ISSN 0361-7688, Programming and Computer Software, 2010, Vol. 36, No. 2, pp. 63–67. © Pleiades Publishing, Ltd., 2010.
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On an Undecidable Problem Related to Difference Equations with Parameters

S. A. Abramov
Dorodnicyn Computing Center, Russian Academy of Sciences, ul. Vavilova 40, Moscow, 119991 Russia
e-mail: sergeyabramov@mail.ru

Received June 1, 2009

Abstract—Linear difference equations with polynomial coefficients depending on parameters are considered. It is proved that the problem of existence of numerical values of parameters for which the given equation has a polynomial solution (alternatively, a solution given by a rational function) is undecidable (similar to the undecidability of the same problem in the differential case proved by J.-A. Weil).

DOI: 10.1134/S0361768810020027

1. INTRODUCTION

Currently, many undecidable problems in computer algebra and related disciplines have become known to the researchers. Some of them are related to ordinary differential equations, especially, algebraic differential equations of the form $f(x, y', y'', \ldots, y^{(\rho)}) = 0$, where f is a polynomial with the coefficients from the filed of rational numbers $\mathbb{Q}$ (see, e.g., [1], which is devoted to algorithmic decidability and undecidability of differential equations of this kind and, in addition to original results, contains an introductory survey of publications on this subject). Linear differential equations with polynomial coefficients are a special case of algebraic differential equations, and many problems related to them seem simpler than those for general algebraic–differential equations. Nevertheless, there still exist undecidable problems related to these linear equations. The following unpublished result related to linear differential equations with polynomial coefficients depending on parameters was proved by J.-A. Weil: there does not exist an algorithm that allows one to learn whether there are numerical values of parameters for which a given equation has a solution given by a nonzero rational function. This result can easily be extended to the case of polynomial solutions of the differential equations under consideration, which is discussed in Section 2. The basic result of the paper is discussed in Section 3. It is shown that algorithmic undecidability similar to that found by Weil takes place for linear difference equations with polynomial coefficients depending on parameters as well. Note that algorithms for searching polynomial and rational solutions to parameter-free equations with polynomial coefficients are well known in computer algebra. The pioneer algorithms of this kind can be found in [2] (differential case) and [3, 4] (difference case). In Section 4, a problem related to equations with one parameter, which seems have not been solved yet, is discussed.

The results of this paper were reported in brief in [5].

2. DIFFERENTIAL EQUATION WITH PARAMETERS

Sometimes, one faces equations (for example, differential) containing parameters. Let operator L in equation $L(y) = 0$ have the form

$$\begin{aligned} & r_\rho(x, t_1, \ldots, t_m)D^\rho \\ & + r_{\rho-1}(x, t_1, \ldots, t_m)D^{\rho-1} + \ldots \\ & + r_0(x, t_1, \ldots, t_m), \end{aligned} \tag{1}$$

where $r_0, r_1, \ldots, r_\rho$ are polynomials of the specified variables, x is an independent variable, and $t_1, t_2, \ldots, t_m$ are parameters.

There arises the following question: Is it possible to find values of the parameters for which equation $L(y) = 0$ has solutions of one or another form or, at least, to learn whether such values exist. The following result was obtained by J.-A. Weil.[1]

Theorem 1. *There does not exist an algorithm that, for an arbitrary operator L of form* (1) *with $r_i(x, t_1, t_2, \ldots, t_m) \in \mathbb{Q}[x, t_1, t_2, \ldots, t_m]$, $i = 0, 1, \ldots, \rho$, answers the question of whether numerical values of parameters $t_1, t_2, \ldots, t_m$ exist for which equation $L(y) = 0$ has a solution that is a nonzero rational function of x.*

[1] Weil did not publish this result. The proof (with reference to Weil) was given in [6], where a number of special cases of the problem of finding exact solutions to equations with parameters were considered.

Proof is based on the Matiyasevich theorem that states that there does not exist an algorithm that, for an arbitrary polynomial $P(t_1, t_2, ..., t_m)$ with integer coefficients, determines whether the equation $P(t_1, t_2, ..., t_m) = 0$ has an integer solution [7, 8] (this theorem yields solution (in the negative sense) to Hilbert's Tenth Problem). Let $P(t_1, t_2, ..., t_m)$ be a polynomial with integer coefficients. Then,

$$F(x, t_1, t_2, \dots, t_m) = e^{xP(t_1, t_2, \dots, t_m)} \times (x-1)^{t_1}(x-2)^{t_2}\dots(x-m)^{t_m} \tag{2}$$

is a rational function of x for some numerical values $\tau_1, \tau_2, ..., \tau_m$ of parameters $t_1, t_2, ..., t_m$ if and only if $P(\tau_1, \tau_2, ..., \tau_m) = 0$ and $\tau_1, \tau_2, ..., \tau_m \in \mathbb{Z}$. For any values of the parameters, function F satisfies the equation

$$y' - \frac{F}{F}y = 0,$$

i.e., equation

$$y' - \left(\frac{t_1}{x-1} + \frac{t_2}{x-2} + \dots + \frac{t_m}{x-m} + P(t_1, t_2, \dots, t_m)\right)y = 0, \tag{3}$$

which turns to an equation with polynomial coefficients if we multiply it by $(x - 1)(x - 2)...(x - m)$. Hence, if an algorithm A existed that answered the question of existence of parameter values for which equation $L(y) = 0$ with operator L of form (1) has a rational solution, then this algorithm would make it possible to learn whether equation $P(t_1, t_2, ..., t_m) = 0$ could be solved in integer numbers for a given $P(t_1, t_2, ..., t_m) \in \mathbb{Z}[t_1, t_2, ..., t_m]$. To this end, it would be sufficient to construct differential equation (3), represent it as an equation with polynomial coefficients depending on $t_1, t_2, ..., t_m$, and to apply algorithm A to the differential equation obtained. This contradicts the Matiyasevich theorem. □

The result by Weil can easily be extended to the case of polynomial solutions of the differential equations discussed: there does not exist an algorithm that, for an arbitrary operator L of form (1) with $r_i(x, t_1, t_2, ..., t_m) \in \mathbb{Q}[x, t_1, t_2, ..., t_m]$, $i = 0, 1, ..., \rho$, answers the question of whether numerical values of parameters $t_1, t_2, ..., t_m$ exist for which equation $L(y) = 0$ has a nonzero polynomial solution. This assertion follows from the fact that there does not exist an algorithm that, for an arbitrary polynomial $P(t_1, t_2, ..., t_m)$ with integer coefficients, determines whether equation $P(t_1, t_2, ..., t_m) = 0$ has a nonnegative integer solution (this fact, in turn, follows from the Matiyasevich theorem, since equation $P(t_1, t_2, ..., t_m) = 0$ has an integer solution if and only if equation

$$P(u_1 - v_1, u_2 - v_2, \dots, u_m - v_m) = 0, \tag{4}$$

where $u_1, v_1, u_2, v_2, ..., u_m, v_m$, are unknowns, has a nonnegative integer solution). As can be seen, function (2) is a polynomial in x for certain numerical values $\tau_1, \tau_2, ..., \tau_m$ of parameters $t_1, t_2, ..., t_m$ if and only if $\tau_1, \tau_2, ..., \tau_m$ are nonnegative integers and $P(\tau_1, \tau_2, ..., \tau_m) = 0$. Applying the same reasoning as before, we find that, if an algorithm A' existed that answered the question of existence of parameter values for which equation $L(y) = 0$ with operator L of form (1) has a nonzero polynomial solution, then this algorithm would make it possible to learn whether equation $P(t_1, t_2, ..., t_m) = 0$ can be solved in positive integer numbers for a given polynomial $P(t_1, t_2, ..., t_m)$.

3. DIFFERENCE EQUATION WITH PARAMETERS

Now, let us consider the difference case. In this case, operator L has the form

$$r_\rho(x, t_1, \dots, t_m)E^\rho + r_{\rho-1}(x, t_1, \dots, t_m)E^{\rho-1} + \dots + r_0(x, t_1, \dots, t_m), \tag{5}$$

where E is a shift operator, $E(F(x)) = F(x + 1)$, $r_0, r_1, ..., r_\rho$ are polynomials of the above-specified variables, and $t_1, t_2, ..., t_m$ are parameters.

A *denominator* of a rational function $F(x) \in \mathbb{Q}(x)$ is defined to be a polynomial $c(x) \in K[x]$, $\mathrm{lc}(c(x)) = 1$, such that the function can be represented as $F(x) = \frac{b(x)}{c(x)}$, where polynomials $b(x) \in \mathbb{Q}[x]$ and $c(x)$ are coprime. In this case, the polynomial $b(x)$ is referred to as the *numerator* of $F(x)$. The denominator of a zero rational function is considered to be polynomial $c(x) = 1$. For any rational function, its numerator and denominator are uniquely determined.

Lemma 1. *Let* $L = x^mE - (x + \tau_1)(x + \tau_2)...(x + \tau_m)$, *where* $\tau_1, \tau_2, ..., \tau_m$ *are some numbers. The following assertions are valid*:

(i) *if equation* $L(y) = 0$ *has a solution given by a nonzero rational function, then* $\tau_1, \tau_2, ..., \tau_m$ *are integers*;

(ii) *if* $\tau_1, \tau_2, ..., \tau_m$ *are integers, then equation* $L(y) = 0$ *has a solution given by the rational function*

$$R(x) = R_1(x)R_2(x)\dots R_m(x),$$

where

$$R_i(x) = \begin{cases} x(x+1)\dots(x+\tau_i-1), & \tau_i \ge 0 \\ \dfrac{1}{(x-1)(x-2)\dots(x-|\tau_i|)}, & \tau_i < 0, \end{cases} \tag{6}$$

$i = 1, 2, ..., m$;

(iii) *if at least one number among integers* $\tau_1, \tau_2, ..., \tau_m$ *is negative, then the denominator of the rational function* $R(x)$ *from item* (ii) *is divisible by* $x - 1$ *and, hence,* $R(x)$ *is not a polynomial.*

Proof. (i) and (ii). Let, for some i, $1 \le i \le m$, τ_i be integer. Then, the substitution of

$$y(x) = z(x)R_i(x)$$

into equation $L(y) = 0$, where $z(x)$ is a new unknown function and the rational function $R_i(x)$ is defined by (6), results in a new equation, which, after simplification, takes the form $L'(z) = 0$, where

$$L' = x^{m-1}E - (x+\tau_1)\dots(x+\tau_{i-1})(x+\tau_{i+1})\dots(x+\tau_m)$$

for $m \ge 2$. If $m = 1$, then $L' = E - 1$, and equation $L'(z) = 0$ has a solution that is identically equal to one, which proves (ii).

If some of the numbers $\tau_1, \tau_2, \dots, \tau_m$ are not integer, then, after several substitutions of the above form with integer τ_i, we obtain equation $\tilde{L}(u) = 0$, where u is a new unknown function,

$$\tilde{L} = x^l E - (x+\sigma_1)(x+\sigma_2)\dots(x+\sigma_l),$$

and all numbers $\sigma_1, \sigma_2, \dots, \sigma_l$ are not integer. Let equation $\tilde{L}(u) = 0$ has a nonzero polynomial solution $p(x)$. Then,

$$x^l p(x+1) = (x+\sigma_1)(x+\sigma_2)\dots(x+\sigma_l)p(x).$$

It is evident that $x|p(x)$. Let h be the greatest integer such that $(x + h)|p(x)$. For some polynomial $q(x)$, we have $p(x) = (x + h)q(x)$ and

$$x^l(x+h+1)q(x+1)$$
$$= (x+\sigma_1)(x+\sigma_2)\dots(x+\sigma_l)(x+h)q(x).$$

Hence, $(x + h + 1)|q(x)$, which implies $(x + h + 1)|p(x)$. The latter contradicts the selection of h. Thus, we arrived at the contradiction with the assumption that equation $\tilde{L}(u) = 0$ has a polynomial solution. In [4], it is proved that, if operator $a_\rho(x)E^\rho + \dots + a_1(x)E + a_0(x)$ with polynomial coefficients, where $a_\rho(x)$ and $a_0(x)$ are nonzero polynomials, is applied to some nonpolynomial rational function resulting in a polynomial, then $a_\rho(x - \rho)$ and $a_0(x)$ are not coprime polynomials. Thus, equation $\tilde{L}(u) = 0$ cannot have nonpolynomial rational solutions either. This proves (i).

(iii) None of rational functions (6) may have multiplier $(x - 1)$ in its numerator. □

Theorem 2. *There does not exist an algorithm that, for an arbitrary operator L of form* (5) *with* $r_i(x, t_1, t_2, \dots, t_m) \in \mathbb{Q}[x, t_1, t_2, \dots, t_m]$, $i = 0, 1, \dots, \rho$, *answers the question of whether numerical values of parameters* $t_1, t_2, \dots, t_m$ *exist for which equation* $L(y) = 0$ *has a solution that is a nonzero rational function of x. This assertion is true for polynomial solutions either.*

Proof. Suppose that there exists an algorithm A that determines whether there exist solutions given by nonzero rational functions. Let $P(t_1, t_2, \dots, t_m)$ be an arbitrary polynomial in the specified variables with integer coefficients. Consider the following difference operator L with parameters $t_1, t_2, \dots, t_m$:

$$x^{m+1}E - (x + Q(t_1, t_2, \dots, t_m))(x+t_1)(x+t_2)\dots(x+t_m),$$

where

$$Q(t_1, t_2, \dots, t_m) = \frac{1}{1 + P^2(t_1, t_2, \dots, t_m)}.$$

By virtue of assertions (i) and (ii) of the above lemma, equation $L(y) = 0$ has a nonzero rational solution if and only if there exist $\tau_1, \tau_2, \dots, \tau_m \in \mathbb{Z}$ such that $Q(\tau_1, \tau_2, \dots, \tau_m) \in \mathbb{Z}$. In this case, $P(\tau_1, \tau_2, \dots, \tau_m) = 0$, since, otherwise, $0 < Q(\tau_1, \tau_2, \dots, \tau_m) < 1$. Thus, with the help of algorithm A, it is possible to learn whether the given algebraic equation $P(t_1, t_2, \dots, t_m) = 0$ has an integer solution (equation $L(y) = 0$ is transformed to an equation with polynomial coefficients by multiplying it by $1 + P^2(t_1, t_2, \dots, t_m)$). Hence, the assumption of existence of an algorithm A is not true.

If there existed an algorithm A' determining existence of nonzero polynomial solutions of algebraic equations with integer coefficients $L(y) = 0$, then, by virtue of assertion (iii) of the above lemma, there would exist an algorithm that determines existence of integer nonnegative solutions of algebraic equations with integer coefficients. Hence, algorithm A' does not exist. □

4. THE CASE OF ONE PARAMETER

In the previous sections, we discussed the case of operators with an arbitrary number of parameters. The fact that there do not exist algorithms in this general case does not imply that there do not exist algorithms for operators with one parameter that answer the question of whether there exists a parameter value for which the corresponding equation has a nonzero polynomial (rational) solution. To the best author's knowledge, the problem of existence of such algorithms is not solved yet both for differential and difference operators. A priori, the assumption of existence of such an algorithm does not contradict the Matiyasevich theorem, since the latter states that there does not exists an algorithm that is capable of determining the existence of integer solutions in the case of an algebraic equation with integer coefficients and an arbitrary number of unknowns. As for equations in one unknown, such an algorithm is well known from high-school mathematics.

The classical example is as follows: the solution of the differential equation

$$(1-x^2)y'' - xy' + t^2 y = 0 \qquad (7)$$

with parameter $t \in \mathbb{N}$ is the Chebyshev polynomial $T_t(x)$. The fact that Eq. (7) has a polynomial solution for any $t \in N$ can be established even if one knows nothing about the Chebyshev polynomials. Suppose that solution to this equation is given by a series (in particular, by a polynomial),

$$y(x) = c_0 + c_1x + c_2x^2 + \dots.$$

Then, it is not difficult to show that its coefficients satisfy the following difference equation:

$$(n+1)(n+2)c_{n+2} + (t^2 - n^2)c_n = 0.$$

The facts that we arrived at a second-order difference equation not containing c_{n+1} and that $t^2 - n^2$ vanishes when $n = t$ and the leading coefficient vanishes when $n + 2 = 0$ and $n + 2 = 1$ allow us to conclude that Eq. (7) actually has a polynomial solution of degree t containing only even degrees of x when t is even and only odd degrees when t is odd. If numerical value of t does not belong to $\mathbb{N}$, then the equation has no polynomial solutions.

Another example is the equation

$$(1 - x^2)y'' - 2xy' + t(t+1)y = 0,$$

the solution of which for $t \in \mathbb{N}$ is the Legendre polynomial $L_t(x)$. The corresponding difference equation in this case takes the form $(n + 1)(n + 2)c_{n+2} + (t(t + 1) - n(n + 1) - n(n + 1))c_n = 0$.

The assumption that there may exist algorithms for arbitrary differential and difference operators of the discussed form with one parameter is substantiated by an example related to systems of first-order algebraic–differential equations. Using the idea of the proof of Theorem 1, it is not difficult to prove that there does not exist an algorithm that recognizes existence of solutions to these systems all components of which are nonzero rational functions. Indeed, let us multiply Eq. (3) by $(x - 1)(x - 2)...(x - m)$, consider y, t_1, t_2, ..., t_m as unknown functions of variable x, and supplement the equation obtained with the additional m equations

$$t_1' = 0, \quad t_2' = 0, \dots, t_m' = 0.$$

By virtue of the discussions in Section 2, this equation has a solution with the components t_1, t_2, ..., t_m, y in the form of nonzero rational functions if and only if equation $P(t_1, t_2, ..., t_m) = 0$ has integer solutions and all of them are not zero. The recognition of existence of such a solution is an undecidable problem, since the problem of existence of arbitrary integer solutions reduces to this problem (for example, equation $P(t_1, t_2, ..., t_m) = 0$ has integer solutions if and only if Eq. (4) has integer solutions and none of them is equal to zero). Thus, for systems of first-order algebraic–differential equations, there does not exists an algorithm that recognizes existence of solutions all components of which are nonzero rational functions. However, for one first-order equation in one unknown function, such an algorithm exists [9]. This algorithm finds all rational solutions, whatever they are, including zero ones. By the results of operation of this algorithm, it can easily be determined whether the equation has a nonzero rational solution. In this example, the equations have no parameters, but this is not important. The important thing is that there exists an algorithm for $m = 1$, whereas no algorithms exist for arbitrary m.

On the other hand, the very fact that undecidability of a certain algorithmic problem for an arbitrary number of variables has been proven with the help of the Matiyasevich theorem gives no grounds to state that the restriction of this problem to the case of one variable will turn out decidable. In paper [10] by Richardson (see also [8, Chapter 9]), it is established that, for the functions of real variables that are constructed from these variables, rational numbers, constant π, operations + and $\cdot$, function sin, and superpositions, the problem of existence of a point in $\mathbb{R}^m$ where the function is positive is undecidable (strictly speaking, in the paper [10], which was published in 1968, when Hilbert's Tenth Problem had not been solved yet, a weaker assertion was proved; however, the use of the Matiyasevich theorem, which was proved in 1970, makes it possible to reformulate Richardson's result in this form). Then, starting from the functions $h(x) = x\sin x$ and $g(x) = x\sin x^3$, Richardson [10] defines functions

$$f_1(x) = h(x), \quad f_2(x) = h(g(x)), \dots, f_m(x)$$
$$= h(g(g(\dots g(g(x))\dots))), \dots,$$

and shows that, for any $m > 1$, the set of points

$$(f_1(x), f_2(x), \dots, f_m(x)), \quad x \in \mathbb{R},$$

is everywhere dense in $\mathbb{R}^m$. Together with the function of m variables $F(x_1, x_2, ..., x_m)$, the function of one variable $F(f_1(x), f_2(x), ..., f_m(x))$ is considered. Since these functions are continuous, we find that, for the function of one real variable x constructed from x, rational numbers, constant π, operations + and $\cdot$, function sin, and superpositions, the problem of existence of a point on a number line where the function is positive is undecidable.

The last two examples demonstrate that the transition from an arbitrary m to $m = 1$ poses a problem the answer to which cannot be found from general considerations.

ACKNOWLEDGMENTS

This work was supported in part by the Russian Foundation for Basic Research, project no. 10-01-00249-a, and by ECONET, project no. 21315ZF.

The author is grateful to J.-A. Weil (Limoges University) and S.P Tsarev (Siberian Federal University) for interesting and helpful discussions.

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