

Procedures for Searching Local Solutions of Linear Differential Systems with Infinite Power Series in the Role of Coefficients

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Construction of Laurent, regular, and formal (exponential–logarithmic) solutions of full-rank linear ordinary differential systems is discussed. The systems may have an arbitrary order, and their coefficients are formal power series given algorithmically. It has been established earlier that the first two problems are algorithmically decidable and the third problem is not decidable. A restricted variant of the third problem was suggested for which the desired algorithm exists. In the paper, a brief survey of algorithms for the above-mentioned decidable problems is given. Implementations of these algorithms in the form of Maple procedures with a uniform interface and data representation are suggested.

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1. INTRODUCTION

Systems of linear ordinary differential equations arise in many fields of mathematics. Infinite power series are used for representing both solutions of the systems and the systems themselves. The problem of representation of infinite series is important for computer algebra. In this paper, like in the previous works [1–5], we use an algorithmic representation: for each series, an algorithm is specified that, given an integer i , finds the coefficient of x^i . Any deterministic algorithms are allowed. Of course, for such a series representation, algorithmically undecidable problems inevitably arise; for example, it is impossible to verify algorithmically the equality of a series to zero (this follows from classical Turing's results [6]).

However, if it is known that a system of m equations with the same number of unknowns has a full rank (which, in the general case, cannot be verified algorithmically), then algorithms can be proposed for finding solutions of various kinds. It should be emphasized that we mean local solutions, i.e., solutions of the system at some point, say, the origin. These solutions have a form of series in x or contain such series as components. All series are assumed to be formal, and their convergence is not considered. Earlier, algorithms for constructing all Laurent and regular solutions were proposed in [2, 7]. As for formal exponential–logarithmic solutions (further, following the established tradition, we will call them simply formal solutions), the situation is more complicated. Such solutions are remarkable by virtue of the fact that a first-order normal system $y' = Ay$, where A is an

$m \times m$ -matrix with entries given by formal infinite Laurent series and y is a vector of m unknown functions, has an m -dimensional space of formal solutions [8]. However, it may happen that a system of any order has no formal (or even any) solutions and it is impossible to verify algorithmically whether or not this is true.

The problem of determining the dimension of the space of formal solutions remains undecidable even in the case where it is known in advance that this dimension is not equal to zero, i.e., the system has nonzero formal solutions [3, 4]. At the same time, it was shown that, if the system is known to have at least N linearly independent formal solutions, then these N solutions can be constructed algorithmically [5]. The rest of the paper is as follows. Some preliminary results are given in Section 2. A brief survey of algorithms for solving the above-mentioned decidable problems is presented in Section 3. An implementation of the algorithms in the form of a Maple package is considered in Section 4.

To our best knowledge, the proposed package is the first complete implementation of algorithms for constructing solutions of systems with the coefficients given by infinite, rather than truncated, series. There are known algorithms that, under certain restrictions, can work with such systems. For example, the algorithm from [9] is capable of finding regular solutions of systems satisfying certain conditions. However, it is additionally assumed that we can determine whether any series involved is zero. The problem of representation of infinite series is not considered. As for the implementations of the proposed algorithms, they were based on the assumption that the coefficients of

the system are polynomials or rational functions rather than infinite series. Our algorithms were earlier published in [2, 5, 7], which also presented prototype versions of the procedures and results of first experiments with them. By now, the procedures have been improved, with the interface and data representation being uniform for all procedures.

The procedures are freely available at the address <http://www.ccas.ru/ca/doku.php/eg>.

2. PRELIMINARIES

2.1. Systems and Operators

Let K be a number field. The ring of polynomials and the field of rational functions of x over K are conventionally denoted as $K[x]$ and $K(x)$, respectively. The ring of formal power series of x over K is denoted as $K[[x]]$, and the field of formal Laurent series, as $K((x))$. For a nonzero element $a(x) = \sum a_i x^i$ from $K((x))$ its *valuation* $\text{val}_x a(x)$ is defined as $\text{val}_x a(x) = \min\{i | a_i \neq 0\}$, with $\text{val}_x 0 = \infty$. Valuation of a vector or matrix with series entries is assumed to be equal to the minimum of valuations of the components.

If R is a ring (in particular, field), then $\text{Mat}_m(R)$ denotes the ring of square matrices of order m with entries from R . I_m denotes the identity matrix of order m , M^T denotes a transposed matrix M , and $M_{i,*}$, $1 \leq i \leq m$, denotes the $(1 \times m)$ -matrix coinciding with the i th row of the $(m \times m)$ -matrix M .

This paper deals with local problems, i.e., with searching for solutions at some point. Without loss of generality, this point may be assumed to be 0. It is convenient to write differential systems in terms of the operation $\theta = x \frac{d}{dx}$ rather than the conventional differentiation operation $\frac{d}{dx}$ (the transition from one notation to the other presents no difficulties). We consider systems of the form

$$A_r(x)\theta^r y + A_{r-1}(x)\theta^{r-1} y + \dots + A_0(x)y = 0, \quad (1)$$

where $y = (y_1, y_2, \dots, y_m)^T$ is a vector of unknown functions of x . As for the coefficients

$$A_0(x), A_1(x), \dots, A_r(x) \quad (2)$$

we assume that $A_i(x) \in \text{Mat}_m(K[[x]])$, $i = 0, 1, \dots, r$, with $A_r(x)$ (the *leading* matrix of the system) being nonzero.

Entries of matrices $A_i(x)$ are called *coefficients of the system*. These are algorithmically specified formal power series: for any element $a(x) = \sum_{i=0}^{\infty} a_i x^i$ of a matrix from (2), there is an algorithm Ξ_a that calcu-

lates $\Xi_a(i) = a_i$ for $i \in \mathbb{Z}_{\geq 0}$. Such series are called *constructive* formal power series.

System (1) can be written as $L(y) = 0$, where operator L has the form

$$A_r(x)\theta^r + A_{r-1}(x)\theta^{r-1} + \dots + A_0(x), \quad (3)$$

r is the *order* of L (notation $r = \text{ord} L$). Operator (3) can be represented by a single operator matrix belonging to $\text{Mat}_m(K[[x]][\theta])$:

$$\begin{pmatrix} L_{11} & \dots & L_{1m} \\ \dots & \dots & \dots \\ L_{m1} & \dots & L_{mm} \end{pmatrix}, \quad (4)$$

$L_{ij} \in K[[x]][\theta]$, $i, j = 1, \dots, m$, with $\max_{i,j} \text{ord} L_{ij} = r$.

Thus, the system can be represented in an operator form by using one of the two operator representation. In what follows, the form of the representation will be selected from convenience considerations.

We say that the rows of operator (4) with numbers i_1, \dots, i_s , $s \leq m$, are *linearly independent* over $K[[x]][\theta]$ if, from the fact that the linear combination of the rows with the left multipliers $f_1, \dots, f_s \in K[[x]][\theta]$ is equal to zero, i.e., $f_1 L_{i_1,*} + \dots + f_s L_{i_s,*} = 0$, it follows that $f_1 = \dots = f_s = 0$.

The matrix A_r is the leading matrix of the system $L(y) = 0$ and the operator L independent of the form of the representation of the system and operator.

If all rows of operator (4) are linearly independent over $K[[x]][\theta]$, then we say that the equations of the corresponding system are independent over $K[[x]][\theta]$. In this case, the operator $L \in \text{Mat}_m(K[[x]][\theta])$, as well as the system $L(y) = 0$, has the *full rank*. We will also call them the operator and system of *full rank*. It is these operators and systems that are considered in this paper.

2.2. Local Solutions

A solution of a differential system the components of which are formal Laurent series is called *Laurent solution*.

A *regular* solution has the form

$$y(x) = x^\lambda w(x), \quad (5)$$

where $\lambda \in K$, $w(x) \in K((x))^m[\ln x]$. Each solution of this kind can be written as

$$x^\lambda \sum_{s=0}^k g_s(x) \frac{\ln^s x}{s!}, \quad (6)$$

where $k \in \mathbb{Z}_{\geq 0}$ and $g_s(x) \in K((x))^m$, $s = 0, 1, \dots, k$. In this case, we say that $y(x)$ admits factor x^λ or that x^λ is an *admissible factor* of the solution $y(x)$. The set

$$x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_p} \tag{7}$$

is called a *complete set* of admissible factors of regular solutions of system S , if (i) no exponents of the elements of set (7) differ by an integer; (ii) each element x^{λ_i} of set (7) is an admissible factor for some nonzero regular solution of system S ; (iii) for every nonzero regular solution of system S , set (7) contains an admissible factor for this solution.

All regular solutions of a given system that admit one and the same factor form a linear space over K .

A *formal* solution can be represented in the parametric form as

$$x = \tau^q, \quad y(x) = e^{Q(1/\tau)} \tau^\lambda w(\tau). \tag{8}$$

The expression $e^{Q(x^{-1/q})}$, where Q is a polynomial over K , is called an *exponential part* of the formal solution, $q \in \mathbb{Z}_{>0}$ is the *ramification index* of the solution. The term $\tau^\lambda w(\tau)$ is called a *regular part*: $\lambda \in K$, $w(\tau) \in K[[\tau]]^m[[\ln \tau]]$. The exponential part can also be written in terms of the parameterizing variable τ , like in (8).

If $q = 1$ and $Q \in K$, the solution is regular (see above); in all other cases, it is *irregular*.

2.3. Systems with Polynomial Coefficients, Induced Recurrence Systems, EG-Algorithms

A particular case of a power series is a polynomial. Consider systems of form (1) with polynomial coefficients. Let the expansion of a Laurent solution in terms of powers of x has coefficients $z(n)$, $n \in \mathbb{Z}$, where $z(n) = (z_1(n), \dots, z_m(n))^T$. Then, the sequence $(z(n))$ satisfies the *induced recurrence* system

$$B_0(n)z(n) + B_{-1}(n)z(n-1) + \dots + B_t(n)z(n+t) = 0, \tag{9}$$

where t is a nonpositive integer, $B_t(n)$, ..., $B_0(n) \in \text{Mat}_m(K[n])$. This system is constructed by the application of the transformation

$$x \rightarrow E^{-1}, \quad \theta \rightarrow n, \tag{10}$$

to the original differential system [10], where E denotes the shift operator: $Ez(n) = z(n+1)$, $E^{-1}z(n) = z(n-1)$.

One can consider the recurrence operator $R \in \text{Mat}_m(K[n][E^{-1}])$ corresponding to system (9). An operator \bar{R} is called an *l-embracing operator* for the operator R if its leading matrix (i.e., the matrix coefficient of the greatest degree of E , or, which is the same,

the least degree of E^{-1} , in the expansion of the operator R) is nonsingular and $\bar{R} = FR$ for some $F \in \text{Mat}_m(K[n][E])$. Accordingly, system $\bar{R}(z) = 0$ is called an *l-embracing system* for the system $R(z) = 0$ if \bar{R} is an *l-embracing operator* for R . The prefix “l” indicates that the leading matrix is nonsingular. It may happen that the l-embracing system has more solutions than the original system of form (9).

The algorithm EG_σ [11–13] transforms system $R(z) = 0$ into the l-embracing system $\bar{R}(z) = 0$. When we consider solutions in the form of sequences, in other words, *sequential* solutions of system (9), the algorithm EG_σ allows us to get rid of all unnecessary solutions. To this end, a finite set of *linear constraints* is additionally constructed. System $R(z) = 0$ is transformed into an l-embracing system $\bar{R}(z) = 0$ by performing a sequence of one-type steps, on each of which one operation is not safe in the sense that it can result in appearance of extra solutions. For example, the i th equation of the system ($1 \leq i \leq m$) can be replaced by a linear combination of all equations of the system, and this linear combination has polynomial coefficients $v_1(n), v_2(n), \dots, v_m(n)$. If n_0 is an integer root $v_i(n)$, then, by virtue of (9), for any solution $y(x) = \sum_{n=v}^{\infty} z(n)x^n$, $v \leq n_0$, of the original system, the following linear constraint must hold

$$(B_0(n_0))_{i,*}z(n_0) + (B_{-1}(n_0))_{i,*}z(n_0-1) + \dots + (B_{-n_0+v}(n_0))_{i,*}z(v) = 0. \tag{11}$$

In Section 3, we apply a special variant of the EG_σ algorithm that makes it possible to work with infinite induced recurrence systems, which come to existence when considering differential systems with the coefficients in the form of series.

Remark 1. A differential variant of the EG algorithm is the algorithm EG_δ , which, given a differential system of full rank with polynomial coefficients, constructs the l-embracing system, i.e., the system with a nonsingular leading matrix, with the set of solutions of the constructed system containing all solutions of the original system [3, 13, 14].

3. ALGORITHMS FOR CONSTRUCTING LOCAL SOLUTIONS

Let us outline key ideas of algorithms for searching for local solutions of the three types indicated in Section 2.2. The basic attention will be paid to the discussion of how to overcome difficulties arising from the fact that the coefficients are infinite series.

3.1. Laurent Solutions: Lower Bounds of Valuations and Sequences of Coefficients

For the original differential system $L(y) = 0$, the induced recurrence system is $R(z) = 0$ with $R = ML$, where transformation M is given by (10). We have

$$R = B_0(n) + B_{-1}(n)E^{-1} + B_{-2}(n)E^{-2} + \dots,$$

and the induced system has the form

$$B_0(n)z(n) + B_{-1}(n)z(n-1) + \dots = 0, \quad (12)$$

where

- $z(n) = (z_1(n), \dots, z_m(n))^T$ is a column vector of unknown sequences such that $z_i(n) = 0$ for all negative n with sufficiently large $|n|$, $i = 1, \dots, m$;
- $B_0(n), B_{-1}(n), \dots \in \text{Mat}_m(K[n])$;
- $B_0(n)$ is a nonzero leading matrix of the operator R and system (12).

It is shown in [2] that the induced system $R(z) = 0$ has a full rank; i.e., the equations of the system $R(z) = 0$ are independent over $K[n][[E^{-1}]]$ if and only if the original differential system $L(y) = 0$ is of full rank. The system $L(y) = 0$ in this case has a Laurent solution $y(x) = u(v)x^v + u(v+1)x^{v+1} + \dots$ if and only if the two-sided sequence

$$\dots, 0, 0, u(v), u(v+1), \dots \quad (13)$$

of column vectors of the coefficients of $y(x)$ satisfies the induced recurrence system

$$\begin{aligned} B_0(v)u(v) &= 0, \\ B_0(v+1)u(v+1) + B_{-1}(v+1)u(v) &= 0, \\ B_0(v+2)u(v+2) + B_{-1}(v+2)u(v+1) \\ &+ B_{-2}(v+2)u(v) = 0, \\ &\dots \end{aligned}$$

If the matrix $B_0(n)$ is nonsingular, then the equation $\det B_0(n) = 0$ may be viewed as an indicial equation of the original differential system: the set of integer roots of this algebraic equation includes the set of all possible valuations of the Laurent solutions of the system $L(y) = 0$. This makes it possible to find the lower bound of valuations of all Laurent solutions of the system.

However, in many cases, the matrix $B_0(n)$ is singular even if the leading matrix $A_r(x)$ of the system $L(y) = 0$ is nonsingular. The following theorem states that this is not a deadlock situation.

Theorem 1 ([2]). Let $R = ML$, where L is an operator of full rank belonging to $\text{Mat}_m(K[[x]])[\theta]$. Let all coefficients of the operator L be constructive power series. Then, there exists an operator $F \in \text{Mat}_m(K[n])[E]$ such that the leading matrix $\bar{B}_0(n)$ of the operator $\bar{R} = FR$

is nonsingular, and this operator can be constructed algorithmically. In addition, a finite set of linear constraints (see Section 2.3) can be constructed to remove redundant solutions arising when going from system $R(z) = 0$ to $\bar{R}(z) = 0$.

The operator F the existence of which is stated in Theorem 1 has, like the operator R , polynomial coefficients. Note that F has a finite order.

The special variant of algorithm EG_σ based on this theorem, which will be referred to as EG_σ^∞ , allows one to construct any number of first terms of the sum (operator)

$$\bar{B}_0(n) + \bar{B}_{-1}(n)E^{-1} + \bar{B}_{-2}(n)E^{-2} + \dots \quad (14)$$

In this sense, we can construct this sum.

The system $\bar{R}(z) = 0$ with a nonsingular leading matrix and the set of linear constraints mentioned in the theorem allow one to find all Laurent solutions of the original differential system $L(y) = 0$ of full rank. Here, it is important that the set of linear constraints is finite and that each of these constraints contains only a finite number of nonzero terms (because we are interested only in solutions $z(n)$ for which $z_i(n) = 0$ for all negative n smaller than the lower bound of valuations of the Laurent solutions of the original differential system).

Now, we introduce a concept to be used in what follows. Let $a(x) = a_0 + a_1x + a_2x^2 + \dots$ be a formal power series and p be a nonnegative integer. Then, the polynomial $a^{(p)}(x) = a_0 + a_1x + a_2x^2 + \dots + a_px^p$ is called a p -truncation of the series $a(x)$ (or *truncation to the power p*). In this case, p is called the *truncation power*.

Let S be a linear differential system of arbitrary order with the coefficients in the form of formal power series. Then, the p -truncation $S^{(p)}$ of system S is a system whose coefficients are p -truncations of the corresponding coefficients of system S . In the same sense, we can consider p -truncations of the operators.

Remark 2. If system S of form (1) is of full rank, then the least integer μ such that $S^{<p>}$ has a full rank for any $p \geq \mu$ is called *width* of S . There exist systems S of full rank and nonnegative integers p such that $S^{(p)}$ has a full rank, whereas $S^{(p+1)}$ has a smaller rank. However, it is proved in [2] that the width is determined for any system of full rank. Under our assumptions about the system, its width can be found algorithmically.

Let V_S denote the space of Laurent solutions of system S and $V_S^{(p)}$, $p \in \mathbb{Z}$, the space the elements of which are p -truncations of the corresponding elements of the space V_S . We consider algorithmic search of Laurent solutions of differential systems with the

coefficients in the form of series as a solution of problem \mathbf{P}_L . It is assumed that a full-rank system S of form (1) and $d_0 \in \mathbb{Z}_{\geq 0}$ are given.

\mathbf{P}_L : Find $p_0 \in \mathbb{Z}$ such that $\dim V_S = \dim V_S^{(p)}$ for all $p \geq p_0$ and a basis of the space $V_S^{(d)}$, where $d = \max\{d_0, p_0\}$.

An algorithm for solving problem \mathbf{P}_L is based on considering the induced recurrence system and reducing it to a “convenient form” by the above-discussed EG_σ^∞ algorithm. System (12) is infinite, and the algorithm cannot work with all matrices B_{-i} , $i = 0, 1, \dots$, simultaneously. To overcome this, lazy calculations and storage of the information about all already performed reductions and shifts are used. This makes it possible to involve into the computational process matrices B_{-i} with the increasing values of i , without need to carry out anew already performed work on matrices with lesser i .

A more detailed description of the algorithm for constructing Laurent solutions is given in [2].

3.2. Regular Solutions:
Generalized Approach by Heffter

For a differential system $L(y) = 0$ and any integer $i \geq 0$, the result of application of L to $g(x) \ln^i(x)/i!$ is

$$G_{ii}(g) \frac{\ln^i x}{i!} + \dots + G_{i1}(g) \frac{\ln x}{1!} + G_{i0}(g),$$

where $G_{i0}, G_{i1}, \dots, G_{ii} \in \text{Mat}_m(K[[x]])[\theta]$ and $G_{00} = L$, $G_{i+j,j} = G_{i0}$ for all $i, j \geq 0$ [16, 17]. Let us denote $L_i = G_{i0}$. Then, $L_i = G_{i+j,j}$ for all $j \geq 0$.

The general scheme of searching regular solutions of the systems under consideration is similar to that proposed in [15] for an arbitrary-order full-rank linear differential systems with polynomial coefficients (see also [13]). The scheme itself is a generalization of the Heffter algorithm [16] and is based on the consideration of a sequence of systems

$$S_0, S_1, \dots, \tag{15}$$

where S_k is the system

$$L_0(g_i) = -\sum_{j=1}^i L_j(g_{i-j}), \quad i = 0, 1, \dots, k \tag{16}$$

(when $i = 0$ in (16), we have $L_0(g_0) = 0$). The following theorem is a generalization of the assertion proved by Heffter for the scalar case.

Theorem 2 ([7, 15, 18]). The set of nonnegative integers k for which system S_k has a Laurent solution

$$\left(g_0(x)^T, g_1(x)^T, \dots, g_k(x)^T\right)^T, \quad g_0(x) \neq 0,$$

is finite; if it is empty, then $L(y) = 0$ has no nonzero solutions in $K((x))^m[[\ln x]]$. If this set is nonempty and \tilde{k} is its greatest element, then any solution of system $L(y) = 0$ belonging to $K((x))^m[[\ln x]]$ can be written as

$$\sum_{s=0}^{\tilde{k}} g_{\tilde{k}-s}(x) \frac{\ln^s x}{s!}, \tag{17}$$

where

$$\left(g_0(x)^T, g_1(x)^T, \dots, g_{\tilde{k}}(x)^T\right)^T, \quad g_0(x) \neq 0, \tag{18}$$

is a Laurent solution of system $S_{\tilde{k}}$. At the same time, any Laurent solution of system $S_{\tilde{k}}$ of form (18) generates solution (17) to system $L(y) = 0$.

This brings us to the following scheme of construction of regular solutions.

1. For a given differential system S , consider the induced recurrence system and, by means of the EG_σ^∞ algorithm discussed in Section 3.1, turn to the recurrence system with the nonsingular leading matrix $\bar{B}_0(n)$. Calculate all roots of the equation $\det \bar{B}_0(n) = 0$. Considering two roots λ and λ' equivalent if $\lambda - \lambda' \in \mathbb{Z}$, construct set Λ containing one representative of each equivalence class.

2. For each $\lambda \in \Lambda$, find regular solutions admitting the factor x^λ . To this end, consider system $S(\lambda)$, the result of substitution of (5) into S and subsequent multiplication by $x^{-\lambda}$. Using the operators L_0, L_1, \dots corresponding to the system $S(\lambda)$, find Laurent solutions of the systems in (15) (until the first system that has no Laurent solutions). This yields regular solutions $y(x)$ in form (17) for the system $S(\lambda)$.

Construction of regular solutions by this scheme is reduced to construction of Laurent solutions of nonhomogeneous differential systems with Laurent right-hand sides. It does not significantly differ from the construction described in Section 3.1. The recurrence system for such a nonhomogeneous differential system is also nonhomogeneous, and its right-hand side is a sequence of the coefficients of the right-hand side of the differential system: each component of this vector sequence is an algorithmically specified sequence of elements of the field K . The EG_σ^∞ algorithm can also be extended to such nonhomogeneous recurrence systems. As a result of application of this algorithm, we obtain a nonhomogeneous recurrence system with the left-hand side given by operator (14) and the right-hand side in the form of a sequence of column vectors. The arising linear constraints are nonhomogeneous, and the number of them is finite. The set containing all integer roots of the algebraic equation $\det \bar{B}_0(n) = 0$ and the valuation of the right-hand side of the recurrence system obtained (if $u(v) \neq 0$ the val-

uation of a sequence of form (13) is the number v ; if all elements of the sequence are zeros, then the valuation is equal (∞) includes valuations of all possible Laurent solutions of the original differential system. This solves problem \mathbf{P}_L modified for the nonhomogeneous case.

Let us give more detail about the representation of regular solutions output by the algorithm. Let $W_S(\lambda)$ denote the space of regular solutions of system S that admit the factor x^λ and $W_S^{(p)}(\lambda)$ denote the space obtained from $W_S(\lambda)$ by the replacement of each element of form (6) with

$$x^\lambda \sum_{s=0}^k g_s^{(p)}(x) \frac{\ln^s x}{s!}.$$

By virtue of finite dimension of the space $W_S(\lambda)$, it is clear that the valuations of the series $g_s(x)$ and $g_s^{(p)}(x)$ are bounded from above. In addition, for all sufficiently large p , the equality $\dim W_S(\lambda) = \dim W_S^{(p)}(\lambda)$ holds. We consider algorithmic search of regular solutions of differential equations with the coefficients in the form of series as solution of problem \mathbf{P}_R . Let a full-rank system S of form (1) and $d_0 \in \mathbb{Z}_{\geq 0}$ be given.

\mathbf{P}_R : Find a complete set of admissible factors of nonzero solutions of system S . Determine a $p_0 \in \mathbb{Z}$ such that, for any x^λ from this set, $W_S(\lambda) = \dim W_S^{(p)}(\lambda)$ for all $p \geq p_0$ and find a basis of the space $W_S^{(d)}(\lambda)$, where $d = \max\{d_0, p_0\}$.

The induced systems arising when applying the above-described scheme and their solutions are infinite. Like in the case of Laurent solutions, the algorithm uses lazy calculations. When searching for solutions of a current system from sequence (15), solutions of the previous systems found earlier are extended if necessary to ensure adequate truncation of the right-hand side of the current system of the sequence. Construction of system $S(\lambda)$ is, in fact, not required, since its induced recurrence system can be obtained from the induced recurrence system for S by replacing n with $n + \lambda$.

A detailed description of the algorithm for constructing regular solutions is given in [7].

Remark 3. According to the traditional definition, a regular solution is a linear combination of solutions of form (5) that may include terms with the values of λ differing by a noninteger number. Our algorithm constructs a basis for each subspace consisting of solutions of form (5) for which the values of λ differ by an integer. The union of all such bases is a basis for the space W_S of regular solutions meant in the traditional sense.

3.3. Formal Solutions:

Decidable Variant of the Search Problem

As has already been mentioned in Section 1, the problem of calculation of dimension of the space of all formal solutions of a full-rank system is algorithmically undecidable. The problems of existence testing of irregular solutions and construction of a basis of all formal solutions are also algorithmically undecidable [3, 4].

It is possible to suggest a decidable variant of the problem of searching formal solutions: if it is known in advance that the system has at least N linearly independent formal solutions, then these N solutions can be constructed algorithmically [5]. Further, we consider just this variant of the problem.

The following theorem is valid.

Theorem 3 ([5]). Let a system S have an irregular solution $\bar{y}(x)$. Then, for all sufficiently large nonnegative integers p , the system $S^{(p)}$ has an irregular solution with the same exponential part as that in $\bar{y}(x)$.

Applying one of the algorithms suggested in [5, 19, 20] to a p -truncation of the original system S (as well as the EG_δ algorithm if needed (see Remark 1); if a transformation from a system with polynomial coefficients to a scalar equation is performed, then algorithms from [21, 22] may be useful for searching exponential parts), we obtain a set of candidates for the role of the exponential part of a solution of S . For each such a candidate $e^{Q(x^{-1/q})}$, we perform in the system S the substitution

$$x = \tau^q, \quad y(x) = e^{Q(1/\tau)} \zeta(\tau),$$

where τ is a new independent variable and $\zeta(\tau)$ is a vector of new unknown functions. Multiplying the results by $e^{-Q(1/\tau)}$, we obtain a system whose coefficients are Laurent series for which the lower bound of the valuation is known. By multiplying this system by τ to the required integer power, we obtain system $S_{q,Q}$ whose coefficients are constructive power series. To the system $S_{q,Q}$, we apply the algorithm for constructing the space $W_{S_{q,Q}}$ of all regular solutions. Then, the dimension of the space of formal solutions of system S with the exponential part $e^{Q(x^{-1/q})}$ is $q \dim W_{S_{q,Q}}$.

Thus, if we know in advance that the dimension of the space of all formal solutions of the original system is not less than N , we can construct a subspace of formal solutions of the dimension not less than N by the p -truncations $S^{(p)}$, by way of increasing p from a certain $p_0 \geq 0$ until the subspace of solutions of the desired dimension is constructed. For p_0 , we can take, for example, the system width (see Remark 2). It is possible to take 0 as well; however, in this case, it is

$$\begin{aligned}
 & > \text{sys} := \text{Matrix}([[x + \text{Sum}(x^k, k = 3..infinity), 0, 0], \\
 & \quad [0, \text{Sum}(x^k, k = 1..infinity), 0], \\
 & \quad [0, 0, \text{Sum}(x^k, k = 1..infinity)]]).\text{theta}(y(x), x, 2) + \\
 & \text{Matrix}([[x^2, 0, 0], [0, 1, 0], [0, 0, 1]]).\text{theta}(y(x), x, 1) + \\
 & \text{Matrix}([[-1-x+x^2-\text{Sum}(x^k, k = 3..infinity), -x^2, -1-x], \\
 & \quad [-x^2, -\text{Sum}(x^k, k = 1..infinity), -1-x^3], \\
 & \quad [-x^3, x, -\text{Sum}(x^k, k = 0..infinity)]]).y(x); \\
 \text{sys} = & \begin{pmatrix} x + \sum_{k=3}^{\infty} x^k & 0 & 0 \\ 0 & \sum_{k=1}^{\infty} x^k & 0 \\ 0 & 0 & \sum_{k=1}^{\infty} x^k \end{pmatrix} \theta(y(x), x, 2) + \begin{pmatrix} x^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \theta(y(x), x, 1) + \begin{pmatrix} -1-x+x^2 - \sum_{k=3}^{\infty} x^k & -x^2 & -1-x \\ -x^2 & -\sum_{k=1}^{\infty} x^k & -1-x^3 \\ -x^3 & x & -\left(\sum_{k=0}^{\infty} x^k\right) \end{pmatrix} y(x)
 \end{aligned}$$

Fig. 1.

required to check whether the rank of the truncated system is full for each value $p \geq 0$ under study.

We consider search of formal solutions of differential equations with the coefficients in the form of series as solution of problem \mathbf{P}_F . It is assumed that a full-rank system S of form (1) and $d_0, N \in \mathbb{Z}_{\geq 0}$ are given.

\mathbf{P}_F : Find set $e^{Q_i(x^{-1/q_i})}$, $i = 1, \dots, k$, of exponential parts of formal solutions of system S such that, for $S_i = S_{q_i, Q_i}$, $i = 1, \dots, k$, the inequality $\dim W_S + q_1 \dim W_{S_1} + \dots + q_k \dim W_{S_k} \geq N$ holds. For each of the systems S, S_1, \dots, S_k and number d_0 , find solution of problem \mathbf{P}_R .

If N is greater than the dimension of the space of formal solutions of S , then the algorithm for construction formal solutions does not terminate. A detailed description of the algorithm can be found in [5].

3.4. On Representation of Series in Solutions

All Laurent series in the solutions are constructed in a truncated form with the number of terms not less than the number requested upon launching the algorithm (sometimes, the number of terms is even greater if this is required to ensure the desired dimension of the solution space). This representation is similar to that in the differential systems themselves, where a series is given by the algorithm for determining the coefficient by its index. Small distinction is related with the fact that our algorithms calculate all coefficients sequentially and, when calculating the coefficient with index i , all nonzero coefficients of the series with indices lesser than i (sometimes, as mentioned

above, even with greater indices) will be calculated. Therefore, we use the representation that includes all calculated coefficients of the series contained in the solutions.

4. PROCEDURES FOR CONSTRUCTING LOCAL SOLUTIONS

Algorithms for construction local solutions under consideration are implemented in the computer algebra system Maple [23] as procedures of package EG¹ [7, 13]).

4.1. System Representation

For a differential system $L(y) = 0$ with the infinite series in the role of the coefficients, representation (1) is used, where

- matrix coefficients $A_0(x), A_1(x), \dots, A_r(x)$ are represented by means of standard objects Matrix of the Maple system;

- elements of a matrix coefficient are generally represented as a sum of a polynomial (initial terms of the series) and an infinite power sum specified by means of the standard object Sum of system Maple with an initial value $k_0 \geq 0$ of the summation index; the coefficients of x^k in such a sum can be given by an arbitrary function of index k ; both the polynomial part and the part in the form Sum may be lacking;

¹ The package and a session of Maple with examples of using the procedures described are available at the address <http://www.ccas.ru/ca/doku.php/eg>

```
> EG:-LaurentSolution(sys, theta, y(x), 0);
      [x_ c_1 + O(x^2), -x_ c_1 + O(x^2), -x_ c_1 + O(x^2)]
```

Fig. 2.

- the multiplier $\theta^l y(x)$ is given as `theta(y(x), x, l)`;
- the product of a matrix coefficient and a vector function is denoted by means of the symbol “.” of the standard operation of matrix multiplication used in Maple.

Example 1. Let $m = 3$ and the system be given in form (1) as

$$\begin{pmatrix} x + \sum_{k=3}^{\infty} x^k & 0 & 0 \\ 0 & \sum_{k=1}^{\infty} x^k & 0 \\ 0 & 0 & \sum_{k=1}^{\infty} x^k \end{pmatrix} \theta^2 y(x) + \begin{pmatrix} x^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \theta y(x) + \begin{pmatrix} -1 - x + x^2 - \sum_{k=3}^{\infty} x^k & -x^2 & -1 - x \\ -x^2 & -\sum_{k=1}^{\infty} x^k & -1 - x^3 \\ -x^3 & x & -1 - \sum_{k=1}^{\infty} x^k \end{pmatrix} y(x) = 0.$$

In Maple, this system in the given representation can be as in Fig. 1. This representation is very similar to the original mathematical notation.

In our earlier implementations [2, 7], we represented the differential system $L(y) = 0$ by the operator matrix (4). The operators L_{ij} were considered as power series in x with the coefficients from $K[\theta]$. Recall that the order of each such a coefficient does not exceed the order of the system $L(y) = 0$. If indices i, j are fixed, the operator L_{ij} is given by a function of an integer argument, for example, k , which calculates the coefficient (as a polynomial in θ) of x^k in this operator. For all pairs of indices i, j , these functions can be defined by procedures. In simple cases, functions `if` or `piecewise` are used. This representation is compact: it is required to specify $m \times m$ functions indepen-

dent of the order of the system. However, this representation has its disadvantages as well: it is not quite visual and the order of the system cannot be seen. When searching Laurent and regular solutions, the order is not required. However, it is explicitly used by the algorithm for searching formal solutions. Therefore, for the sake of unification of the system specification by all three algorithms discussed, we turn to the representation described above.

4.2. Laurent Solutions

The algorithm for searching Laurent solutions is implemented as procedure `EG[LaurentSolution]`. In addition to the system, which is given as described in Section 4.1, the procedure has the following three additional arguments:

θ is the name of the operator $x \frac{d}{dx}$,

var is the desired vector function $y(x)$,

d_0 is the least necessary degree of truncation of series in the solution.

Applying the procedure, we obtain a solution of problem \mathbf{P}_L in accordance with its statement in Section 3.1.

Example 2. Let us apply the procedure to the system from Example 1 (see Fig. 2). It can be seen that, in the given case, the degree of the truncation is greater than the requested one (1 rather than 0), since, otherwise, the basis of the solutions could not be determined in accordance with problem \mathbf{P}_L .

4.3. Regular Solutions

The algorithm for searching regular solutions is implemented as procedure `EG[RegularSolution]`. In addition to the system, which is given as described in Section 4.1, the procedure has the same three additional arguments as those in `EG[LaurentSolution]`.

Applying the procedure, we obtain a solution of problem \mathbf{P}_R in accordance with its statement in Section 3.2.

Example 3. Let us apply the procedure to the system from Example 1 (see Fig. 3). Like in Example 2, the degree of the truncation is greater than the requested one (1 rather than 0), since, otherwise, the basis of the solutions could not be determined in accordance with problem \mathbf{P}_R . The Laurent solutions

```
> EG:-RegularSolution(sys,theta,y(x),0);
[ln(x)(x_c1+O(x^2))+x_c2+O(x^2),ln(x)(-x_c1+O(x^2))+_c1+x(-_c2+2_c1)+O(x^2),
ln(x)(-x_c1+O(x^2))-x_c2+O(x^2)]
```

Fig. 3.

```
> Res:=EG:-FormalSolution(sys,theta,y(x),t,'solution_dimension'=6):
Res[1]; Res[2]; Res[3];
[x=t,[ln(t)(t_c1+O(t^2))+t_c2+O(t^2),ln(t)(-t_c1+O(t^2))+_c1+t(-_c2+2_c1)+O(t^2),
ln(t)(-t_c1+O(t^2))-t_c2+O(t^2)]]
[x=t,e^(1/t)[ln(t)O(t^3)-t_c3+O(t^3),ln(t)(t^2_c3+O(t^3))+t^2_c4-t_c3+O(t^3),
ln(t)O(t^3)-t^2_c3-t_c3+O(t^3)]]
[x=t^2,e^(2/t)[sqrt(t)(_c5+O(t)),sqrt(t)O(t),sqrt(t)O(t)]]
```

Fig. 4.

found in Example 2 are contained in the regular solutions found for $_c_1 = 0$.

4.4. Formal Solutions

The algorithm for searching formal solutions is implemented as procedure EG[FormalSolution]. The first three arguments of the procedure are the same as in procedures EG[LaurentSolution] and EG[RegularSolution], and the fourth argument is

τ , name of variable that parameterizes formal solutions (8).

There are also two optional arguments. The order of the optional arguments is not fixed. They are specified by equalities with a key word: 'truncate_solution' = d_0 specifies the minimal necessary degree of series truncation in the solution (by default, $d_0 = 0$); 'solution_dimension' = N determines the lower bound for the dimension of the space of formal solutions (by default, $N = 1$).

Applying the procedure, we obtain a solution of problem P_F in accordance with its statement in Section 3.3.

Example 4. Let us apply the procedure to the system from Example 1 (see Fig. 4). This will result in a list of three elements. The first element of the list Res[1] is the space of regular solutions of the system,

which coincides with that found in Example 3. The dimension of this space is two, the number of arbitrary constants $_c_1, _c_2$. The second element is the space of formal solutions with the exponential part $e^{1/x}$; its dimension is also two. The third element is two spaces of formal solutions with the exponential parts $e^{-2/\sqrt{x}}$ and $e^{2/\sqrt{x}}$, with the dimension of each space being equal to one. The overall dimension of the space of formal solutions found is equal to six, as was requested by means of the argument 'solution_dimension'.

In Section 3, it was shown how the search of regular solutions is used in the construction of formal solutions and how the search of Laurent solutions is used in the construction of regular solutions. Note that the procedure of construction of all formal solutions constructs also all regular, in particular, Laurent solutions. Actually, one procedure EG[FormalSolution] is sufficient in order to obtain solutions of all three types. However, if it is required to construct, say, only Laurent solutions, then it is advantageous to use procedure EG[LaurentSolution], because it will construct them considerably faster, even if the original system has no formal solutions but the Laurent ones. For this reason, we propose three procedures for searching solutions of various types.

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