On the Structure of Multivariate Hypergeometric Terms

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Abstract

Wilf and Zeilberger conjectured in 1992 that a hypergeometric term is proper-hypergeometric if and only if it is holonomic. We prove a slightly modified version of this conjecture in the case of several discrete variables.

1 Introduction

Let K be a field of characteristic zero, $n_1, \ldots, n_d$ variables ranging over the nonnegative integers, and E_i the corresponding shift operators, acting on functions of $n_1, \ldots, n_d$ by $E_i f(n_1, \ldots, n_i, \ldots, n_d) = f(n_1, \ldots, n_i + 1, \ldots, n_d)$. A K-valued function $T(n_1, \ldots, n_d)$ is a *hypergeometric term* if there are rational functions $F_i \in K(n_1, \ldots, n_d)$ (called the *certificates* of T) such that $E_i T = F_i T$, for $i = 1, \ldots, d$. $T(n_1, \ldots, n_d)$ is *holonomic* if partial derivatives of its generating function $\sum_{n_1,\ldots,n_d \geq 0} T(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d}$ lie in a finite-dimensional vector space over the rational function field $K(x_1, \ldots, x_d)$. A holonomic sequence satisfies a system of homogeneous linear recurrences of a special form. If T is holonomic then its definite sums w.r.t. some of the variables are still holonomic as functions of the remaining variables. If T is also hypergeometric then the holonomic recurrences satisfied by these sums can be found efficiently by means of Zeilberger's Creative Telescoping algorithm [21, 22, 19].

A hypergeometric term is *proper* if it can be expressed as a product of a polynomial, several factorials of linear forms with integer coefficients, their reciprocals, and exponential functions. In [20] it is proved that proper hypergeometric terms are holonomic. Wilf and Zeilberger conjectured [19, p. 585] that a hypergeometric term is proper if and only if it is holonomic. Their conjecture concerns hypergeometric terms which depend on several discrete and continuous variables. We prove a slightly modified version of their conjecture in the discrete case, namely that every holonomic hypergeometric term is *conjugate* to a proper term (meaning that the two terms have the same certificates). This modification is necessary as shown, e.g., by the bivariate hypergeometric term $T(n, k) = |n - k|$ which is holonomic since its generating function $\sum_{n,k \geq 0} |n - k| x^n y^k = (x/(1-x)^2 + y/(1-y)^2)/(1 - xy)$ is rational, but T is not proper (see Example 6).

Our proof of the modified Wilf-Zeilberger conjecture is based on the *Ore-Sato Theorem* (as it is called in [5]) which states essentially that for every hypergeometric term T there is a rational function R and a proper term T' such that $(E_i T)/T = (E_i(RT'))/(RT')$ for all i. This was proved in the bivariate case by Ore using elementary means [11, 12], and in the multivariate case by Sato using homological algebra [15, Appendix]. We give an elementary proof of the multivariate Ore-Sato Theorem. The necessary tools that are useful also for other purposes are developed in Section 3 (normal forms of rational functions) and Section 4 (shift-invariant and pairwise shift-invariant polynomials). The certificates F_i of a hypergeometric term clearly satisfy the *compatibility conditions* $(E_j F_i) / F_i = (E_i F_j) / F_j$. In Section 5 we give an algorithm which, given compatible rational functions $F_1, \ldots, F_d$, computes compatible rational functions $F'_1, \ldots, F'_d$, and a rational function R such that $F_i = (E_i R/R) F'_i$, and the numerators and denominators of F'_i factor

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into integer-linear factors (i.e., polynomials of the form $a_1x_1 + \cdots + a_dx_d + c$ where the a_i's are integers). In Section 6 we use this structure theorem to prove the Ore-Sato Theorem (Corollary 4). In Section 7 we show that a rational sequence is holonomic if and only if its denominator factors into integer-linear factors. Together with the Ore-Sato Theorem, this yields our main result: Every holonomic hypergeometric term is conjugate to a nontrivial proper term (Theorem 14). In these results, hypergeometric terms are treated as algebraic objects. But in applications hypergeometric terms are functions which take on specific values, therefore it is important to deal also with the questions of their zeros and of singularities of their certificates – which have received little attention in the literature referred to above. To overcome these problems we introduce the notion of *nonvanishing rising factorials* (Section 1), and two equivalence relations among nontrivial hypergeometric terms, namely *equality modulo an algebraic set*, and *conjugacy* between solutions of a first-order system of recurrences with polynomial coefficients (Section 2).

After we had obtained our results in the bivariate case [3, 4], it was brought to our attention that the bivariate Wilf-Zeilberger conjecture has been proved independently, and at almost the same time, also by Hou [8, 9].

Throughout the paper, K is a field of characteristic zero, and $\mathbb{N}$ denotes the set of nonnegative integers. We write $\boldsymbol{u} = (u_1, u_2, \ldots, u_d)$ for d-tuples of numbers or indeterminates, $\boldsymbol{u} \geq \boldsymbol{v}$ when $u_i \geq v_i$ for $1 \leq i \leq d$, and $\boldsymbol{u}^T\boldsymbol{v} = \sum_{i=1}^d u_iv_i$. If $\boldsymbol{u}^T\boldsymbol{v} = 0$ then $\boldsymbol{u}$ and $\boldsymbol{v}$ are called orthogonal. We denote by $\boldsymbol{e}_i$ the d-tuple whose components are zero except the i-th one which is 1, and by $\boldsymbol{s}$ the d-tuple with all components equal to s. The monomial $x_1^{u_1} \cdots x_d^{u_d}$ is denoted by $\boldsymbol{x}^{\boldsymbol{u}}$. Following [7], we write $p \perp q$ to indicate that polynomials $p, q \in K[\boldsymbol{x}]$ are relatively prime. By a *factor of a rational function* $f \in K(\boldsymbol{x})$ we mean any polynomial factor of either p or q where $f = p/q$, $p, q \in K[\boldsymbol{x}]$, and $p \perp q$. We use E_i to denote the operator that shifts the i-th variable by 1. In particular, if $T : \mathbb{N}^d \to K$ is a d-variate sequence then $E_iT(n_1, \ldots, n_i, \ldots, n_d) = T(n_1, \ldots, n_i + 1, \ldots, n_d)$, and if $f \in K(x_1, x_2, \ldots, x_d)$ is a rational function then $E_iT(x_1, \ldots, x_i, \ldots, x_d) = T(x_1, \ldots, x_i + 1, \ldots, x_d)$.

We define the *rising factorial* $(\alpha)_n$ for all $\alpha \in K$ and $n \in \mathbb{Z}$ by

$$(\alpha)_n = \begin{cases} \prod_{i=0}^{n-1} (\alpha + i), & n \geq 0, \\ \prod_{i=1}^{-n} \frac{1}{\alpha - i}, & n < 0 \text{ and } \alpha \neq 1, 2, \ldots, -n, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Z(\alpha)$ be the set of all $n \in \mathbb{Z}$ such that $(\alpha)_n = 0$. Obviously,

$$Z(\alpha) = \begin{cases} \{n \in \mathbb{Z};\ n + \alpha \leq 0\}, & \alpha \in \mathbb{Z} \text{ and } \alpha > 0, \\ \{n \in \mathbb{Z};\ n + \alpha > 0\}, & \alpha \in \mathbb{Z} \text{ and } \alpha \leq 0, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{1}$$

Note that $(\alpha + n)_{-n}$ serves as a kind of a *pseudo-inverse* for $(\alpha)_n$, in the following sense:

- if $(\alpha)_n \neq 0$ then $(\alpha + n)_{-n} = 1/(\alpha)_n$,
- if $(\alpha)_n = 0$ then $(\alpha + n)_{-n} = 0$.

It is easy to verify that the sequence $(\alpha)_n$ satisfies the first-order recurrence

$$(n + \alpha)(\alpha)_{n+1} - (n + \alpha)^2(\alpha)_n = 0 \tag{2}$$

for all $n \in \mathbb{Z}$. We will also need another solution of (2) which is nonzero for all $\alpha \in K$ and $n \in \mathbb{Z}$. We call it the *nonvanishing rising factorial* and denote it $(\alpha)_n^*$. It is defined as the usual rising factorial, except that zero factors are omitted wherever they appear:

$$(\alpha)_n^* = \begin{cases} (\alpha)_n, & (\alpha)_n \neq 0, \\ (\alpha)_{1-\alpha}(0)_{\alpha+n}, & \alpha \in \mathbb{Z} \text{ and } \alpha > 0 \text{ and } \alpha + n \leq 0, \\ (\alpha)_{-\alpha}(1)_{\alpha+n-1}, & \alpha \in \mathbb{Z} \text{ and } \alpha \leq 0 \text{ and } \alpha + n > 0. \end{cases}$$

Now we have $(\alpha + n)_{-n}^* = 1/(\alpha)_n^*$ for *all* $n \in \mathbb{Z}$.

Example 1 According to our definitions,

$$(1)_n = \begin{cases} n!, & n \geq 0, \\ 0, & n \leq -1 \end{cases}, \qquad (0)_n = \begin{cases} 0, & n \geq 1, \\ \frac{(-1)^n}{(-n)!}, & n \leq 0 \end{cases},$$

$$(1)_n^* = \begin{cases} n!, & n \geq 0, \\ \frac{(-1)^{n+1}}{(-n-1)!}, & n \leq -1 \end{cases}, \qquad (0)_n^* = \begin{cases} (n-1)!, & n \geq 1, \\ \frac{(-1)^n}{(-n)!}, & n \leq 0 \end{cases}.$$

Remark 1 *Proper hypergeometric terms are usually defined in terms of factorials of complex argument, with $z!$ denoting $\Gamma(z+1)$ and $1/z!$ defined to be zero when z is a negative integer. If n is an integer variable and $\alpha \in \mathbb{C}$, we can rewrite the sequence $(n+\alpha)!$ with rising factorials as*

$$(n+\alpha)! = \begin{cases} \alpha!\,(\alpha+1)_n, & \alpha \notin \mathbb{Z}, \\ (1)_{n+\alpha}, & \alpha \in \mathbb{Z} \end{cases} \tag{3}$$

whenever the left-hand side is defined (i.e., $n+\alpha$ is not a negative integer), and its reciprocal as

$$\frac{1}{(n+\alpha)!} = \begin{cases} \frac{(n+\alpha+1)_{-n}}{\alpha!}, & \alpha \notin \mathbb{Z}, \\ (n+\alpha+1)_{-(n+\alpha)}, & \alpha \in \mathbb{Z}, \end{cases} \tag{4}$$

where ordinary factorials (or the Γ-function) are applied only to constants on the right-hand side. The advantage of rising factorials over ordinary ones is that the former do not rely on the Γ-function and are well defined in any field of characteristic zero.

Wilf and Zeilberger [18] *associate with $(n+\alpha)!$ its* shadow

$$\frac{(-1)^n}{(-n-\alpha-1)!}$$

which satisfies the same first-order recurrence w.r.t. n. When $\alpha \notin \mathbb{Z}$ the shadow is just a constant-factor multiple of $(n+\alpha)!$ (the constant being $-(\sin\alpha\pi)/\pi$), while for $\alpha \in \mathbb{Z}$ the shadow is complementary to $(n+\alpha)!$, in the sense that the latter is defined when $n+\alpha \geq 0$, and the former when $n+\alpha < 0$. If we replace the rising factorials in the right-hand side of (3) by their nonvanishing counterparts, nothing changes for $\alpha \notin \mathbb{Z}$, but for $\alpha \in \mathbb{Z}$ we have instead of $(1)_{n+\alpha}$

$$(1)_{n+\alpha}^* = \begin{cases} (1)_{n+\alpha} & = & (n+\alpha)!, & n+\alpha \geq 0, \\ (0)_{n+\alpha+1} & = & \frac{(-1)^{n+\alpha+1}}{(-n-\alpha-1)!}, & n+\alpha < 0. \end{cases}$$

Thus rewriting factorials in terms of the nonvanishing rising factorials, we either get the factorial itself or its shadow (perhaps with the opposite sign), whichever is defined.

2 Multivariate sequences

By a *sequence* $T(\boldsymbol{n})$ we mean a function $T : \mathbb{N}^d \to K$. We call a set $A \subseteq \mathbb{N}^d$ *algebraic* if there is a polynomial $p \in K[\boldsymbol{x}] \setminus \{0\}$ which vanishes on A. Clearly, if A is algebraic and B is not, then $B \setminus A$ is not algebraic. Also, a finite union of algebraic sets is algebraic.

Proposition 1 *Let $F, G \in K(\boldsymbol{x})$ be rational functions which agree on a non-algebraic set $B \subseteq \mathbb{N}^d$. Then $F = G$.*

Proof: Let $F = p/q$, $G = u/v$, where $p, q, u, v \in K[\boldsymbol{x}]$. The polynomial $pv - qu$ vanishes on the non-algebraic set B, hence it is the zero polynomial, and so $F = G$. □

Definition 1 (equality modulo an algebraic set) *We write $T =_a T'$ if the set $\{\boldsymbol{n} \in \mathbb{N}^d;\ T(\boldsymbol{n}) \neq T'(\boldsymbol{n})\}$ is algebraic. A sequence $T(\boldsymbol{n})$ is* trivial *if $T =_a 0$.*

Equality modulo an algebraic set is clearly an equivalence relation. It is also a congruence because $T_1 =_a T_2$ and $T_1' =_a T_2'$ imply $T_1 + T_1' =_a T_2 + T_2'$ and $T_1 T_1' =_a T_2 T_2'$. Trivial sequences can be described as those with algebraic support. Note however that a nontrivial sequence can vanish on a non-algebraic set.

Example 2 The sequence $T(n,k) = \binom{n}{k} = (n-k+1)_k (k+1)_{-k}$ is nontrivial because $\operatorname{supp} T = \{(n,k) \in \mathbb{N}^2;\ n \geq k\}$ is not algebraic. But neither is its complement $\{(n,k) \in \mathbb{N}^2;\ n < k\}$.

Definition 2 (hypergeometric term, conjugate hypergeometric terms) *A sequence $T(\boldsymbol{n})$ is a* hypergeometric term *if there are polynomials $p_i, q_i \in K[\boldsymbol{x}] \setminus \{0\}$ such that*

$$p_i(\boldsymbol{n})(E_i T(\boldsymbol{n})) \quad = \quad q_i(\boldsymbol{n}) T(\boldsymbol{n}) \tag{5}$$

for all $\boldsymbol{n} \in \mathbb{N}^d$ and $1 \leq i \leq d$. Two hypergeometric terms T_1, T_2 are conjugate *if they satisfy (5) with the same p_i, q_i. In this case we write $T_1 \simeq T_2$.*

Proposition 2 (i) *The product of two hypergeometric terms is a hypergeometric term.*

(ii) *If $T_1 \simeq T_2$ and $T_1' \simeq T_2'$ then $T_1 T_1' \simeq T_2 T_2'$.*

We omit the straightforward proofs.

Proposition 3 *If T is a hypergeometric term and $T' =_a T$ then T' is a hypergeometric term and $T' \simeq T$.*

Proof: Let T satisfy (5) and let $p(\boldsymbol{n})T'(\boldsymbol{n}) = p(\boldsymbol{n})T(\boldsymbol{n})$ for all $\boldsymbol{n} \in \mathbb{N}^d$. Then

$$p(\boldsymbol{n})(E_i p(\boldsymbol{n})) p_i(\boldsymbol{n})(E_i T'(\boldsymbol{n})) \quad = \quad p(\boldsymbol{n})(E_i p(\boldsymbol{n})) q_i(\boldsymbol{n}) T'(\boldsymbol{n}) \tag{6}$$

for all $\boldsymbol{n} \in \mathbb{N}^d$ and $1 \leq i \leq d$, hence T' is a hypergeometric term. Clearly $T(\boldsymbol{n})$ also satisfies (6), so $T' \simeq T$. □

The converse of Proposition 3 is of course not true, because conjugate hypergeometric terms may differ everywhere, as any constant multiple of a term is clearly conjugate to it.

Example 3 The "patchwork" sequence

$$\begin{aligned} T(n,k) &= \begin{cases} 2(n-2k)!, & n > 2k, \\ 3^k, & n = 2k, \\ 7\frac{(-1)^n}{(2k-n-1)!}, & n < 2k \end{cases} \\ &= \begin{cases} 2\,(1)_{n-2k}, & n > 2k, \\ 3^k, & n = 2k, \\ 7\,(-1)^n (2k-n)_{n-2k+1}, & n < 2k \end{cases} \quad = \begin{cases} 2\,(1)^*_{n-2k}, & n > 2k, \\ 3^k, & n = 2k, \\ -7\,(1)^*_{n-2k}, & n < 2k \end{cases} \end{aligned}$$

is a hypergeometric term because it satisfies the recurrences

$$(n-2k)(n-2k+1)T(n+1,k) \quad = \quad (n-2k)(n-2k+1)^2 T(n,k), \tag{7}$$

$$(n-2k-2)(n-2k-1)^2(n-2k)^2 T(n,k+1) \quad = \quad (n-2k-2)(n-2k-1)(n-2k)T(n,k) \tag{8}$$

for all $n, k \in \mathbb{N}$. Clearly, $T(n,k)$ is conjugate to the hypergeometric terms $T_1(n,k) = (1)_{n-2k}$, $T_2(n,k) = (-1)^{n-2k}(2k-n)_{n-2k+1}$, and $T_3(n,k) = (1)^*_{1,n-2k}$ which satisfy the same first-order recurrences (7) and (8), but it is not equal to either of them modulo an algebraic set.

Identification of multivariate sequences which agree outside an algebraic set is consistent with identification of univariate sequences which agree outside a finite set (cf. [16]). Such identification enables us to regard every rational function $R \in K(\boldsymbol{x})$ as a sequence $R(\boldsymbol{n})$, without actually having to specify its values at the singular points of R. Therefore, if T is a hypergeometric term satisfying (5), we can write

$$E_i T(\boldsymbol{n}) =_a F_i(\boldsymbol{n}) T(\boldsymbol{n}) \tag{9}$$

where $F_i = q_i/p_i$, for $1 \leq i \leq d$. Sometimes these rational functions are called the *certificates* of T.

Example 4 For the term $T(n,k)$ defined in Example 3, we have $T(n+1,k) =_a (n-2k+1)T(n,k)$ and $(n-2k-1)(n-2k)T(n,k+1) =_a T(n,k)$, so its certificates are $n-2k+1$ and $1/((n-2k-1)(n-2k))$.

It is clear that the certificates of a hypergeometric term satisfy certain compatibility conditions.

Definition 3 (compatible rational functions) *Rational functions $F_1, F_2, \ldots, F_d \in K(\boldsymbol{x})$ are* compatible *if they satisfy*

$$(E_j F_i)\, F_j = (E_i F_j)\, F_i \tag{10}$$

for all $1 \le i < j \le d$.

Proposition 4 *Let $T(\boldsymbol{n})$ be a hypergeometric term which satisfies (9). If $T \neq_a 0$ then*

(i) *$F_1, F_2, \ldots, F_d$ are compatible,*

(ii) *$F_1, F_2, \ldots, F_d$ are unique.*

Proof: (i) From (9) we have

$$\begin{aligned} E_i E_j T(\boldsymbol{n}) &= (E_j F_i(\boldsymbol{n}))(E_j T(\boldsymbol{n})) = (E_j F_i(\boldsymbol{n})) F_j(\boldsymbol{n}) T(\boldsymbol{n}) \\ &= (E_i F_j(\boldsymbol{n}))(E_i T(\boldsymbol{n})) = (E_i F_j(\boldsymbol{n})) F_i(\boldsymbol{n}) T(\boldsymbol{n}) \end{aligned}$$

for $\boldsymbol{n}$ outside some algebraic set A. Hence $(E_j F_i(\boldsymbol{n})) F_j(\boldsymbol{n}) = (E_i F_j(\boldsymbol{n})) F_i(\boldsymbol{n})$ on $\operatorname{supp} T \setminus A$. As this is a non-algebraic set, Proposition 1 implies that $(E_j F_i) F_j = (E_i F_j) F_i$.

(ii) Assume that in addition to (9), $E_i T(\boldsymbol{n}) =_a G_i(\boldsymbol{n}) T(\boldsymbol{n})$ for $1 \le i \le d$. Then $F_i(\boldsymbol{n}) = G_i(\boldsymbol{n})$ on $\operatorname{supp} T \setminus A$, for some algebraic set A. By Proposition 1, $F_i = G_i$. □

From Proposition 4(ii) it follows that nontrivial hypergeometric terms T_1 and T_2 are conjugate if and only if they have the same certificate.

Obviously every hypergeometric term is conjugate to the zero term, and also to every trivial term. But when restricted to nontrivial terms, this relation is transitive, and hence an equivalence relation:

Proposition 5 *Let T_1, T_2, T_3 be hypergeometric terms such that $T_1 \simeq T_2$, $T_2 \simeq T_3$. If $T_2 \neq_a 0$ then $T_1 \simeq T_3$.*

Proof: This follows from Proposition 4(ii). □

Definition 4 (holonomic) *Let $K((\boldsymbol{x}))$ denote the field of fractions of the formal power series ring $K[[\boldsymbol{x}]]$. A sequence $T(\boldsymbol{n})$ is* holonomic *if the set of all partial derivatives of its generating function $\sum_{\boldsymbol{n}\ge\boldsymbol{0}} T(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$ spans a finite-dimensional subspace of $K((\boldsymbol{x}))$ over the subfield of rational functions $K(\boldsymbol{x})$.*

Theorem 1 [10, Thm. 3.7] *A sequence $T(\boldsymbol{n})$ is holonomic if and only if there is an $s \in \mathbb{N}$ such that*

(i) *for each $i \in \{1, 2, \ldots, d\}$, there is a nonempty set $H_i \subseteq \{0, \ldots, s\}^d$ and a set of univariate polynomials $\{p_{\boldsymbol{h},i} \in K[x] \setminus \{0\};\ \boldsymbol{h} \in H_i\}$ such that*

$$\sum_{\boldsymbol{h}\in H_i} p_{\boldsymbol{h},i}(n_i)\, T(\boldsymbol{n}-\boldsymbol{h}) = 0 \tag{11}$$

for all $\boldsymbol{n} \ge \boldsymbol{s}$, and

(ii) *if $d \ge 2$, each $(d-1)$-variate sequence $a_{i,k}(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_d) = T(n_1, \ldots, n_{i-1}, k, n_{i+1}, \ldots, n_d)$ with $1 \le i \le d$ and $0 \le k \le s-1$ is holonomic.*

Note that if the coefficients in (11) are constant we may use the same recurrence for all $i \in \{1, 2, \ldots, d\}$.

Example 5 The term $T(n,k) = \binom{n}{k}$ is holonomic because it satisfies condition (i) of Theorem 1 with the constant-coefficient recurrence $T(n,k) - T(n-1,k) - T(n-1,k-1) = 0$ valid for $n, k \ge s = 1$. Condition (ii) is satisfied as well because $T(n,0) - T(n-1,0) = 0$ for $n \ge 1$, and $T(0,k) = 0$ for $k \ge 1$.

The term $T(n,k)$ from Example 3 is also holonomic, because it satisfies condition (i) of Theorem 1 with the constant-coefficient recurrence $T(n,k-2) - 4\,T(n-2,k-3) + 3\,T(n-4,k-4) = 0$ valid for $n, k \ge s = 4$. Condition (ii) is obviously satisfied as well.

Theorem 2 *The product of two holonomic sequences is holonomic.*

For a proof, see [10, Thm. 3.8(i)] or [20, Prop. 3.2'].

Definition 5 (factorial term) *A sequence* $T(\boldsymbol{n})$ *is a* factorial term *if there are* $\boldsymbol{u} \in K^d$, $p, q \in \mathbb{N}$, $\boldsymbol{\alpha} \in K^{p+q}$, *and* $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p \in \mathbb{Z}^d$ *such that*

$$T(\boldsymbol{n}) = \boldsymbol{u}^{\boldsymbol{n}} \prod_{i=1}^{p} (\alpha_i)_{\boldsymbol{a}_i^T \boldsymbol{n}} \prod_{i=p+1}^{p+q} (\alpha_i + \boldsymbol{a}_i^T \boldsymbol{n})_{-\boldsymbol{a}_i^T \boldsymbol{n}} \tag{12}$$

for all $\boldsymbol{n} \in \mathbb{N}^d$.

Definition 6 (proper term) *A sequence* T *is a* proper term *if there is a polynomial* $P \in K[\boldsymbol{x}]$ *and a factorial term* T' *such that*

$$T = PT'. \tag{13}$$

Note that the definitions of hypergeometric, holonomic, factorial, and proper terms are all symmetric in the variables $n_1, n_2, \ldots, n_d$. Hence if $T(\boldsymbol{n})$ has one of these properties, then so does $T(\pi(\boldsymbol{n}))$ where π is any permutation of $\boldsymbol{n}$.

Theorem 3 *Every proper term is hypergeometric and holonomic.*

Proof: Let $T(\boldsymbol{n})$ be a proper term. Then $T(\boldsymbol{n}) = P(\boldsymbol{n})T'(\boldsymbol{n})$ where $P \in K[\boldsymbol{x}]$ is a polynomial and $T'(\boldsymbol{n})$ is of the form (12). As a rational function, $P(\boldsymbol{n})$ is clearly hypergeometric. By using (2) repeatedly, each factor on the right-hand side of (12) is hypergeometric as well. Hence, by Proposition 2, $T(\boldsymbol{n})$ is hypergeometric.

Similarly, each factor of $T(\boldsymbol{n})$ satisfies a recurrence with constant coefficients: If $r = \deg_{n_1} P(\boldsymbol{n})$ then $\Delta_{n_1}^{r+1} P(\boldsymbol{n}) = 0$. Clearly, $\boldsymbol{u}^{\boldsymbol{n}+\boldsymbol{1}} = \boldsymbol{u}^{\boldsymbol{1}} \boldsymbol{u}^{\boldsymbol{n}}$. If $f(\boldsymbol{n}) = (\alpha)_{\boldsymbol{a}^T \boldsymbol{n}+c}$ or $f(\boldsymbol{n}) = (\alpha + \boldsymbol{a}^T \boldsymbol{n})_{-(\boldsymbol{a}^T \boldsymbol{n})}$ then $f(\boldsymbol{n}+\boldsymbol{h}) = f(\boldsymbol{n})$ where $\boldsymbol{h}$ is any nonzero integer vector orthogonal to $\boldsymbol{a}$. The same is true of the factors of each $(d-1)$-variate sequence $T(n_1, \ldots, n_{i-1}, k, n_{i+1}, \ldots, n_d)$ where $k \in \mathbb{N}$. Thus by Theorem 1, each factor of $T(\boldsymbol{n})$ is holonomic, hence by Theorem 2, so is $T(\boldsymbol{n})$. □

For factorial terms, this result can be found in [17], and for proper terms in [20, 19, 14].

Wilf and Zeilberger conjectured [19, p. 585] that the converse of Theorem 3 holds as well. Taken literally, this is not true as we show in Example 6 of Section 3. However, we prove in Theorem 14 a slightly modified version of their conjecture, namely that over an algebraically closed field, every holonomic hypergeometric term is conjugate to a nontrivial proper term.

3 A normal form for rational functions

In this section E denotes the shift operator corresponding to x, so that $Ef(x) = f(x+1)$ for every $f \in K(x)$.

Theorem 4 *For every rational function* $F \in K(x)$ *there are polynomials* $a, b, c \in K[x]$ *such that*

(i) $F = \frac{a}{b} \cdot \frac{Ec}{c}$,

(ii) $a \perp E^k b$ *for all* $k \in \mathbb{N}$,

(iii) $a \perp c$ *and* $b \perp Ec$.

For a proof, see [13] or [14]. The original version of this theorem (without (iii)) is due to Gosper [6].

Definition 7 (PNF) *If* a, b, c, F *satisfy* (i) *and* (ii) *of Theorem 4, then* (a, b, c) *is a* polynomial normal form *or* PNF *of* F. *A PNF which satisfies* (iii) *of Theorem 4 is* strict.

Lemma 1 *If* (a, b, c) *is a strict PNF of* p/q *where* $p, q \in K[x]$, *then* $a \mid p$ *and* $b \mid q$.

Proof: We have $pbc = aqEc$, hence $a \,|\, pbc$ and $b \,|\, aqEc$. By (ii) and (iii), $a \perp bc$ and $b \perp aEc$, so $a \,|\, p$ and $b \,|\, q$. □

In place of (ii), we need the stronger property that $a \perp E^k b$ for all $k \in \mathbb{Z}$. To achieve this we allow c to be a rational function.

Definition 8 (shift-reduced) *A rational function $u \in K(x)$ is* shift-reduced *if there are $a, b \in K[x]$ such that $u = a/b$ and $a \perp E^k b$ for all $k \in \mathbb{Z}$.*

Theorem 5 *For every rational function $F \in K(x)$ there are rational functions $u, v \in K(x)$ such that*

(i) $F = u \cdot \frac{Ev}{v}$,

(ii) *u is shift-reduced.*

Proof: If $F = 0$ take $u = 0$ and $v = 1$. Otherwise let (a, b, c) be a PNF of F, and (a_1, b_1, c_1) a strict PNF of b/a. We claim that taking $u = b_1/a_1$, $v = c/c_1$ satisfies (i) and (ii). Indeed,

$$u \cdot \frac{Ev}{v} \;=\; \frac{b_1}{a_1} \cdot \frac{c_1}{Ec_1} \cdot \frac{Ec}{c} \;=\; \frac{a}{b} \cdot \frac{Ec}{c} \;=\; F,$$

proving (i). Because $a_1 \perp E^k b_1$ for $k \geq 0$, we have $b_1 \perp E^k a_1$ for $k \leq 0$. By Lemma 1, $a_1 \,|\, b$ and $b_1 \,|\, a$. As $a \perp E^k b$ for $k \geq 0$, it follows that $b_1 \perp E^k a_1$ for $k \geq 0$ as well, proving (ii). □

Definition 9 (RNF) *If u, v, F are as in Theorem 5, (u, v) is a* rational normal form*, or* RNF*, of F. We denote the set of all* RNF*'s of F by* $\mathrm{RNF}_x(F)$.

Note that together with an algorithm for computing strict PNF (to be found in [13] or [14]), the proof of Theorem 5 provides an algorithm for computing an element of $\mathrm{RNF}_x(F)$.

Theorem 6 *Let (u, v) and (u_1, v_1) be two* RNF*'s of $F \in K(x) \backslash \{0\}$. Write $u = zp/q$ and $u_1 = z_1 p_1/q_1$ where $z, z_1 \in K$, $p, q, p_1, q_1 \in K[x]$ are monic, $p \perp q$, and $p_1 \perp q_1$. Then $z = z_1$, $\deg p = \deg p_1$, and $\deg q = \deg q_1$.*

For a proof, see [4].

Example 6 Let $T(n, k) = |n - k|$. Then $(n - k)T(n + 1, k) - (n - k + 1)T(n, k) = 0$ and $(n - k)T(n, k + 1) - (n - k - 1)T(n, k) = 0$ for all $n, k \in \mathbb{N}$, so $T(n, k)$ is a hypergeometric term. It is also holonomic as it satisfies condition (i) of Theorem 1 with the constant-coefficient recurrence $T(n, k) - T(n - 1, k - 1) = 0$ valid for $n, k \geq s = 1$, and condition (ii) is obviously satisfied as well.

We claim that $|n - k|$ is not equal to any proper term, not even modulo an algebraic set. To prove this, assume on the contrary that $|n - k| =_a T'(n, k)$ where $T'(n, k)$ is a proper term. Let $Q \in K[x, y]$ be a nonzero polynomial such that $|n - k| \, Q(n, k) = T'(n, k) \, Q(n, k)$ for all $n, k \in \mathbb{N}$. Write

$$T'(n, k) = P(n, k) \; u^n v^k \prod_{i=1}^{p} (\alpha_i)_{a_i n + b_i k} \prod_{j=1}^{q} (\beta_j + c_j n + d_j k)_{-(c_j n + d_j k)}$$

where $P \in K[x, y]$, $u, v, \alpha_i, \beta_j \in K$, and $a_i, b_i, c_j, d_j \in \mathbb{Z}$. If $(\alpha_i)_{a_i n_0 + b_i k_0} = 0$ for some $n_0, k_0 \in \mathbb{N}$ then, by (1), $\alpha_i \in \mathbb{Z}$ and either $\alpha_i > 0$ and $\alpha_i + a_i n_0 + b_i k_0 \leq 0$, or $\alpha_i \leq 0$ and $\alpha_i + a_i n_0 + b_i k_0 > 0$. In the former case, $a_i < 0$ or $b_i < 0$, so $(\alpha_i)_{a_i n + b_i k}$ vanishes on the non-algebraic set $\{(n, k) \in \mathbb{N}^2;\ \alpha_i + a_i n + b_i k \leq 0\}$. In the latter case, $a_i > 0$ or $b_i > 0$, so $(\alpha_i)_{a_i n + b_i k}$ vanishes on the non-algebraic set $\{(n, k) \in \mathbb{N}^2;\ \alpha_i + a_i n + b_i k > 0\}$. In either case, $T'(n, k)$, and hence $|n - k| \, Q(n, k)$ would vanish on a non-algebraic set, which is false. Hence $(\alpha_i)_{a_i n + b_i k} \neq 0$ for all $n, k \in \mathbb{N}$. In the same way we see that $(\beta_j + c_j n + d_j k)_{-(c_j n + d_j k)} \neq 0$ for all $n, k \in \mathbb{N}$. Therefore we can write

$$T'(n, k) = P(n, k) \; u^n v^k \, \frac{\prod_{i=1}^{p} (\alpha_i)_{a_i n + b_i k}}{\prod_{j=1}^{q} (\beta_j)_{c_j n + d_j k}}.$$

Pick $n_0, k_0 \in \mathbb{N}$ such that $n_0 < k_0$ and $Q(n_0, k_0) \neq 0$. Such n_0, k_0 certainly exist, for otherwise the univariate polynomial $Q(n_0, k)$ would be identically zero for each n_0, as it would vanish for all $k > n_0$, and hence Q itself would be the zero polynomial. Let $t(n) = T'(n, k_0)\, Q(n, k_0) = |n - k_0|\, Q(n, k_0)$. This is a univariate hypergeometric term which can be written in the form

$$t(n) = p(n)\, w^n \frac{\prod_{i=1}^{p'} (\gamma_i)_n}{\prod_{j=1}^{q'} (\delta_j)_n}, \quad \text{for all } n \in \mathbb{N}, \tag{14}$$

where $p \in K[x]$, $w, \gamma_i, \delta_j \in K$, and $(\gamma_i)_n, (\delta_j)_n \neq 0$ for all $n \in \mathbb{N}$. If $\gamma_i - \delta_j \in \mathbb{Z}$ then $(\gamma_i)_n/(\delta_j)_n$ is a rational function of n, hence we can rewrite (14) as

$$t(n) = r(n)\, w^n\, t'(n), \quad \text{for all } n \in \mathbb{N},$$

where $r \in K(x)$ is a rational function, and $t'(n)$ is a nonvanishing univariate hypergeometric term whose certificate $f'(n) = t'(n+1)/t'(n)$ is a shift-reduced rational function. Let $f(n) = t(n+1)/t(n) = |n + 1 - k_0|\, Q(n+1, k_0)/(|n - k_0|\, Q(n, k_0)) =_a (n+1-k_0)\, Q(n+1, k_0)/((n - k_0)\, Q(n, k_0))$. Then both $(w\, f'(n), r(n))$ and $(1, (n-k_0)\, Q(n, k_0))$ belong to $\mathrm{RNF}_n(f)$. It follows from Theorem 6 that $w\, f'(n) = 1$, hence $t'(n) = c/w^n$ for all $n \in \mathbb{N}$, where $c \in K \setminus \{0\}$ is a constant, so $t(n) = c\, r(n)$ for all $n \in \mathbb{N}$. But $t(n) = |n - k_0|\, Q(n, k_0) =_a (n - k_0)\, Q(n, k_0)$, therefore by Proposition 1, the two rational functions $c\, r(n)$ and $(n - k_0)\, Q(n, k_0)$ are identical, and $t(n) = (n - k_0)\, Q(n, k_0)$ for all $n \in \mathbb{N}$. Thus we have $|n - k_0|\, Q(n, k_0) = (n - k_0)\, Q(n, k_0)$ for all $n \in \mathbb{N}$, and in particular, $|n_0 - k_0|\, Q(n_0, k_0) = (n_0 - k_0)\, Q(n_0, k_0)$. As $Q(n_0, k_0) \neq 0$, it follows that $|n_0 - k_0| = n_0 - k_0$, contrary to our choice of $n_0 < k_0$. This contradiction shows that $|n - k|$ is not equal to any proper term, not even modulo an algebraic set. Note however that $|n - k|$ is *conjugate* to the nontrivial proper term $n - k$, as well as to any term T'' of the form

$$T''(n, k) = \begin{cases} a\,(n - k), & n \geq k, \\ b\,(n - k), & n < k \end{cases}$$

where $a, b \in K$ are arbitrary.

4 Shift invariance and integer linearity

Definition 10 (shift-invariant, pairwise shift-invariant, integer-linear) *A rational function $f \in K(\boldsymbol{x})$ is* shift-invariant *if there is a nonzero integer vector $\boldsymbol{a} \in \mathbb{Z}^d$ such that $f(\boldsymbol{x} + \boldsymbol{a}) = f(\boldsymbol{x})$. A rational function $f \in K(\boldsymbol{x})$ is* pairwise shift-invariant *if for each pair of indices i, j, $1 \leq i < j \leq d$, there are $h_{ij}, h_{ji} \in \mathbb{Z}$, not both zero, such that $E_i^{h_{ij}} E_j^{h_{ji}} f(\boldsymbol{x}) = f(\boldsymbol{x})$. A polynomial $p \in K[\boldsymbol{x}]$ is* integer-linear *if $p(\boldsymbol{x}) = u \cdot (\boldsymbol{a}^T \boldsymbol{x}) + v$ where $\boldsymbol{a} \in \mathbb{Z}^d$ and $u, v \in K$.*

Note the following facts:

- If $d = 2$, the notions of shift invariance and pairwise shift invariance coincide.
- Any constant polynomial is integer-linear (take $u = 0$).
- Over an algebraically closed field, any univariate polynomial factors into integer-linear factors.

Lemma 2 *Let $f \in K(x)$, $a \in K$, $a \neq 0$. If $f(x + a) = f(x)$ then $f(x) = c \in K$.*

Proof: Write $f(x) = p(x)/q(x)$ where $p, q \in K[x]$. Let $x_0 \in K$ be such that $q(x_0 + ka) \neq 0$ for all $k \in \mathbb{N}$. By induction on k, $f(x_0 + ka) = f(x_0)$ for all $k \in \mathbb{N}$. Write $c = f(x_0)$. Then $r(x) = p(x) - c\, q(x) \in K[x]$ vanishes on $\{x_0 + ka;\ k \in \mathbb{N}\}$. In characteristic zero this is an infinite set, hence r is the zero polynomial, and $f(x) = c$ as claimed. □

Lemma 3 *Let $f \in K(\boldsymbol{x})$, $\boldsymbol{a} \in K^d$, $a_d \neq 0$. If $f(\boldsymbol{x} + \boldsymbol{a}) = f(\boldsymbol{x})$ then there is a $(d-1)$-variate rational function $h \in K(x_1, x_2, \ldots, x_{d-1})$ such that*

$$f(\boldsymbol{x}) = h(x_1 - \frac{a_1}{a_d} x_d, \ldots, x_{d-1} - \frac{a_{d-1}}{a_d} x_d).$$

Furthermore, if $f \in K[\boldsymbol{x}]$ then $h \in K[x_1, x_2, \ldots, x_{d-1}]$.

Proof: Define

$$h(\boldsymbol{x}) = f(x_1 + \frac{a_1}{a_d}x_d, \ldots, x_{d-1} + \frac{a_{d-1}}{a_d}x_d, x_d).$$

Then $f(\boldsymbol{x}) = h(x_1 - \frac{a_1}{a_d}x_d, \ldots, x_{d-1} - \frac{a_{d-1}}{a_d}x_d, x_d)$ and $h(x_1, \ldots, x_{d-1}, x_d + a_d) = h(\boldsymbol{x})$. Considering h as an element of $K(x_1, x_2, \ldots, x_{d-1})(x_d)$, Lemma 2 implies that, in fact, $h \in K(x_1, x_2, \ldots, x_{d-1})$. □

Proposition 6 *A d-variate rational function $f \in K(\boldsymbol{x})$ is shift-invariant if and only if there are nonzero integer vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{d-1} \in \mathbb{Z}^d$ and a $(d-1)$-variate rational function $g \in K(x_1, x_2, \ldots, x_{d-1})$ such that*

$$f(\boldsymbol{x}) = g(\boldsymbol{v}_1^T\boldsymbol{x},\, \boldsymbol{v}_2^T\boldsymbol{x},\, \ldots,\, \boldsymbol{v}_{d-1}^T\boldsymbol{x}). \tag{15}$$

Furthermore, if $f \in K[\boldsymbol{x}]$ then $g \in K[x_1, x_2, \ldots, x_{d-1}]$.

Proof: Let $\boldsymbol{a} \in \mathbb{Z}^d$ be a nonzero vector such that $f(\boldsymbol{x} + \boldsymbol{a}) = f(\boldsymbol{x})$. W.l.g. assume that $a_d \neq 0$. By Lemma 3, there is a $(d-1)$-variate rational function $h \in K(x_1, x_2, \ldots, x_{d-1})$ such that

$$f(\boldsymbol{x}) = h(x_1 - \frac{a_1}{a_d}x_d, \ldots, x_{d-1} - \frac{a_{d-1}}{a_d}x_d).$$

Then $f(\boldsymbol{x}) = g(a_d x_1 - a_1 x_d, a_d x_2 - a_2 x_d, \ldots, a_d x_{d-1} - a_{d-1}x_d) = g(\boldsymbol{v}_1^T\boldsymbol{x},\, \boldsymbol{v}_2^T\boldsymbol{x},\, \ldots,\, \boldsymbol{v}_{d-1}^T\boldsymbol{x})$ where $g(x_1, x_2, \ldots, x_{d-1}) = h(x_1/a_d, x_2/a_d, \ldots, x_{d-1}/a_d)$, and $\boldsymbol{v}_i = a_d\boldsymbol{e}_i - a_i\boldsymbol{e}_d \neq \boldsymbol{0}$.

Conversely, assume that f is of the form (15). Let $\boldsymbol{a} \in \mathbb{Z}^d$ be a nonzero integer vector such that $\boldsymbol{v}_1^T\boldsymbol{a} = \boldsymbol{v}_2^T\boldsymbol{a} = \cdots = \boldsymbol{v}_{d-1}^T\boldsymbol{a} = 0$. Then $f(\boldsymbol{x} + \boldsymbol{a}) = f(\boldsymbol{x})$. □

Proposition 7 *A d-variate rational function $f \in K(\boldsymbol{x})$ is pairwise shift-invariant if and only if there is a nonzero integer vector $\boldsymbol{v} \in \mathbb{Z}^d$ and a univariate rational function $g \in K(x)$ such that*

$$f(\boldsymbol{x}) = g(\boldsymbol{v}^T\boldsymbol{x}).$$

Furthermore, if $f \in K[\boldsymbol{x}]$ then $g \in K[x]$.

Proof: First let f be pairwise shift-invariant. We prove by induction on d that $f(\boldsymbol{x}) = g(\boldsymbol{v}^T\boldsymbol{x})$.

- $d = 1$: The assertion holds vacuously.
- $d > 1$: Consider f as an element of $K(x_d)(x_1, \ldots, x_{d-1})$. By the induction hypothesis, there are $g' \in K(x_d)(x)$ and $v'_1, \ldots, v'_{d-1} \in \mathbb{Z}$, not all zero, such that

  $$f(x_d)(x_1, \ldots, x_{d-1}) = g'(x_d)(u) \tag{16}$$

  where $u = v'_1x_1 + \cdots + v'_{d-1}x_{d-1}$. W.l.g. assume that $v'_1 \neq 0$. Regarding g' as an element of $K(x_d, x)$, we can write (16) as $f(\boldsymbol{x}) = g'(x_d, u)$. Now

  $$g'(x_d, u) = f(\boldsymbol{x}) = E_1^{h_{1d}}E_d^{h_{d1}}f(\boldsymbol{x}) = f(x_d + h_{d1})(x_1 + h_{1d}, x_2, \ldots, x_{d-1}) = g'(x_d + h_{d1}, u + v'_1h_{1d}).$$

  By Proposition 6 applied to the bivariate rational function g', there are $g \in K(x)$ and $a, b \in \mathbb{Z}$ not both zero such that

  $$f(\boldsymbol{x}) = g'(x_d, u) = g(au + bx_d) = g(\boldsymbol{v}^T\boldsymbol{x})$$

  where $\boldsymbol{v} = (av'_1, \ldots, av'_{d-1}, b) \neq \boldsymbol{0}$.

Conversely, let $f(\boldsymbol{x}) = g(\boldsymbol{v}^T\boldsymbol{x})$ and $1 \leq i < j \leq d$. If $v_i = v_j = 0$ then set $h_{ij} = h_{ji} = 1$, otherwise set $h_{ij} = v_j$, $h_{ji} = -v_i$. In both cases $E_i^{h_{ij}}E_j^{h_{ji}}f(\boldsymbol{x}) = f(\boldsymbol{x})$. □

Corollary 1 *Assume that K is algebraically closed. If $p \in K[\boldsymbol{x}]$ is irreducible and pairwise shift-invariant then p is integer-linear.*

Proof: By Proposition 7, there is $q \in K[x]$ and a nonzero integer vector $\boldsymbol{a} \in \mathbb{Z}^d$ such that $p(\boldsymbol{x}) = q(\boldsymbol{a}^T\boldsymbol{x})$. As p is irreducible, so is q, hence $\deg q \leq 1$. Thus there are $c, d \in K$ such that $q(x) = cx + d$ and, consequently, $p(\boldsymbol{x}) = c \cdot (\boldsymbol{a}^T\boldsymbol{x}) + d$. □

Lemma 4 *Fix a pair of indices i, j, $1 \le i < j \le d$. If for every irreducible factor p of $P \in K[\boldsymbol{x}]$ with $\deg_{x_i} p, \deg_{x_j} p > 0$ there are $a, b \in \mathbb{Z}$, $a > 0$, such that ${E_i}^a {E_j}^b p \,|\, P$, then for every irreducible factor p of P with $\deg_{x_i} p, \deg_{x_j} p > 0$ there are $A, B \in \mathbb{Z}$, $A > 0$, such that ${E_i}^A {E_j}^B p = p$.*

Proof: Pick any irreducible factor p_0 of P such that $\deg_{x_i} p_0, \deg_{x_j} p_0 > 0$. Construct a sequence of nonconstant irreducible factors p_l of P such that $p_{l+1} = {E_i}^{a_l} {E_j}^{b_l} p_l$ where $a_l, b_l \in \mathbb{Z}$ and $a_l > 0$, for $l \ge 0$. As $K[\boldsymbol{x}]$ is a unique factorization domain, there are indices $l_0 < l_1$ such that $p_{l_0} = p_{l_1}$. By definition of p_l, it follows that

$$p_{l_0} = {E_i}^A {E_j}^B p_{l_0} \tag{17}$$

where $A = a_{l_0} + a_{l_0+1} + \cdots + a_{l_1-1} > 0$ and $B = b_{l_0} + b_{l_0+1} + \cdots + b_{l_1-1}$ are integers. We have additionally

$$p_{l_0} = {E_i}^{A'} {E_j}^{B'} p_0, \tag{18}$$

where $A' = a_0 + a_1 + \cdots + a_{l_0-1} > 0$ and $B' = b_0 + b_1 + \cdots + b_{l_0-1}$ are integers. Applying ${E_i}^{-A'} {E_j}^{-B'}$ to (17) and using (18) we obtain $p_0 = {E_i}^A {E_j}^B p_0$. □

Theorem 7 *Let K be algebraically closed, and $P \in K[\boldsymbol{x}]$. If for each irreducible factor p of P and for each pair of indices i, j, $1 \le i < j \le d$ with $\deg_{x_i} p, \deg_{x_j} p > 0$ there are $a, b \in \mathbb{Z}$, $a > 0$, such that ${E_i}^a {E_j}^b p \,|\, P$, then P factors into integer-linear factors.*

Proof: Lemma 4 implies that each irreducible factor of P is pairwise shift-invariant. Hence by Corollary 1, each irreducible factor of P is integer-linear. □

For the case $d = 2$, a different proof of Theorem 7 using algebraic functions is given in [2, Lemma 3].

5 Compatible rational functions

Theorem 8 [1] *Let $a, b, u, v \in K[x] \setminus \{0\}$, $u \perp v$, $r = u/v$, p an irreducible factor of v, and*

$$a(x) r(x+1) = b(x) r(x). \tag{19}$$

Then there are $m, n \in \mathbb{N}$, $m \ge 1$, $n \ge 0$, such that $p(x+m)$ divides $a(x)$ and $p(x-n)$ divides $b(x)$.

Proof: Rewrite (19) as

$$a(x) u(x+1) v(x) = b(x) u(x) v(x+1). \tag{20}$$

Let $m \in \mathbb{N}$, $m \ge 1$, be such that $p(x+m-1)$ divides $v(x)$ but $p(x+m)$ does not. Then (20) implies that $p(x+m) \,|\, a(x) u(x+1) v(x)$. As $p(x+m) \perp u(x+1) v(x)$, it follows that $p(x+m) \,|\, a(x)$.

Let $n \in \mathbb{N}$, $n \ge 0$, be such that $p(x-n)$ divides $v(x)$ but $p(x-n-1)$ does not. Then (20) implies that $p(x-n) \,|\, b(x) u(x) v(x+1)$. As $p(x-n) \perp u(x) v(x+1)$, it follows that $p(x-n) \,|\, b(x)$. □

The following property of divisibility in $K[\boldsymbol{x}]$ will be used freely.

Proposition 8 *Let $p, q \in K[\boldsymbol{x}]$, p irreducible, $\deg_{x_d} p \ne 0$. Then $p \,|\, q$ in $K[\boldsymbol{x}]$ if and only if $p \,|\, q$ in $K(x_1, \ldots, x_{d-1})[x_d]$.*

Proof: Divisibility in $K[\boldsymbol{x}]$ obviously implies divisibility in $K(x_1, \ldots, x_{d-1})[x_d]$.

Conversely, let $q = pr$ where $r \in K(x_1, \ldots, x_{d-1})[x_d]$. As p is irreducible in $K[\boldsymbol{x}]$ and $\deg_{x_d} p \ne 0$, p is primitive when considered as an element of $K[x_1, \ldots, x_{d-1}][x_d]$. Write $r = (\alpha/\beta) r'$, $q = \gamma q'$ where $\alpha, \beta, \gamma \in K[x_1, \ldots, x_{d-1}]$ and $q', r' \in K[x_1, \ldots, x_{d-1}][x_d]$ are primitive. Then

$$\beta \gamma q' = \alpha p r'.$$

By Gauss's Lemma $p r'$ is primitive, hence $\alpha = \beta \gamma$. It follows that $r = \gamma r' \in K[\boldsymbol{x}]$, hence $p \,|\, q$ in $K[\boldsymbol{x}]$. □

Theorem 9 *Let $F, G \in K(x,y)$ be compatible rational functions. Let (G', R) be an* RNF *of G, considered as a rational function of y over $K(x)$, and $F'(x,y) = F(x,y)R(x,y)/R(x+1,y)$. Then*

(i) $F(x,y) = F'(x,y)\frac{R(x+1,y)}{R(x,y)}$,

(ii) $G(x,y) = G'(x,y)\frac{R(x,y+1)}{R(x,y)}$,

(iii) *F', G' are compatible rational functions,*

(iv) *each irreducible factor $p \in K[x,y]$ of either F' or G' is shift-invariant.*

Proof: Properties (i) and (ii) follow from the definitions of F' and G', respectively. The compatibility condition (10) for F, G implies that

$$F'(x,y)\,G'(x+1,y) = F'(x,y+1)\,G'(x,y), \tag{21}$$

so F', G' are compatible. It remains to prove (iv). Write

$$F'(x,y) = \frac{s(x,y)}{t(x,y)}, \quad G'(x,y) = \frac{u(x,y)}{v(x,y)} \tag{22}$$

where $s,t,u,v \in K[x,y]$, $s(x,y) \perp t(x,y)$, and $u(x,y) \perp v(x,y+m)$ for all $m \in \mathbb{Z}$.

Let $p \in K[x,y]$ be an irreducible factor of s, t, u, or v. If $\deg_x p = 0$ or $\deg_y p = 0$ then p is trivially shift-invariant. In the case $\deg_x p, \deg_y p > 0$ we use two lemmas.

Lemma 5 *Let F', G', s,t,u,v be as in (21), (22). If $p \in K[x,y]$ is an irreducible factor of uv, $\deg_y p \neq 0$, then there are $A, B \in \mathbb{Z}$, $A > 0$, such that $p(x+A, y+B)$ divides st.*

Proof: a) If $p \,|\, v$ rewrite (21) as

$$s(x,y)t(x,y+1)G'(x+1,y) = s(x,y+1)t(x,y)G'(x,y).$$

By Theorem 8, there is $m \in \mathbb{Z}$, $m \geq 1$, such that

$$p(x+m,y) \,|\, s(x,y)t(x,y+1).$$

Then $p(x+m,y) \,|\, s(x,y)$ or $p(x+m,y-1) \,|\, t(x,y)$. Take $(A,B) = (m,0)$ in the former case, $(A,B) = (m,-1)$ in the latter.

b) If $p \,|\, u$ rewrite (21) as

$$s(x,y+1)t(x,y)\frac{1}{G'(x+1,y)} = s(x,y)t(x,y+1)\frac{1}{G'(x,y)}.$$

By Theorem 8, there is $m \in \mathbb{Z}$, $m \geq 1$, such that

$$p(x+m,y) \,|\, s(x,y+1)t(x,y).$$

Then $p(x+m,y-1) \,|\, s(x,y)$ or $p(x+m,y) \,|\, t(x,y)$. Take $(A,B) = (m,-1)$ in the former case, $(A,B) = (m,0)$ in the latter. □

Lemma 6 *Let F', G', s,t,u,v be as in (21), (22) where $G'(x,y)$ is shift-reduced w.r.t. y. If $q \in K[x,y]$ is an irreducible factor of st and $\deg_x q \neq 0$, then there is $C \in \mathbb{Z}$ such that $q(x,y+C)$ divides uv.*

Proof: a) If $q \,|\, t$ rewrite (21) as

$$u(x,y)v(x+1,y)F'(x,y+1) = u(x+1,y)v(x,y)F'(x,y).$$

By Theorem 8, there are $m, n \in \mathbb{Z}$ such that

$$q(x, y+m) \mid u(x,y)v(x+1,y) \quad \text{and} \quad q(x, y-n) \mid u(x+1,y)v(x,y).$$

Since u/v is shift-reduced w.r.t. y it follows that $q(x,y+m) \mid u(x,y)$ or $q(x,y-n) \mid v(x,y)$. Take $C = m$ in the former case, $C = -n$ in the latter.

b) If $q \mid s$ rewrite (21) as

$$u(x+1,y)v(x,y)\frac{1}{F'(x,y+1)} = u(x,y)v(x+1,y)\frac{1}{F'(x,y)}.$$

By Theorem 8, there are $m, n \in \mathbb{Z}$ such that

$$q(x, y+m) \mid u(x+1,y)v(x,y) \quad \text{and} \quad q(x, y-n) \mid u(x,y)v(x+1,y).$$

Since u/v is shift-reduced w.r.t. y it follows that $q(x,y+m) \mid v(x,y)$ or $q(x,y-n) \mid u(x,y)$. Take $C = m$ in the former case, $C = -n$ in the latter. □

Proof of Thm. 9 (cont'd): If p is an irreducible factor of uv then by Lemma 5 there are $A, B \in \mathbb{Z}$, $A > 0$, such that $p(x+A, y+B)$ divides st. By Lemma 6, there is $C \in \mathbb{Z}$ such that $p(x+A, y+B+C)$ divides uv. Hence by Lemma 4, all irreducible factors of uv are shift-invariant.

If p is an irreducible factor of st then by Lemma 6 there is $C \in \mathbb{Z}$ such that $p(x, y+C)$ divides uv. By Lemma 5, there are $A, B \in \mathbb{Z}$, $A > 0$, such that $p(x+A, y+B+C)$ divides st. By Lemma 4, all irreducible factors of st are shift-invariant. □

Corollary 2 *Let $F, G \in K(x,y)$ be compatible rational functions over an algebraically closed field K. Then $F', G' \in K(x,y)$ mentioned in Theorem 9 factor into integer-linear factors.*

Proof: By Theorem 9 and Corollary 1. □

Theorem 10 *Let $F_1, F_2, \ldots, F_d \in K(\boldsymbol{x})$ be compatible rational functions over an algebraically closed field K. Then there are compatible rational functions $F'_1, F'_2, \ldots, F'_d \in K(\boldsymbol{x})$ which factor into integer-linear factors, and a rational function $R \in K(\boldsymbol{x})$ such that $F_i = F'_i \cdot (E_i R)/R$, for $i = 1, 2, \ldots, d$.*

Proof: We present an algorithm for computing $F'_1, F'_2, \ldots, F'_d$, and R with desired properties.

Algorithm Multi-RNF

`input:` compatible functions $F_1, F_2, \ldots, F_d \in K(\boldsymbol{x})$;
`output:` $R, F'_1, F'_2, \ldots, F'_d \in K(\boldsymbol{x})$ satisfying Theorem 10;

$R_1 := 1$;
`for` $i = 1, \ldots, d$ `do`
  $F_i^{(1)} := F_i$;
`for` $k = 2, \ldots, d$ `do`
  `select` $(F_k^{(k)}, S_k) \in \mathrm{RNF}_{x_k}(F_k^{(k-1)})$;
  $R_k := S_k R_{k-1}$;
  `for` $i = 1, \ldots, d,\ i \neq k,$ `do`
    $F_i^{(k)} := F_i^{(k-1)} S_k / (E_i S_k)$;
`return` $R_d, F_1^{(d)}, \ldots, F_d^{(d)}$.

We claim that for $k = 1, 2, \ldots, d$:

(i) $F_1^{(k)}, F_2^{(k)}, \ldots, F_d^{(k)}$ are compatible,

(ii) for $i = 1, 2, \ldots, d$ we have $F_i = F_i^{(k)} \frac{E_i R_k}{R_k}$,

(iii) each irreducible factor of any of $F_1^{(k)}, \ldots, F_d^{(k)}$ is pairwise shift-invariant as a polynomial in $x_1, \ldots, x_k$.

The proof of this claim is by induction on k.

- $k = 1$: In this case, (i) – (iii) hold trivially.
- $k > 1$: Assume that (i) – (iii) hold at $k - 1$.
  (i) Multiplying $F_i^{(k-1)}(E_i F_j^{(k-1)}) = (E_j F_i^{(k-1)}) F_j^{(k-1)}$ by $S_k/(E_i E_j S_k)$ we obtain $F_i^{(k)}(E_i F_j^{(k)}) = (E_j F_i^{(k)}) F_j^{(k)}$.
  (ii) $F_i = F_i^{(k-1)} \cdot \frac{E_i R_{k-1}}{R_{k-1}} = F_i^{(k-1)} \cdot \frac{S_k}{E_i S_k} \cdot \frac{E_i S_k}{S_k} \cdot \frac{E_i R_{k-1}}{R_{k-1}} = F_i^{(k)} \cdot \frac{E_i R_k}{R_k}$.
  (iii) Let p be an irreducible factor of $F_i^{(k)}$. By construction, p is a shift of some irreducible factor q of $F_i^{(k-1)}$ or $F_k^{(k-1)}$. By the induction hypothesis, for each pair of indices u, v, $1 \leq u < v \leq k-1$, there are $a, b \in \mathbb{Z}$, not both zero, such that $E_u^a E_v^b q = q$. As p is a shift of q, $E_u^a E_v^b p = p$ as well. Now let $1 \leq u < k$. By Theorem 9 applied to $F_k^{(k-1)}$ as a rational function of $x = x_u$, $y = x_k$, and considering all the other x_i as parameters, there are $a, b \in \mathbb{Z}$, not both zero, such that $E_u^a E_k^b p = p$. This shows that p is pairwise shift-invariant as a polynomial in $x_1, \ldots, x_k$.

This finishes the proof of our claim. As K is algebraically closed, it follows from Corollary 1 that each irreducible factor of $F_1^{(d)}$ is an integer-linear polynomial in $x_1, \ldots, x_d$, hence the claim at $k = d$ implies the correctness of Algorithm Multi-RNF and thus the assertion of the theorem. □

6 The structure of hypergeometric terms

Definition 11 (Z-term) *A hypergeometric term $T(\boldsymbol{n})$ is a* Z-term *if its certificates F_i in* (9) *factor into integer-linear factors, for $i = 1, 2, \ldots, d$.*

Theorem 11 *Let $T(\boldsymbol{n})$ be a hypergeometric term over an algebraically closed field K. Then there is a rational function $R \in K(\boldsymbol{x})$ and a Z-term $T'(\boldsymbol{n})$ such that $T =_a RT'$.*

Proof: Let $F_i \in K(\boldsymbol{x})$, $i = 1, \ldots, d$, be such that $E_i T =_a F_i T$, $i = 1, \ldots, d$, and let $R, F_1', \ldots, F_d' \in K(\boldsymbol{x})$ be the rational functions associated with $F_1, \ldots, F_d$ by Theorem 10. Take any hypergeometric term T' such that

$$T' =_a \frac{T}{R}.$$

Then $T =_a RT'$, and

$$E_i T' =_a \frac{E_i T}{E_i R} =_a F_i \frac{R}{E_i R} \cdot \frac{T}{R} =_a F_i' T', \quad \text{for } 1 \leq i \leq d.$$

As $F_1', \ldots, F_d'$ factor into integer-linear factors, T' is a Z-term. □

Definition 12 (uniform term) *Let $a_1, a_2, \ldots, a_d$ be relatively prime integers. A Z-term $T(\boldsymbol{n})$ is* uniform of type $\boldsymbol{a} = (a_1, a_2, \ldots, a_d)$ *if there are univariate rational functions $F_i \in K(x)$ such that*

$$E_i T(\boldsymbol{n}) =_a F_i(\boldsymbol{a}^T \boldsymbol{n}) T(\boldsymbol{n}) \tag{23}$$

for $1 \leq i \leq d$.

Proposition 9 *For every integer vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_d)$ there is an integer matrix $\boldsymbol{A} \in \mathbb{Z}^{d \times d}$ such that the first row of $\boldsymbol{A}$ is $\boldsymbol{a}$, and* $\det \boldsymbol{A} = \gcd(a_1, a_2, \ldots, a_d)$.

Proof: By induction on d.

- $d = 1$: This is clear, assuming $\gcd(a) = a$.
- $d > 1$: Write $d = \gcd(a_1, a_2, \ldots, a_d)$ and $d' = \gcd(a_1, a_2, \ldots, a_{d-1})$. There are $u, v \in \mathbb{Z}$ such that $ud' - va_d = \gcd(d', a_d) = d$. By the induction hypothesis there is a matrix $\boldsymbol{A}' \in \mathbb{Z}^{(d-1)\times(d-1)}$ whose first row equals $(a_1, a_2, \ldots, a_{d-1})$ while $\det \boldsymbol{A}' = \gcd(a_1, a_2, \ldots, a_{d-1})$. Let

$$\boldsymbol{A} \quad = \quad \left[\begin{array}{c|c} & a_d \\ \cline{2-2} \boldsymbol{A}' & 0 \\ \hline \boldsymbol{a}' & u \end{array}\right]$$

where $\boldsymbol{a}' = (v/d')(a_1, a_2, \ldots, a_{d-1})$. Then the first row of $\boldsymbol{A}$ is $\boldsymbol{a}$, and

$$\det \boldsymbol{A} = u \det \boldsymbol{A}' + (-1)^{d+1} a_d (v/d')(-1)^{d-2} \det \boldsymbol{A}' = ud' - va_d = d.$$

□

Theorem 12 *If K is algebraically closed, any uniform term $T(\boldsymbol{n})$ is conjugate to a nontrivial factorial term.*

Proof: Let $T(\boldsymbol{n})$ be a uniform term of type $\boldsymbol{a}$. By Proposition 9, there is a unimodular integer matrix $\boldsymbol{A} \in \mathbb{Z}^{d\times d}$ whose first row is $\boldsymbol{a}$. Using (23) repeatedly, we find that for fixed $u \in \mathbb{N}$,

$$E_i^u T(\boldsymbol{n}) \quad =_a \quad F_{i,u}(\boldsymbol{a}^T\boldsymbol{n}) T(\boldsymbol{n}), \tag{24}$$
$$E_i^{-u} T(\boldsymbol{n}) \quad =_a \quad F_{i,-u}(\boldsymbol{a}^T\boldsymbol{n}) T(\boldsymbol{n}), \tag{25}$$

where $F_{i,u}(\boldsymbol{a}^T\boldsymbol{n}) = \prod_{j=0}^{u-1} E_i^j F_i(\boldsymbol{a}^T\boldsymbol{n})$ and $F_{i,-u}(\boldsymbol{a}^T\boldsymbol{n}) = 1/\prod_{j=1}^{u} E_i^{-j} F_i(\boldsymbol{a}^T\boldsymbol{n})$. Let

$$T'(\boldsymbol{n}) = T(\boldsymbol{A}^{-1}\boldsymbol{n}) \tag{26}$$

be a K-valued function defined on the integer cone $\boldsymbol{A}^{-1}\boldsymbol{n} \geq \boldsymbol{0}$. Write $\boldsymbol{n}' = \boldsymbol{A}^{-1}\boldsymbol{n}$. Then

$$\begin{aligned} E_i T'(\boldsymbol{n}) \quad &= \quad E_i T(\boldsymbol{A}^{-1}\boldsymbol{n}) \\ &=_a \quad T(\boldsymbol{A}^{-1}\boldsymbol{n} + \tilde{\boldsymbol{a}}^{(i)}) \; = \; T(\boldsymbol{n}' + \tilde{\boldsymbol{a}}^{(i)}) \end{aligned} \tag{27}$$

where $\tilde{\boldsymbol{a}}^{(i)}$ is the i-th column of $\boldsymbol{A}^{-1}$. As $\boldsymbol{a}^T\boldsymbol{n}' = \boldsymbol{a}^T\boldsymbol{A}^{-1}\boldsymbol{n} = n_1$, we obtain from (27) using (24), (25) that

$$E_i T'(\boldsymbol{n}) = f_i(n_1) T'(\boldsymbol{n})$$

for $1 \leq i \leq d$, where $f_i(n_1) = \prod_{j=1}^{d} F_{j,\tilde{a}_j^{(i)}}(n_1 + s_j)$ and $s_j = \sum_{k=j+1}^{d} a_k \tilde{a}_k^{(i)}$.

From the compatibility condition (10) applied to F_i and F_1 it follows that $f_i(n_1)$ is constant for $2 \leq i \leq d$. Factoring $f_1(x)$ over K we can write

$$\begin{aligned} f_1(x) \quad &= \quad v_1 \prod_{i=1}^{p} (x + \alpha_i) \prod_{i=p+1}^{p+q} (x + \alpha_i)^{-1}, \\ f_i(x) \quad &= \quad v_i, \end{aligned}$$

where $v_i \in K$, $p, q \in \mathbb{N}$, and $\alpha_i \in K$. Then the sequence

$$H'(\boldsymbol{n}) \;=\; \boldsymbol{v}^{\boldsymbol{n}} \prod_{i=1}^{p} (\alpha_i)_{n_1} \prod_{i=p+1}^{p+q} (\alpha_i + n_1)_{-n_1} \tag{28}$$

(defined on $\boldsymbol{A}^{-1}\boldsymbol{n} \geq \boldsymbol{0}$) satisfies the same hypergeometric recurrences as $T'(\boldsymbol{n})$. Using the inverse substitution of (26), we see that $T(\boldsymbol{n}) = T'(\boldsymbol{A}\boldsymbol{n})$ is conjugate to $H(\boldsymbol{n}) = H'(\boldsymbol{A}\boldsymbol{n})$. But

$$H(\boldsymbol{n}) \;=\; \boldsymbol{u}^{\boldsymbol{n}} \prod_{i=1}^{p} (\alpha_i)_{\boldsymbol{a}^T\boldsymbol{n}} \prod_{i=p+1}^{p+q} (\alpha_i + \boldsymbol{a}^T\boldsymbol{n})_{-\boldsymbol{a}^T\boldsymbol{n}}$$

(where $u_i = \boldsymbol{v}^{\tilde{\boldsymbol{a}}^{(i)}}$) is a factorial term. In (28), when $\alpha_i \in \mathbb{Z}$ we are free to replace $(\alpha_i)_{n_1}$ by $(1)_{\alpha_i+n_1-1}$ or by $(0)_{\alpha_i+n_1}$. We can do likewise with its pseudoinverse $(\alpha_i + n_1)_{-n_1}$. By a judicious choice between these alternatives we can always make $H(\boldsymbol{n})$ nontrivial. □

Corollary 3 *If K is algebraically closed, any Z-term $T(\boldsymbol{n})$ is conjugate to a nontrivial factorial term.*

Proof: W.l.g. assume that T is nontrivial. Let $T(\boldsymbol{n})$ be a Z-term such that $E_iT(\boldsymbol{n}) =_a F_i(\boldsymbol{n})T(\boldsymbol{n})$. Let $F_i(\boldsymbol{x}) = F_1^{(i)}(\boldsymbol{x})F_2^{(i)}(\boldsymbol{x})\cdots F_m^{(i)}(\boldsymbol{x})$ be a factorization of F_i such that $F_k^{(i)}F_k^{(j)}$ is a uniform rational function for all $1 \leq k \leq m$ and $1 \leq i \leq j \leq d$, while $F_k^{(i)}F_l^{(i)}$ where $1 \leq i \leq d$ and $1 \leq k < l \leq m$ is not (unless one of $F_k^{(i)}$, $F_l^{(i)}$ is constant). It follows from the unique factorization of polynomials in $K[\boldsymbol{x}]$ that $F_k^{(1)}, F_k^{(2)}, \ldots, F_k^{(d)}$ are compatible for each k. It can be shown that there are uniform terms $T_k(\boldsymbol{n})$ satisfying $E_iT_k(\boldsymbol{n}) =_a F_k^{(i)}(\boldsymbol{n})T_k(\boldsymbol{n})$. Then $T(\boldsymbol{n}) \simeq \prod_{k=1}^m T_k(\boldsymbol{n})$. As in the proof of Theorem 12, we can achieve that $T(\boldsymbol{n})$ will be nontrivial. Since products of factorial terms are factorial, the claim follows from Theorem 12. □

Corollary 4 *(Ore-Sato Theorem) If K is algebraically closed, any hypergeometric term $T(\boldsymbol{n})$ is conjugate to $R(\boldsymbol{n})T'(\boldsymbol{n})$ where $R \in K(\boldsymbol{x}) \setminus \{0\}$ is a rational function and $T'(\boldsymbol{n})$ is a nontrivial factorial term.*

Proof: W.l.g. assume that T is nontrivial. By Theorem 11, $T =_a RT''$ where $R \in K(\boldsymbol{x})$ and T'' is a Z-term. By Proposition 3, this implies that $T \simeq RT''$. By Corollary 3, $T'' \simeq T'$ where T' is a nontrivial factorial term. Then $RT'' \simeq RT'$. As $RT'' \neq_a 0$, it follows by Proposition 5 that $T \simeq RT'$. □

7 Holonomic hypergeometric terms

Theorem 13 *Assume that K is algebraically closed. If a rational sequence $R(\boldsymbol{n})$ is conjugate to a nontrivial holonomic hypergeometric term $T(\boldsymbol{n})$ then the denominator of R factors into integer-linear factors.*

Proof: We prove this by induction on d.

- $d = 1$: Every univariate polynomial over an algebraically closed field factors into integer-linear factors.
- $d > 1$: Write $R = P/Q$ where $P, Q \in K[\boldsymbol{x}]$ and $P \perp Q$. Let $Q = VW$ where $V, W \in K[\boldsymbol{x}]$ and V is irreducible. We wish to show that V is integer-linear. Denote $T' = TW$ and $R' = RW = P/V$. Then T' is holonomic hypergeometric, $T' \neq_a 0$, and $T' \simeq R'$. Hence there are $F_i \in K(\boldsymbol{x})$ such that both T' and R' satisfy (9). By Proposition 1, $F_i = (E_iR')/R'$. Thus for $1 \leq i \leq d$,

$$E_iT'(\boldsymbol{n}) \quad =_a \quad \frac{E_iR'(\boldsymbol{n})}{R'(\boldsymbol{n})}T'(\boldsymbol{n}). \tag{29}$$

  We claim that

$$E_1^{-a_1}\cdots E_d^{-a_d}T'(\boldsymbol{n}) =_a \frac{E_1^{-a_1}\cdots E_d^{-a_d}R'(\boldsymbol{n})}{R'(\boldsymbol{n})}T'(\boldsymbol{n}) \tag{30}$$

  for all $a_1, \ldots, a_d \geq 0$. The proof is by induction on $a_1+\cdots+a_d$. If $a_1+\cdots+a_d = 0$ then $a_1 = \cdots = a_d = 0$ and the claim is trivial. If $a_1 + \cdots + a_d > 0$ assume w.l.g. that $a_1 > 0$. Then

$$E_1^{-a_1}\cdots E_d^{-a_d}T'(\boldsymbol{n}) \quad =_a \quad \frac{E_1^{-a_1}\cdots E_d^{-a_d}R'(\boldsymbol{n})}{E_1^{-(a_1-1)}E_2^{-a_2}\cdots E_d^{-a_d}R'(\boldsymbol{n})}E_1^{-(a_1-1)}E_2^{-a_2}\cdots E_d^{-a_d}T'(\boldsymbol{n})$$

$$
\begin{aligned}
&=_a \quad \frac{E_1^{-a_1}\cdots E_d^{-a_d}R'(\boldsymbol{n})}{E_1^{-(a_1-1)}E_2^{-a_2}\cdots E_d^{-a_d}R'(\boldsymbol{n})}\,\frac{E_1^{-(a_1-1)}E_2^{-a_2}\cdots E_d^{-a_d}R'(\boldsymbol{n})}{R'(\boldsymbol{n})}T'(\boldsymbol{n}) \\
&=_a \quad \frac{E_1^{-a_1}\cdots E_d^{-a_d}R'(\boldsymbol{n})}{R'(\boldsymbol{n})}T'(\boldsymbol{n}),
\end{aligned}
$$

using (29) and the induction hypothesis.

As T' is holonomic, Theorem 1(i) implies that there is an $s \in \mathbb{N}$, a nonempty set $H_1 \subseteq \{0,\ldots,s\}^d$, and univariate polynomials $p_{\boldsymbol{h},1} \in K[x] \setminus \{0\}$ for each $\boldsymbol{h} \in H_1$ such that

$$\sum_{\boldsymbol{h}\in H_1} p_{\boldsymbol{h},1}(n_1)\,T'(\boldsymbol{n}-\boldsymbol{h}) \quad = \quad 0$$

for all $\boldsymbol{n} \geq \boldsymbol{s}$. Using (30) we see that there is an algebraic set A such that

$$\sum_{\boldsymbol{h}\in H_1} p_{\boldsymbol{h},1}(n_1)\,R'(\boldsymbol{n}-\boldsymbol{h}) \quad = \quad 0$$

on $\operatorname{supp} T' \setminus A$. As this is non-algebraic, Proposition 1 and $R' = P/V$ imply that

$$\sum_{\boldsymbol{h}\in H_1} p_{\boldsymbol{h},1}(n_1)\,\frac{P(\boldsymbol{n}-\boldsymbol{h})}{V(\boldsymbol{n}-\boldsymbol{h})} = 0. \tag{31}$$

Pick $\boldsymbol{h}_0 \in H_1$ and clear denominators in (31). The factor $V(\boldsymbol{n}-\boldsymbol{h}_0)$ appears explicitly in every term except the one with $\boldsymbol{h} = \boldsymbol{h}_0$. Hence $V(\boldsymbol{x}-\boldsymbol{h}_0)$ which is irreducible divides

$$p_{\boldsymbol{h}_0,1}(x_1)\,P(\boldsymbol{x}-\boldsymbol{h}_0)\prod_{\substack{\boldsymbol{h}\in H_1\\ \boldsymbol{h}\neq\boldsymbol{h}_0}} V(\boldsymbol{x}-\boldsymbol{h}).$$

If it divides $p_{\boldsymbol{h}_0,1}(x_1)$ then $V(\boldsymbol{x}) \in K[x_1]$. As it is irreducible, V is integer-linear. Next, $V(\boldsymbol{x}-\boldsymbol{h}_0)$ cannot divide $P(\boldsymbol{x}-\boldsymbol{h}_0)$ because $V \mid Q$ and $P \perp Q$, hence it divides one of $V(\boldsymbol{x}-\boldsymbol{h})$ where $\boldsymbol{h} \neq \boldsymbol{h}_0$. But then $V(\boldsymbol{x}) = V(\boldsymbol{x}+\boldsymbol{a})$ where $\boldsymbol{a} = \boldsymbol{h}_0 - \boldsymbol{h} \neq \boldsymbol{0}$. W.l.g. assume that $a_d \neq 0$. Then by Lemma 3, there is a $(d-1)$-variate polynomial $h \in K[x_1,\ldots,x_{d-1}]$ such that

$$V(\boldsymbol{x}) = h(x_1 - \frac{a_1}{a_d}x_d, \ldots, x_{d-1} - \frac{a_{d-1}}{a_d}x_d). \tag{32}$$

Define $R''(x_1,\ldots,x_{d-1}) := R'(x_1,\ldots,x_{d-1},0)$ and $T''(n_1,\ldots,n_{d-1}) := T'(n_1,\ldots,n_{d-1},0)$. Then R'',T'' are hypergeometric terms and $R'' \simeq T''$. By Theorem 1(ii), T'' is holonomic. Therefore by the induction hypothesis, the denominator $V(x_1,\ldots,x_{d-1},0)$ of R'' factors into integer-linear factors. From (32) we find

$$V(x_1,\ldots,x_{d-1},0) = h(x_1,\ldots,x_{d-1}),$$

hence

$$h(x_1,\ldots,x_{d-1}) = \prod_{i=1}^{r}\left(u_i\sum_{j=1}^{d-1}c_{ij}x_j + v_i\right)$$

for some $r \in \mathbb{N}$, $u_i, v_i \in K$, and $c_{ij} \in \mathbb{Z}$. Now it follows from (32) that

$$V(\boldsymbol{x}) = \prod_{i=1}^{r}\left(u_i\sum_{j=1}^{d-1}c_{ij}\left(x_j - \frac{a_j}{a_d}x_d\right) + v_i\right) = \frac{1}{a_d^r}\prod_{i=1}^{r}\left(u_i\sum_{j=1}^{d-1}c_{ij}\left(a_d x_j - a_j x_d\right) + a_d v_i\right).$$

But V is irreducible, so $r = 1$ and V is integer-linear.

□

Example 7 In the literature, rational sequences such as $1/(n^2+k^2)$ [19, p. 586], $1/(n^2+k)$ [10, p. 358] and $1/(nk+1)$ [7, Exer. 5.107] are shown to be nonholonomic by various *ad hoc* arguments. Using Theorem 13, nonholonomicity of these sequences follows from the fact that their denominators do not factor into integer-linear factors. Likewise, the trivariate rational sequence $T(n,m,k) = 1/((n-m)(k-m)+1)$ is not holonomic by Theorem 13. Note that $T(n,m,k)$ satisfies condition (i) of Theorem 1 with the constant-coefficient recurrence $T(n,m,k) - T(n-1,m-1,k-1) = 0$ valid for $n,m,k \geq s = 1$, but condition (ii) is not satisfied as the bivariate sequence $T(n,0,k) = 1/(nk+1)$ is not holonomic. □

Lemma 7 *If $Q \in K[\boldsymbol{x}]\setminus\{0\}$ factors into integer-linear factors then the rational sequence $1/Q(\boldsymbol{n})$ is conjugate to a nontrivial factorial term.*

Proof: Write $Q(\boldsymbol{n}) = u\prod_{i=1}^{p}\left(\boldsymbol{a}_i^T\boldsymbol{n}+\alpha_i\right)$ where $u \in K\setminus\{0\}$, $p \in \mathbb{N}$, $\alpha_i \in K$, and $\boldsymbol{a}_i \in \mathbb{Z}^d$ for $1 \leq i \leq p$. W.l.g. assume that each $\boldsymbol{a}_i$ has at least one positive component. Then

$$\frac{1}{\alpha_i+\boldsymbol{a}_i^T\boldsymbol{n}} \simeq \begin{cases} (1)_{\boldsymbol{a}_i^T\boldsymbol{n}+\alpha_i-1}(\boldsymbol{a}_i^T\boldsymbol{n}+\alpha_i+1)_{-(\boldsymbol{a}_i^T\boldsymbol{n}+\alpha_i)}, & \text{if } \alpha_i\in\mathbb{Z} \text{ and } \boldsymbol{a}_i\geq\boldsymbol{0},\\ (\alpha_i)_{\boldsymbol{a}_i^T\boldsymbol{n}}(\boldsymbol{a}_i^T\boldsymbol{n}+\alpha_i+1)_{-(\boldsymbol{a}_i^T\boldsymbol{n}+1)}, & \text{otherwise}\end{cases}$$

where the right-hand side is conjugate to a nontrivial factorial term. It follows that $1/Q(\boldsymbol{n})$ is conjugate to a nontrivial factorial term as well. □

Theorem 14 *If K is algebraically closed, any holonomic hypergeometric term $T(\boldsymbol{n})$ is conjugate to a nontrivial proper term.*

Proof: W.l.g. assume that T is nontrivial. By Corollary 4, $T \simeq RT_1$ where $R \in K(x,y)\setminus\{0\}$ and T_1 is a nontrivial factorial term. By changing all rising factorials in T_1 into their nonvanishing counterparts, we obtain a conjugate holonomic sequence T_2 which is nowhere zero. Then $T \simeq RT_2$ and $1/T_2$ is also holonomic. So $R \simeq T/T_2$. Note that T/T_2 is nontrivial, and holonomic by Theorem 2. Write $R = P/Q$ where $P,Q \in K[x,y]$ and $P \perp Q$. By Theorem 13, Q factors into integer-linear factors. By Lemma 7, $1/Q$ is conjugate to a nontrivial proper term T_3. Thus $T \simeq PT_2T_3$ which is a nontrivial proper term. □

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