $i = 0, \dots, r-1$, one has

$$A_r^{-1} A_i = \tilde{A}_r^{-1} \tilde{A}_i + O(x^{-p+h+1}).$$

Let

$$\tilde{\gamma} = \min_{0 \leqslant i \leqslant r-1} (\mathrm{val}\,(\tilde{A}_r^{-1} \tilde{A}_i)), \;\; \gamma = \min_{0 \leqslant i \leqslant r-1} (\mathrm{val}\,(A_r^{-1} A_i)).$$

It follows that if $h - p \geqslant \tilde{\gamma}$ then $\gamma = \tilde{\gamma}$. We obtain using (16) that, under the conditions

$$h - p \geqslant \tilde{\gamma}, \;\; l \geqslant mr \max(-\tilde{\gamma}, 0) + \tilde{\gamma} + \mathrm{val}\,(\det \tilde{A}_r),$$

the indicial polynomial $I(\lambda)$ of S coincides with the indicial polynomial of $S^{\langle l \rangle}$. Moreover, the sequence (17) is the same for the two systems S and $S^{\langle l \rangle}$. We thus have proven the following

Proposition 11 *Let $\tilde{S}$ be a system of the form*

$$\tilde{A}_r(x) \vartheta^r y + \tilde{A}_{r-1}(x) \vartheta^{r-1} y + \cdots + \tilde{A}_0(x) y = 0$$

with polynomial matrices $\tilde{A}_0(x), \tilde{A}_1(x), \dots, \tilde{A}_r(x)$. Let its leading matrix $\tilde{A}_r(x)$ be strongly non-singular. Let

$$d = \deg \tilde{A}_r, \;\; p = -\mathrm{val}\,\tilde{A}_r^{-1}, \;\; h = d - p, \;\; \gamma = \min_{0 \leqslant i \leqslant r-1} (\mathrm{val}\,(\tilde{A}_r^{-1} \tilde{A}_i)), \;\; q = \max(-\gamma, 0)$$

and

$$h - p - \gamma \geqslant 0. \tag{18}$$

Let l be an integer such that

$$l \geqslant mrq + \gamma + \mathrm{val}\,(\det \tilde{A}_r). \tag{19}$$

Denote by $I(\lambda)$ the indicial polynomial of $\tilde{S}$. Let the set if integer roots of $I(\lambda)$ be non-empty, and e_*, e^* be the minimal and maximal integer roots of $I(\lambda)$. Let S be of the form (14) and $S^{\langle l \rangle} = \tilde{S}$. Let k satisfies the equality

$$\max\{e^* - e_* + 1, k + (rm - 1)q\} = l - rmq - \gamma - \mathrm{val}\,\det A_r(x). \tag{20}$$

Then for any $e \in \mathbb{Z}$ and column vectors $c_e, c_{e+1}, \dots, c_{e+k-1} \in K^m$, the system S possesses a solution

$$y(x) = c_e x^e + c_{e+1} x^{e+1} + \cdots + c_{e+k-1} x^{e+k-1} + O(x^{e+k}),$$

if and only if, the system $\tilde{S}$ possesses a solution $\tilde{y}(x) \in K((x))^m$ such that $\tilde{y}(x) - y(x) = O(x^{e+k})$.

Example 8 Let

$$\tilde{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1-x \end{pmatrix}, \ \tilde{A}_0 = \begin{pmatrix} 0 & -1 \\ -x+2x^2+2x^3+2x^4 & -2+4x \end{pmatrix}.$$

For the first-order differential system $\tilde{S}$

$$\tilde{A}_1(x)\vartheta y + \tilde{A}_0(x)y = 0$$

we have

$$d=1, \ p=0, \ h=1, \ \gamma=0, \ I(\lambda)=\lambda(\lambda-2), \ e^*-e_*+1=3.$$

The conditions of Proposition 11 are satisfied.

The general solution of $\tilde{S}$ is

$$\tilde{y}_1 = C_1 - C_1x + C_2x^2 - C_2x^3 + 0x^4 + \frac{2C_1}{15}x^5 + \frac{C_1}{30}x^6 + \left(\frac{C_1}{210} + \frac{2C_2}{35}\right)x^7 + \dots,$$

$$\tilde{y}_2 = \quad -C_1x + 2C_2x^2 - 3C_2x^3 + 0x^4 + \frac{2C_1}{3}x^5 + \frac{C_1}{5}x^6 + \left(\frac{C_1}{30} + \frac{2C_2}{5}\right)x^7 + \dots,$$

where C_1, C_2 are arbitrary constants. We can put $l = 4$ in (20), because $\deg\tilde{A}_0 = 4$, $\deg\tilde{A}_1 < 4$, and (18) holds. Then (20) has the form $\max\{3, k\} = 4$, thus $k = 4$. This means that all Laurent series solutions of any system S of the form

$$A_1(x)\vartheta y + A_0(x)y = 0 \qquad (21)$$

with non-singular matrix A_1 and such that $S^{\langle 4\rangle} = \tilde{S}$ are power series solutions having the form

$$y_1 = C_1 - C_1x + C_2x^2 - C_2x^3 + O(x^4),$$
$$y_2 = \quad -C_1x + 2C_2x^2 - 3C_2x^3 + O(x^4),$$

where C_1, C_2 are arbitrary constants. Consider, e.g., the first-order differential system S of the form (21) with

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1-x \end{pmatrix},$$

$$A_0 = \begin{pmatrix} 0 & -1 \\ -x+2x^2+2x^3+2x^4+2x^5+2x^6+x^7+x^8+\dots & -2+4x \end{pmatrix}.$$

Its general solution is

$$y_1 = C_1 - C_1 x + C_2 x^2 - C_2 x^3 + 0x^4 + 0x^5 + 0x^6 + \frac{C_1}{35} x^7 + \dots,$$

$$y_2 = \quad - C_1 x + 2C_2 x^2 - 3C_2 x^3 + 0x^4 + 0x^5 + 0x^6 + \frac{C_1}{5} x^7 + \dots,$$

which corresponds to the forecast and expectations. □

The following example shows that if the condition *'strong non-singularity of the leading matrix of the truncated system'* of Proposition 11 is not satisfied then it may happen that the correspondence between the Laurent solutions of $\tilde{S}$ and S as described in that proposition do not occur.

Example 9 Consider the first-order differential system S:

$$A_1(x)\vartheta y + A_0(x) y = 0,$$

where

$$A_1 = \begin{pmatrix} x & x^3 \\ 1 & x \end{pmatrix} \quad A_0 = \begin{pmatrix} 0 & -x^4 + 3x^3 \\ 0 & -x^3 + 3x \end{pmatrix}.$$

Its general solution is

$$y_1(x) = C_1 + C_2 \ln(x), \; y_2(x) = \frac{C_2}{x^3}$$

where C_1, C_2 are arbitrary constants.

The truncated systems $S^{\langle l \rangle}$ for $l = 1, 2$ coincide and have the leading matrix

$$\begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix}$$

which is non-singular but not strongly non-singular. The general solution of $S^{\langle 2 \rangle}$ is

$$y_1(x) = C_1, \; y_2(x) = \frac{C_2}{x^3}.$$

The truncated system $S^{\langle 3 \rangle}$ has the leading matrix A_1 which is strongly non-singular. Note that $(d, p, h, \gamma, q) = (3, 2, 1, 0, 0)$ and the condition (11), i.e., $h - p - \gamma \geqslant 0$ of Proposition 11 is not satisfied. The general solution of $S^{\langle 3 \rangle}$ is

$$y_1(x) = C_1 + C_2 \int \frac{e^{-x}}{(x-1)^2} \, dx, \; y_2(x) = C_2 \frac{e^{-x}}{x^3 (x-1)}.$$

The expansions of $y_1(x)$ and $y_2(x)$ at $x = 0$ are respectively given by

$$y_1(x) = C_1 + C_2 (x + \frac{1}{2} x^2 + O(x^3)),$$

$$y_2(x) = C_2(-x^{-3} - \frac{1}{2}x^{-1} - \frac{1}{3} - \frac{3}{8}x - \frac{11}{30}x^2 + O(x^3)).$$

Let now consider, instead of S, the first-order system R :

$$B_1(x)\vartheta y + B_0(x)y = 0$$

where $B_1 = A_1$ and

$$B_0 = \begin{pmatrix} 0 & -x^5 + 3x^3 \\ 0 & -x^3 + 3x \end{pmatrix},$$

so that $R^{\langle 3 \rangle} = S^{\langle 3 \rangle}$. We find that the general solution of R is

$$y_1(x) = C_1, \;\; y_2(x) = \frac{C_2 \, e^{\frac{1}{2}x^2}}{x^3}.$$

It has no logarithmic term, and the statement of Proposition 11 holds. □

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