# The Fourth Conference "Computer Algebra" in Moscow

Sergei A. Abramov [1], Anna Ryabenko [1],
Leonid A. Sevastianov [2,3], Yvette Zonn [1], Min Wu [4]

[1] Dorodnicyn Computing Centre, Federal Research Center "Computer Science and Control"
Russian Academy of Sciences, Vavilova 40, Moscow 119333, Russia
[2] Peoples' Friendship University of Russia
Miklukho-Maklaya 6, Moscow 117198, Russia
[3] Bogoliubov Laboratory of Theoretical Physics Joint Institute for Nuclear Research
Joliot-Curie 6, Dubna, Moscow 141980, Russia
[4] Shanghai Laboratory of Trustworthy Computing
East China Normal University, Shanghai 200062, China

sergeyabramov@mail.ru, anna.ryabenko@gmail.com
leonid.sevast@gmail.com, zonn@ccas.ru, mwu@sei.ecnu.edu.cn

The International Conference "Computer Algebra"was held online via Zoom on June 28-29, 2021 and the conference web-site is http://www.ccas.ru/ca/conference. Co-organized by the Dorodnicyn Computing Center of Federal Research Center "Computer Science and Control" of Russian Academy of Science and the Peoples' Friendship University of Russia, the conference was devoted to inspiring discussions on computer algebra and related topics. Researchers from different countries presented talks on their latest research work.

This is the fourth edition of the conference, and the previous three were in 2016, 2017 and 2019, respectively.

## 1 Invited Talk Abstracts

### Level Lines of a Polynomial in the Plane

*A. D. Bruno (Keldysh Institute of Applied Mathematics of RAS, Russia), A. B. Batkhin (1. Keldysh Institute of Applied Mathematics of RAS, Russia; 2. Moscow Institute of Physics and Technology, Russia)*

We propose a method for computing the position of all level lines of a real polynomial in the real plane. To do this, it is necessary to compute its critical points and critical lines (there are finite number of them), and then its critical values of the polynomial. Now finite number of critical levels and one representative of noncritical level corresponding to a value between two neighboring critical ones enough to compute. Software for these computations is discussed. A nontrivial example is considered.

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### What's New in Maple 2021

*J. Gerhard (Maplesoft, Canada)*

We will give an overview over the new features of Maple 2021, including limits and asymptotic expansions, automatic plotting domain and range selection, a new Student:-ODEs package, approximate polynomial algebra, and improved LaTeX export.

## 2 Contributed Talk Abstracts

### On Infinite Sequences and Difference Operators

*S. A. Abramov (1. Dorodnicyn Computing Center, FRC CSC of RAS, Russia; 2. Faculty of Computational Mathematics and Cybernetics, Moscow State University, Russia), M. Barkatou (Université de Limoges, CNRS, France), M. Petkovšek (University of Ljubljana, Slovenia)*

Some properties of linear difference operators whose coefficients have the form of infinite two-sided sequences over a field of characteristic zero are considered. In particular, it is found that such operators are deprived of some properties that are natural for differential operators over differential fields. In addition, we discuss decidability of certain problems arising in connection with the algorithmic representation of infinite sequences.

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### Quadratization of ODE Systems

*F. Alauddin (Trinity School, New York, USA), A. Bychkov (Higher School of Economics, Moscow, Russia), G. Pogudin (LIX, CNRS, École Polytechnique, Institute Polytechnique de Paris, Palaiseau, France)*

Quadratization, that is a transformation of an ODE system with polynomial right-hand side into an ODE system with at most quadratic right-hand side via the introduction of new variables, has been recently used for model order reduction, synthesis of chemical reaction networks, and numeric algorithms. We will discuss some recent results on optimal (i.e. with the smallest number of variables) quadratizations: a practical algorithm for finding optimal monomial quadratizations and some results on arbitrary quadratizations of scalar ODEs. We will conclude with a list of open problems.

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### Asymptotic Expansions of Solutions to the Hierarchy of the Fourth Painlevé Equation

*V. I. Anoshin (National Research University "Higher School of Economics", Russia), A. D. Beketova (National Research University "Higher School of Economics", Russia), A. V. Parusnikova (National Research University "Higher School of Economics", Russia)*

In this paper, we study the asymptotic expansions of the solutions of the hierarchies of the fourth Painlevé equation using power geometry methods. To find these expansions, we need to build a Newton polygon and find solutions to the truncated equations using the rules given below. Hierarchies of Painlevé equations are used in physics, geometry, and other fields.

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### Algorithms for Solving a Polynomial Equation in One or Two Variables

*A. B. Batkhin (1. Keldysh Institute of Applied Mathematics of RAS, Russia; 2. Moscow Institute of Physics and Technology, Russia), A. D. Bruno (Keldysh Institute of Applied Mathematics of RAS, Russia)*

Here we demonstrate two new methods of solution of polynomial equations, based on constructing a convex polygon, and provide description of corresponding software. The first method allows to find approximate roots of a polynomial by means of the Hadamard polygon. The second one allows to compute branches of an algebraic curve near its singular point and near infinity by means of the Newton polygon and to draw sketches of real algebraic curves in the plane. Computer algebra algorithms are specified, which allow to investigate any complex cases.

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**Role of Monomial Orderings in Efficient Gröbner Basis Computation in Parameter Identifiability Problem**

*M. Bessonov (Department of Mathematics, CUNY NYC College of Technology, New York, USA), I. Ilmer (Ph.D. Program in Computer Science, CUNY Graduate Center, New York, USA), T. Konstantinova (Department of Mathematics, CUNY Queens College, New York, USA), A. Ovchinnikov (1. Ph.D. Program in Computer Science, CUNY Graduate Center, New York, USA; 2. Department of Mathematics, CUNY Queens College, New York, USA; 3. Ph.D. Program in Mathematics, CUNY Graduate Center, New York, USA), G. Pogudin (LIX, CNRS, École Polytechnique, Institute Polytechnique de Paris, France)*

We present empirical runtime and memory use improvements for computing Gröbner bases of ideals generated by polynomials that appear in solving the parameter identifiability problems for ODE models by the SIAN algorithm. Such differential-algebraic systems may also occur some other in prolongation-based algorithms and efficiently computing Gröbner bases can be critical. The main speed-up is achieved by automatically choosing problem-specific monomial orderings.

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**Amoebas of Multivariate Hypergeometric Polynomials**

*D. V. Bogdanov (Moscow Center of Technological Modernization of Education, Russia), T. M. Sadykov (Plekhanov Russian University, 125993, Moscow, Russia)*

With any integer convex polytope $P \subset \mathbb{R}^n$ we associate a multivariate hypergeometric polynomial whose set of exponents is $\mathbb{Z}^n \cap P$. This polynomial is defined uniquely up to a constant multiple and satisfies a holonomic system of partial differential equations of Horn's type. We prove that under certain nondegeneracy conditions the zero locus of any such polynomial is optimal in the sense that the topology of its amoeba is as complex as it could possibly be. Using this, we derive optimal properties of several classical families of multivariate hypergeometric polynomials.

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**Symbolic Integration of Differential Forms**

*S. Chen (KLMM, Academy of Mathematics and Systems Science, CAS, China)*

Symbolic integration of differential forms is a higher dimensional analogue of symbolic integration of elementary functions. This note will present a Liouville-style theorem for integration of closed differential forms with rational-function coefficients.

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**A Maple Implementation of the Finite Element Method for Solving Metastable State Problems for Systems of Second-Order Ordinary Differential Equations**

*G. Chuluunbaatar (1. Joint Institute for Nuclear Research, Dubna, Russia; 2. Peoples' Friendship University of Russia (RUDN University), Moscow, Russia), A. A. Gusev (1. Joint Institute for Nuclear Research, Dubna, Russia; 2. Dubna State University, Dubna, Russia), V. L. Derbov (N. G. Chernyshevsky Saratov National Research State University, Saratov, Russia), S. I. Vinitsky (1. Joint Institute for Nuclear Research, Dubna, Russia; 2. Peoples' Friendship University of Russia (RUDN University), Moscow, Russia), O. Chuluunbaatar (1. Joint Institute for Nuclear Research, Dubna, Russia; 2. Institute of Mathematics and Digital Technology, Mongolian Academy of Sciences, Ulaanbaatar, Mongolia)*

We present a new algorithm for systems of second-order ordinary differential equations to calculate metastable states with complex eigenvalues of energy or to find bound states with homogeneous boundary conditions depending on a spectral parameter. The boundary-value problems is discretized by means of the FEM using the Hermite interpolation polynomials with arbitrary multiplicity of the nodes, which preserves the continuity of derivatives of the desired solutions. For the solution of the relevant algebraic problems the Newton iteration scheme is implemented.

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**From Prony's Exponential Fitting to Sub-Nyquist Signal Processing Using Computer Algebra Techniques**

*A. Cuyt (1. University of Antwerpen, Belgium; 2. Shenzhen University, Shenzhen, China), W.-S. Lee (1. University of Antwerpen, Belgium; 2. University of Stirling, UK)*

The classical Prony's exponential fitting method is closely related to sparse polynomial interpolation. Using techniques from computer algebra, a sub-Nyquist version of Prony's method has been developed, which offers new potentials in signal processing, including an application in antenna design.

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**The Integrability Condition in the Normal Form Method**

*V. F. Edneral (1. Skobeltsyn Institute of Nuclear Physics of Lomonosov Moscow State University, Russia; 2. Peoples' Friendship University of Russia (RUDN University), Russia)*

The purpose of this report is to demonstrate the search for integrability conditions by the normal form method. We have chosen an equation of the Liénard-type as the object of the demonstration. We presented the equation as a dynamical system and parameterized it. We constructed polynomial equations in the system parameters which should be satisfied for the integrable cases.

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**Deciding Cryptographically Important Properties of Finite Quasigroups**

*A. V. Galatenko (Lomonosov Moscow State University, Russia), A. E. Pankratiev (Lomonosov Moscow State University, Russia), V.ZM. Staroverov (Lomonosov Moscow State University, Russia)*

Finite quasigroups are a promising structure for design of various cryptographic primitives. Due to security reasons it makes sense to select quasigroups with a number of properties, such as polynomial completeness or at least non-affinity and poor quasigroup structure. We describe algorithms that decide these properties, analyse algorithm complexity and outline the results of practical efficiency analysis.

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**Symbolic Implementation of Multivector Algebra in Julia Language**

*M. N. Gevorkyan (Peoples' Friendship University of Russia (RUDN University), Russian Federation), D. S. Kulyabov (1. Peoples' Friendship University of Russia (RUDN University), Russian Federation; 2. Joint Institute for Nuclear Research, Russian Federation), A. V. Korolkova (Peoples' Friendship University of Russia (RUDN University), Russian Federation), A. V. Demidova (Peoples' Friendship University of Russia (RUDN University), Russian Federation), T. R. Velieva (1. Peoples' Friendship University of Russia (RUDN University), Russian Federation; 2. Plekhanov Russian University of Economics, Russian Federation)*

In this work, we will briefly present the main facts from the theory of polyvectors and multivectors, generalizing the known mathematical constructions. In the report, we will consider the solution of a number of geometric examples using the Grassmann.jl library. This is implementation of the multivector algebra in the Julia language. It mixes computer algebra calculations with numerical calculations for better performance.

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**Application of Symbolic Computations for Investigation of the Equilibria of the System of Connected Bodies Moving on a Circular Orbit**

*S. A. Gutnik (1. Moscow State Institute of International Relations (MGIMO University), Russia; 2. Moscow Institute of Physics and Technology, Russia), V. A. Sarychev (Keldysh Institute of Applied Mathematics RAS, Russia)*

Computer algebra methods were used to solve a system of 12 algebraic equations that determines the equilibrium orientations for a system of two bodies, connected by a spherical hinge, that moves on a circular orbit around the Earth. To determine the equilibria the algebraic system was decomposed using algorithms for Gröbner basis construction. Evolution of the conditions for equilibria existence in the dependence of the parameter of the problem was investigated. The effectiveness of the algorithms for Gröbner basis construction was analyzed depending on the number of parameters of the problem.

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**Revisiting Geometric Integrators in Mechanics**

*A. Hamdouni (La Rochelle University, France), V. Salnikov (1. CNRS – National Center for Scientific Research, France; 2. La Rochelle University, France)*

We address the question of efficient construction of geometric integrators – numerical methods preserving some internal geometric structure of the system of equations. Such methods are of particular importance for modelling and simulation of mechanical systems, where these structures permit to control the conservation of physically relevant quantities. We focus our attention on the so called generalized geometry, for which we present an approach to design higher order Runge–Kutta style numerical methods.

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**Automatic Confirmation of Exhaustive Use of Information on a Given Equation**

*D. E. Khmelnov (Dorodnicyn Computing Center, Federal Research Center, Computer Science and Control of RAS, Russia), A. A. Ryabenko (Dorodnicyn Computing Center, Federal Research Center, Computer Science and Control of RAS, Russia), S. A. Abramov (1. Dorodnicyn Computing Center, Federal Research Center, Computer Science and Control of RAS, Russia; 2. Faculty of Computational Mathematics and Cybernetics, Moscow State University, Russia)*

Algorithms were previously proposed that allow one to find truncated Laurent solutions to linear differential equations with coefficients in the form of truncated formal power series. Below are suggested some automatic means of confirming the impossibility of obtaining a larger number of terms of such solutions without some additional information on a given equation. The confirmation has the form of a counterexample to the assumption about the possibility of obtaining some additional terms of the solution.

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**On Semantics of Names in Formulas and References in Object-Oriented Languages**

*A. V. Klimov (Keldysh Institute of Applied Mathematics of RAS, Russia)*

The long-standing problem of a formal semantics of free and bounded names in mathematical formulas, as well as the semantics of references in object-oriented languages, is discussed. We review the history of the topic and related works including recent contributions to the theory of names. An outline of a constructive denotational semantics of a functional language with a new/fresh name generator is presented.

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**Subsystems in Finite Quantum Mechanics**

*V. V. Kornyak (Laboratory of Information Technologies, Joint Institute for Nuclear Research, Dubna, Russia)*

Any Hilbert space with composite dimension can be represented as a tensor product of smaller Hilbert spaces. This allows to decompose a quantum system into subsystems. We propose a model based on finite quantum mechanics for the constructive study of such decompositions.

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**About Big Matrix Inversion**

*G. Malaschonok (National University of Kyiv-Mohyla Academy, Ukraine), I. Tchaikovsky (Lviv Polytechnic National University, Ukraine)*

We consider three types of inverse matrices: inverse, pseudoinverse, and generalized inverse. And we discuss algorithms, which are applicable for commutative domains. The research is motivated by modern problems of supercomputing.

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**Solving the Hyperbolic Equation in Elementary Functions**

*M. D. Malykh (Peoples' Friendship University of Russia, Russia), M. K. Yu (Lomonosov Moscow State University, Skobeltsyn Institute of Nuclear Physics (SINP MSU))*

The classical problem about oscillation of music strings is considered. Its solution can be represented as Fourier series, and also as elementary functions. Such a representation is valid only over the field of real numbers and doesn't belong Liouville theory. This circumstance lids to several difficulties at work with the solutions in CAS. A typical example from CAS Maple is considered.

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**On a Machine-Checked Proof for an Optimized Method to Multiply Polynomials**

*S. D. Meshveliani (A. K. Ailamazyan Program Systems Institute of RAS, Russia)*

The Karatsuba method to multiply univariate polynomials is simple to program and has essentially smaller run-time cost order than the simplest method of "multiply each monomial by each and sum". Here it is shortly described the design for a provable program in the Agda language for this Karatsuba method. In this program, computing the polynomial product is expressed in the same function together with the machine-checked proof for that the method is equivalent to the simplest multiplication. This is a part of the library DoCon-A of provable programs for algebra developed by the author.

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**Symbolic Computation in Studying the Stability of Periodic Motion of the Swinging Atwood Machine**

*A.N. Prokopenya (Warsaw University of Life Sciences – SGGW, Poland)*

The swinging Atwood machine under consideration is a conservative Hamiltonian system with two degrees of freedom. Although its equations of motion are essentially nonlinear one can construct a periodic solution in the form of power series in a small parameter if the masses difference is sufficiently small. We have investigated perturbations of periodic solution and proved its stability in linear approximation. All tedious symbolic computations are performed with the aid of the computer algebra system Wolfram Mathematica.

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**Quadrature Formula for the Double Layer Potential**

*I. O. Reznichenko (1. Moscow State University, Russia; 2. Keldysh Institute of Applied Mathematics, Russia), P. A. Krutitskii (Keldysh Institute of Applied Mathematics, Russia)*

In this work, a quadrature formula for the double layer potential is derived in the case of the Helmholtz equation with continuous density given on a smooth closed or open surface. This quadrature formula gives higher computational accuracy than standard quadrature formula, which is confirmed by numerical tests. The advantage of the new quadrature formula is especially noticeable near the surface, where the standard quadrature formula diverges rapidly, while the new formula provides acceptable accuracy for points that are distant from the surface at distances comparable to the integration step and more.

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**A Plain Note on Binary Solutions to Large Systems of Linear Equations**

*A. V. Seliverstov (Institute for Information Transmission Problems of RAS (Kharkevich Institute), Russia)*

A generic-case algorithm is proposed to recognize systems of linear equations without any binary solution, when the number of equations is close to the number of unknowns. This problem corresponds to a well-known optimization problem, i.e., the multidimensional knapsack problem. In 1994 Nikolai Kuzyurin discovered an average-case polynomial-time optimization algorithm. His proof is based on binomial tail bounds. Contrariwise, our algebraic approach allows to specify the structure of the set of inconvenient inputs. For any fixed dimension, this set is included in the set of zeros of an explicit nonzero multivariate polynomial.

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**Refinements on Bounds for Polynomial Roots**

*D. Ştefănescu*[1] *(Department of Theoretical Physics and Mathematics, University of Bucharest, Romania)*

We discuss the efficiency of the computation of bounds for polynomial roots.

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**Hypergeometric Type Power Series**

*B. Teguia Tabuguia (Mathematics and Natural Sciences, University of Kassel, Germany), W. Koepf (Mathematics and Natural Sciences, University of Kassel, Germany)*

For three decades now, the second author proposed a symbolic general-purpose approach to compute formal power series. This was originally presented in three main steps that reduce the problem to solving a holonomic recurrence equation (RE) for the coefficients of the power series sought. However, for linear combinations of Laurent-Puiseux series having hypergeometric term coefficients, one needs to compute so-called $m$-fold hypergeometric term solutions of holonomic REs. We give an overview of a new algorithm, called **mfoldHyper**, that achieves this purpose and allows linearity in computing hypergeometric type power series.

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**High Accuracy Trigonometric Approximations of the Real Bessel Functions of the First Kind**

*M. Wu (Shanghai Laboratory of Trustworthy Computing, East China Normal University, Shanghai, China)*

We construct high accuracy trigonometric interpolants from equidistant evaluations of the Bessel functions $J_n(x)$ of the first kind and integer order. The trigonometric models are cosine or sine based depending on whether the Bessel function is even or odd. The main novelty lies in the fact that the frequencies in the trigonometric terms modelling $J_n(x)$ are also computed from the data in a Prony-type approach. Hence the interpolation problem is a nonlinear problem. Some existing compact trigonometric models for the Bessel functions $J_n(x)$ are hereby rediscovered and generalized.

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**Machine Learning for Bratu's Problem: Solution and Parameter Estimation**

*M. Youssef (Institute of Applied Analysis and Numerical Simulation, University of Stuttgart, Germany), R. Pulch (Institute of Mathematics and Computer Science, University of Greifswald, Germany)*

We investigate a parametric nonlinear Bratu equation. Our aim is to determine cheap and efficient approximations of the solution and the parameter values. We define the truth solution as the Poly-Sinc collocation solution. We arrange a set of samples including basis coefficients of the truth solution. The target is to approximate the mapping from the parameter domain to the basis coefficients and vice versa. We apply machine learning with artificial neural networks for these approximations. We present results of numerical computations for both the forward problems and inverse problems. The calculations include one-dimensional and higher-dimensional models.

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**On the Structure of Polynomial Solutions of Gosper's Key Equation**

[1] Our friend and colleague, a wonderful Romanian mathematician and Professor of the University of Bucharest, Doru Stefanesc passed away on May 9, 2021. Doru planned to take part in our conference and sent the title of his talk on April 26, 2021 by e-mail from a Berlin hospital.

*E. V. Zima (Wilfrid Laurier University, Waterloo, Canada)*

The structure of polynomial solutions to the Gosper's key equation is analyzed. A method for rapid "extraction" of simple high-degree factors of the solution is given. It is shown that in cases when equation corresponds to a summable non-rational hypergeometric term the Gosper's algorithm can be accelerated by removing non-essential dependency of its running time on the value of dispersion of its rational certificate.