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Polynomial Relations for Bounds on the Exponents in Solutions to Operator Equations

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Abstract—A general approach to finding the indicial polynomials for differential, difference, and q-difference operators is discussed. The structure of such a polynomial corresponding to the product of operators is considered.

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1. INTRODUCTION

The roots of indicial polynomials of linear ordinary differential, difference, and, q-difference operators (equations) provide important information about the solutions to these operators themselves. This information includes the valuation of the Laurent solution, i.e., the lowest power of x appearing in the series with a nonzero coefficient, the degree of the polynomial solution, etc. The set of roots of an appropriately chosen indicial polynomial gives a finite number of candidates for the corresponding role.

We consider the most common cases of such operators—differential, difference and q-difference ones—and show that they can be considered from a unified point of view and propose a unified approach to obtaining the indicial polynomial. This is achieved by including in the class of objects under consideration the so-called induced recurrent operators, the direct purpose of which is to construct a sequence of coefficients of series that are solutions or components of solutions to the original differential, difference or q-difference equations. It is found that, in addition to this unified approach to obtaining the indicial polynomial, the use of induced operators opens up the possibility of proving various properties of indicial polynomials.

The paper is organized as follows.

Section 2 provides some background on the literature discussion of the possibilities of using indicial polynomials. Section 3 is devoted to induced operators, and the possibility of their use for obtaining indicial polynomials is also discussed here. The indicial polynomials themselves obtained by this approach are described by Theorem 1 in Section 4.

In Section 5, we discuss one of the possible applications of the approach under consideration to constructing indicial polynomials in proofs of their properties. As an example, a proof of the multiplicative property of indicial polynomials is given. For the differential case, this property was previously established in [1]. In this paper, Theorem 2 is proved, which covers three cases: differential, difference, and q-difference ones.

We use come conventional notation: $K[x]$ is the ring of polynomials, $K[[x]]$ is the ring of formal power series, and $K((x))$ is the field or ring of formal Laurent series of x over the given field (or ring) K. Another conventional term in the literature is indicial equation, and this equation itself has the form $f(n) = 0$, where f is a polynomial over a ring or field. As applied to f, we use the term indicial polynomial.

2. ON INDICIAL EQUATIONS AT THE POINTS 0 AND ∞

2.1. Point 0

For a linear ordinary differential operator L for which 0 is a regular singular point, ([2], Chapter IV, Section 2), the Frobenius method ([2], Chapter IV, Section 8 and [3], Section 5.3.3) makes it possible to construct a polynomial f and algorithmically describe all power series $g(x)$ with nonzero free terms such that, for any constant λ, it holds that

$$L(x^{\lambda} g(x)) = f(\lambda) x^{\lambda}.$$

In particular, this allows us to find all solutions of the form

$$x^{\lambda} g(x), \tag{1}$$

in which the values of λ satisfy the indicial equation $f(\lambda) = 0$.

For $L(y) = 0$, we call *Laurent* (including *formal Laurent*) solutions that have the form of the series $\sum_{n=\lambda}^{\infty} c_n x^n \in K((x))$, $\lambda \in \mathbb{Z}$, $c_\lambda \neq 0$. Here λ is called the *valuation* of the Laurent solution. The Laurent solutions are a particular case of solutions (1), and they are constructed using the same indicial equation $f(\lambda) = 0$ as in case (1): if a differential equation has a Laurent solution with a valuation λ, then λ is a root of the indicial equation. Thus, the set of all integer roots of the indicial equation includes the set of valuations of all Laurent solutions to the differential equation.

2.2. Point ∞

In [4], the problem of finding polynomial solutions to the equation $L(y) = 0$, where L is a linear differential or difference operator with polynomial coefficients is considered. That is, L has the form

$$a_d(x)D^d + \ldots + a_1(x)D + a_0(x), \tag{2}$$

or

$$a_d(x)\Delta^d + \ldots + a_1(x)\Delta + a_0(x), \tag{3}$$

where, we have, for a field K of characteristic 0, $a_0(x)$, ..., $a_d(x) \in K[x]$, $a_d(x) \neq 0$, $D = \frac{d}{dx}$, and $\Delta y(x) = y(x + 1) - y(x)$. For solving this problem, we introduce an integer m and a nonzero polynomial $I(n)$ such that the application of the operator L to the polynomial $s(x)$ of degree $\deg s(x)$ gives a polynomial $L(s)$ for which

$$\deg L(s) \leq \deg s(x) + m. \tag{4}$$

Here the coefficient of $x^{\deg s(x) + m}$ is $\mathrm{lc}\, s(x) I(\deg s(x))$, where $\mathrm{lc}\, s(x)$ denotes the coefficient in the polynomial $s(x)$ of the highest degree of x, i.e., the coefficient of $x^{\deg s(x)}$. Thus, for the polynomial solution $y(x)$ of degree $k = \deg y(x)$ to the equation $L(y) = 0$, it holds that $I(k) = 0$. If $I(n)$ has no nonnegative integer roots, then $L(y) = 0$ has no polynomial solutions. Otherwise, the maximum integer root of the polynomial $I(n)$ can be used as an upper bound on the degree of the solution $y(x)$. Formulas for calculating the number m and the polynomial $I(n)$ given the coefficients of the operator L can be found in [4].

In [5], the same approach is used for finding the polynomial solutions to a difference equation with the only distinction that the difference operator L is specified as a polynomial of the shift operator σ: $\sigma y(x) = y(x + 1)$. That is, L has the form

$$\tilde{a}_d(x)\sigma^d + \ldots + \tilde{a}_1(x)\sigma + \tilde{a}_0(x), \tag{5}$$

where $\tilde{a}_0(x), \ldots, \tilde{a}_d(x) \in K[x]$. It is possible to pass from (5) to (3) and then construct $I(n)$. In [5], an algorithm for constructing $I(n)$ given the coefficients $\tilde{a}_0(x)$, ..., $\tilde{a}_d(x)$ is described, which allows us to reduce the calculations that occur when passing from (5) to (3).

In Chapter 2, Section 6 in [6], the differential and difference cases of the equation, as well as the q-difference case are considered in the problem of constructing polynomial solutions: $L(y) = 0$, where L has the form

$$a_d(x)Q^d + \ldots + a_1(x)Q + a_0(x). \tag{6}$$

The operator Q is defined by $Q(y(x)) = y(qx)$, where q is a nonzero element in K such that $|q^n| \neq 1$ for any integer n or $K = K_1(q)$ and the element q of K is transcendental over the field K_1.

In [6], for these three cases, the number m occurring in (4), is denoted by ω and is called the *increment* of the operator L, and $I(n)$ is called the *indicial polynomial* of L. In Section 7.11 of the same book, for the case of differential equations it is noted that the same polynomial $I(n)$ can be used in the consideration of solutions in $K((x^{-1}))$, i.e., of solutions $y(x) = c_\mu x^\mu + c_{\mu-1}x^{\mu-1} + \ldots$, where $c_\mu \neq 0$ ($\mathrm{val}_{x^{-1}} y(x) = \mu$ is called the *valuation of the series in* x^{-1}). If the equation $L(y) = 0$ has a solution with $\mathrm{val}_{x^{-1}} y(x) = \mu$, then μ is a root of $I(n)$, which is denoted by $I_\infty(n)$ in connection with this problem. For the increment, we use the notation ω_∞.

3. INDUCED OPERATORS

3.1. Compatible Bases

For finding polynomial solutions to $L(y) = 0$, it is proposed in [7] to use a polynomial basis compatible with the operator L. A sequence $\mathcal{B} = \langle P_n \rangle_{n=0}^{\infty}$ of polynomials in $K[x]$ such that $\deg P_n = n$ and $P_n | P_m$ for $0 \leq n < m$ is said to be a basis in the space $K[x]$ *compatible* with the operator $L: K[x] \to K[x]$ if there exist $A, B \in \mathbb{Z}_{\geqslant 0}$ such that, for any $n = 0, 1, \ldots$, it holds that $LP_n = \sum_{i=-A}^{B} \alpha_i(n) P_{n+i}$, where $\alpha_i(n) \in K$ for $n = 0, 1, \ldots$, $i = -A, -A+1, \ldots, B$, $\alpha_{-A}(n) \not\equiv 0$ and $P_n = 0$ for $n < 0$.

The set of formal sums $s(x) = \sum_{n=0}^{\infty} c_n P_n$, where $c_n \in K$ for $n \in \mathbb{Z}_{\geqslant 0}$, is a ring, which we denote by $K[[\mathcal{B}]]$. Let $\mathcal{B}$ be a basis compatible with L. For $s(x) \in K[[\mathcal{B}]]$, it holds that $L(s) = 0$ if and only if $\sum_{i=-B}^{A} \alpha_{-i}(n+i) c_{n+i} = 0$ for all $n \geq 0$ under the condition $c_n = 0$ for $n < 0$. It was proved in [7] that a basis compatible with L for the differential and q-difference operator is $\langle x^n \rangle_{n=0}^{\infty}$, and for the difference operator it is $\left\langle \binom{x}{n} \right\rangle_{n=0}^{\infty}$. Formulas for constructing $\alpha_i(n)$ given the coefficients of L for these three cases are presented. Thus, the problem of constructing polynomial solutions $\sum_{n=0}^{\deg y} c_n P_n$ (and solutions in the form of a formal series $\sum_{n=0}^{\infty} c_n P_n$) to the equation $L(y) = 0$ is reduced to calculating the sequence $\langle c_n \rangle_{n=0}^{\infty}$ satisfying the recurrence

$$b_l(n)c_{n+l} + b_{l-1}(n)c_{n+l-1} + \ldots + b_t(n)c_{n+t} = 0, \qquad (7)$$

where $l \geqslant t$, $b_l(n) \neq 0$, $b_t(n) \neq 0$. In the calculation of c_n, it is assumed that $c_n = 0$ for $n < 0$, and in the construction of polynomial solutions, it is also assumed that $c_n = 0$ for all sufficiently large n. The equation $b_l(n - t) = 0$ is indicial in the problem of constructing polynomial solutions (the set of its integer roots contains all degrees of the polynomial solutions to $L(y) = 0$). Here the role of the indicial polynomial is played by $b_l(n - t)$.

3.2. Inducibility with Respect to a Basis

In [8], recurrence (7) is called *induced* with respect to the basis $\mathcal{B}$ for $L(y) = 0$. The *induced operator* for L is the operator

$$L^{\circledR} = b_l(n)E^l + b_{l-1}(n)E^{l-1} + \ldots + b_t(n)E^t, \qquad (8)$$

corresponding to (7), where E is the shift operator: $Ec_n = c_{n+1}$, $E^{-1}c_n = c_{n-1}$. The nonzero coefficients $b_l(n)$ and $b_t(n)$ are called the leading and trailing coefficients of the operator $L^{\circledR}$ and of the relation $L^{\circledR}(c) = 0$, while l and t are called the leading and trailing orders, respectively.

It was shown in [8] the set of all operators compatible with $\mathcal{B}$, which is denoted by $\mathcal{L}_{\mathcal{B}}$, is a K-algebra, and the inducibility transformation, which assigns to $L \in \mathcal{L}_{\mathcal{B}}$ its induced operator $L^{\circledR}$, is an isomorphism; i.e., for any two operators $L_1, L_2 \in \mathcal{L}_{\mathcal{B}}$ and any $a, b \in K$, it holds that

$$(aL_1 + bL_2)^{\circledR} = aL_1^{\circledR} + bL_2^{\circledR} \quad \text{and} \quad (L_1L_2)^{\circledR} = L_1^{\circledR}L_2^{\circledR}.$$

This implies that, if $\mathcal{B}$ is a basis compatible with the operator ξ: $K[x] \to K[x]$ and compatible with the operator x (i.e., with the operator of multiplication by the independent variable x), then $\mathcal{B}$ is compatible with any $L \in K[x, \xi]$ and the transition to the induced operator is specified by two rules for ξ and x.

The induced recurrences can be used for constructing solutions of the form $y(x) = \sum_{n=\nu}^{\infty} c_n P_n$, $\nu \in \mathbb{Z}$, $c_\nu \neq 0$ if it turns out possible to extend the polynomial basis to negative n. In the differential and q-difference cases, the two-sided sequence of rational functions $\langle x^n \rangle_{n\in\mathbb{Z}}$ is suitable for this purpose. It was shown in [9, 10] that in the difference case the sequence $\langle x^{\underline{n}} \rangle_{n\in\mathbb{Z}}$ in falling powers may be used:

$$x^{\underline{n}} = \begin{cases} x(x-1)\ldots(x-n+1), & \text{if} \quad n > 0, \\ 1, & \text{if} \quad n = 0, \\ \dfrac{1}{(x+1)(x+2)\ldots(x+|n|)}, & \text{if} \quad n < 0. \end{cases}$$

The space of all series in $\mathcal{B} = \langle P_n \rangle_{n\in\mathbb{Z}}$ over K, i.e., the series $\sum_{n\in\mathbb{Z}} c_n P_n$, $c_n \in K$, and $c_n = 0$ for all negative n that are sufficiently large in absolute value, will be denoted by $K((\mathcal{B}))$. For the elements of this space, we introduce the concepts of valuation val_+ and the leading coefficient lc_+ similarly to the case of Laurent power series, i.e., to the case $P_n = x^n$, $n \in \mathbb{Z}$. A solution in the form of a series from $K((\mathcal{B}))$ will be called a Laurent solution.

Suppose that, for the equation $L(y) = 0$ with polynomial coefficients and a two-sided basis $\mathcal{B}$ compatible with it, the induced operator is (8). The equation $b_l(n - l) = 0$, where $b_l(n)$ is the leading coefficient of the induced operator and l is its leading order, is said to be the *indicial* equation (and $b_l(n - l)$ is called the indicial polynomial) in the problem of constructing solutions in $K((\mathcal{B}))$. The set of all integer roots of the indicial equation contains all valuations of such solutions.

Similarly, the set of all series $\sum_{n\in\mathbb{Z}} c_n P_n$, $c_n \in K$ such that $c_n = 0$ for all sufficiently large n is denoted by $K((\mathcal{B}))^-$. For the elements of this set, the concepts of valuation val_- and leading coefficient lc_- are introduced. The equation $b_t(n - t) = 0$ is called the *indicial* equation (and $b_l(n - l)$ the indicial polynomial) in the problem of constructing solutions in $K((\mathcal{B}))^-$.

Example 1. For the ring of differential operators in $K[x, D]$ and the basis $\langle x^n \rangle_{n\in\mathbb{Z}}$, the transformation to the induced operator is specified by the rules

$$x \to E^{-1}, \quad D \to (n+1)E.$$

For $L = (-x^2 + 1)D^2 - 2xD + 12$, we obtain using these rules $L^{\circledR} = (-E^{-2} + 1)(n + 1)E(n + 1)E - 2E^{-1}(n + 1)E + 12 = -(n - 1)nc_n + (n + 1)(n + 2)c_{n+2} - 2nc_n + 12c_n = (n + 1)(n + 2)c_{n+2} - (n + 4)(n - 3)c_n$.

Example 2. For the ring of q-difference operators in $K(q)[x, Q]$ and the basis $\langle x^n \rangle_{n\in\mathbb{Z}}$, the transformation to the induced operator is carried out using the rules

$$x \to E^{-1}, \quad Q \to q^n.$$

For $L = (q^6x^3+1)Q^2 - q^{14}$, we obtain $L^{\mathrm{i}} = (q^{2n} - q^{14}) + q^{2n}E^{-3}$.

Example 3. For the difference operators in $K[x,\sigma]$ and the basis $\langle x^n\rangle_{n\in\mathbb{Z}}$, the transformation to the induced operator is carried out using the rules

$$x \to n + E^{-1}, \quad \sigma \to 1 + (n+1)E.$$

For instance, for $L = (x+6)\sigma - (x+1)$, we obtain $L^{①} = (n+6)(n+1)E + (n+5)$.

4. INDUCED RECURRENCES AND INDICIAL POLYNOMIALS

Below, we consider the equation $L(y) = 0$, where $L \in K[x,\xi]$, $\xi \in \{D, Q, \sigma\}$, and the two-sided basis compatible with ξ basis $\mathcal{B} = \langle P_n\rangle_{n\in\mathbb{Z}}$ consisting of rational functions. That is, the conditions formulated in [7–9] are satisfied:

- $P_n \in K[x]$ and $\deg P_n = n$ for $n \geqslant 0$;
- $P_n | P_m$ for $0 \le n < m$;
- $P_n^{-1} \in K[x]$ and $\deg P_n^{-1} = -n$ for $n < 0$;
- $P_n^{-1} | P_m^{-1}$ for $m < n < 0$;
- there exist $A, B \in \mathbb{Z}_{\geqslant 0}$ such that, for any $n \in \mathbb{Z}$, it holds that

$$\xi P_n = \sum_{i=-A_1}^{B_1} \alpha_{1,i}(n) P_{n+i},$$

where $\alpha_{1,i}(n) \in K$ for $n \in \mathbb{Z}$, $i = -A_1, -A_1+1, \ldots, B_1$, $\alpha_{-A_1}(n) \not\equiv 0$.

$$xP_n = \sum_{i=-A_2}^{B_2} \alpha_{2,i}(n) P_{n+i},$$

where $\alpha_{2,i}(n) \in K$ for $n \in \mathbb{Z}$, $i = -A_2, -A_2+1, \ldots, B_2$, $\alpha_{-A_2}(n) \not\equiv 0$.

Examples 1–3 show two-sided bases compatible with $\xi \in \{D, Q, \sigma\}$.

Theorem 1. *Let $\mathcal{B} = \langle P_n\rangle_{n\in\mathbb{Z}}$ be a two-sided basis compatible with ξ and $L \in K[x,\xi]$. Then, there exist*

(*i*) $\omega_{L,+} \in \mathbb{Z}$ *and* $I_{L,+} \in K[\lambda]$ *such that, for an arbitrary* $s(x) \in K((\mathcal{B}))$, $\mathrm{val}_+ s(x) = \nu$, $\mathrm{tc}_+ s(x) = c_\nu$, *and it holds that*

$$L(s) = c_\nu \cdot I_{L,+}(\nu) P_{\nu+\omega_{L,+}} + \ldots, \tag{9}$$

where the ellipsis denotes a series in $K((\mathcal{B}))$ *with a valuation* val_+ *greater than* $\nu + \omega_{L,+}$;

(*ii*) $\omega_{L,-} \in \mathbb{Z}$ *and* $I_{L,-} \in K[\lambda]$ *such that, for any* $u(x) \in K((\mathcal{B}))^-$, $\mathrm{val}_- u(x) = \mu$, $\mathrm{tc}_- u(x) = d_\mu$, *and it holds that*

$$L(u) = d_\mu \cdot I_{L,-}(\mu) P_{\mu+\omega_{L,-}} + \ldots, \tag{10}$$

where the ellipsis denotes a series in $K((\mathcal{B}))^-$ *with a valuation* val_- *less than* $\mu + \omega_{L,-}$.

Proof. Let the induced operator $L^{①}$ for L with respect to the compatible basis $\mathcal{B}$ have form (8). The application of L to the two-sided series $s(x) = \sum_{n=-\infty}^{\infty} c_n P_n$ gives the series $\bar{s}(x) = \sum_{n=-\infty}^{\infty} \bar{c}_n P_n$. The two-sided sequence $\bar{c}_n$ is obtained by applying the operator $L^{①}$ to the sequence c_n:

$$\bar{c}_n = b_l(n)c_{n+l} + \cdots + b_t(n)c_{n+t}, \quad n \in \mathbb{Z}.$$

Thus, we have

$$L(s) = \sum_{n=-\infty}^{\infty} (L^{①}(c))_n P_n.$$

Let $\mathrm{val}_+ s(x) = \nu$ and $\mathrm{tc}_+ s(x) = c_\nu$. Since $c_n = 0$ for $n < \nu$, we have $(L^{①}(c))_n = 0$ for all $n < \nu - l$ and $(L^{①}(c))_{\nu-l} = b_l(\nu - l)c_\nu$. That is, assertion (i) of the theorem at $\omega_{L,+} = -l$, $I_{L,+}(\lambda) = b_l(\lambda + \omega_{L,+})$ is proved.

In the case $u(x) = u_\mu P_\mu + u_{\mu-1}P_{\mu-1} + \ldots$, $u_\mu \ne 0$, since $d_n = 0$ for $n > \mu$, $(L^{①}(d))_n = 0$ for all $n > \mu - t$ and $(L^{①}(c))_{\mu-t} = b_t(\mu - t)c_\mu$. This proves assertion (ii) of the theorem for $\omega_{L,-} = -t$, $I_{L,-}(\lambda) = b_t(\lambda + \omega_{L,-})$.

The search for Laurent solutions with respect to the basis compatible with ξ is also valid in the case of an operator with coefficients in the form of series: $L \in K[[x]][\xi]$. In the general case, the induced operator $L^{①}$ has an infinite order and has the form

$$b_l(n)E^l + b_{l-1}(n)E^{l-1} + \ldots.$$

For such an operator, there exist $\omega_{L,+} \in \mathbb{Z}$ and $I_{L,+} \in K[\lambda]$ as in Theorem 1 (i).

Multiplication by x^{-1} is an operator compatible with $\langle x^n\rangle_{n\in\mathbb{Z}}$: $x^{-1} \to E$. Therefore, for a ξ compatible with $\langle x^n\rangle_{n\in\mathbb{Z}}$ (e.g., $\xi \in \{D, Q\}$), one can search Laurent solutions in $K((x^{-1}))$ for the equation $L(y) = 0$, where $L \in K((x^{-1}))[\xi]$. In this case, the induced operator $L^{①}$ generally has the form

$$\ldots + b_{t+1}(n)E^{t+1} + b_t(n)E^t,$$

and there exist $\omega_{L,-} \in \mathbb{Z}$ and $I_{L,-} \in K[\lambda]$ as in Theorem 1 (ii).

The proof given for the case of L with polynomial coefficients is also valid in these two cases.

Pay attention to the fact that the polynomial $I_{L,+}(\lambda)$ in (9) (and $I_{L,-}(\lambda)$) in (10)) is the same for any $s(x)$ ($u(x)$) of the form indicated above, and this polynomial is completely determined by the operator L and the basis $\mathcal{B}$ compatible with ξ.

The following definition essentially generalizes the definitions given in [2, 4, 6–8].

Definition 1. The polynomial $I_{L,+}(\lambda)$ in (9) is said to be the *indicial polynomial* of the operator L for the problem of finding solutions in $K((\mathcal{B}))$. The polynomial $I_{L,-}(\lambda)$ in (10) is called the *indicial polynomial* for the operator L for the problem of finding solutions in $K((\mathcal{B}))^-$.

Example 4. For the differential operator L in Example 1 in the problem of constructing solutions to $L(y) = 0$ in $K((x))$, the indicial equation is $\lambda(\lambda - 1) = 0$. The set $\{0, 1\}$ of its integer roots contains all valuations val_+ of solutions. The coefficients of these solutions are successively calculated using $L^{①}(c) = 0$ beginning from the lower bound of valuations for any given n; i.e., in this example $c_0, c_1, \ldots, c_n$ are successively calculated. Let us write out the solution up to x^4:

$$C_1 + C_2 x - 6C_1 x^2 - \frac{5}{3}C_2 x^3 + 3C_1 x^4 + \ldots,$$

where C_1, C_2 are arbitrary constants.

In the problem of constructing solutions in $K((x^{-1}))$, the indicial equation is $-(\lambda - 3)(\lambda + 4) = 0$. The set $\{-4, 3\}$ of its integer roots contains all valuations val_- of solutions. The coefficients of these solutions are successively calculated using $L^{①}(c) = 0$ beginning from the upper bound of valuations up to any given n; i.e., $c_3, c_2, \ldots, c_n$ are successively calculated. Let us write out the solutions up to x^{-4} for this equation:

$$C_1 x^3 - \frac{3}{5}C_1 x + C_2 x^{-4} + \ldots,$$

where C_1, C_2 are arbitrary constants. The equation $L(y) = 0$ also has polynomial solutions $C_1 x^3 - \frac{3}{5}C_1 x$.

Example 5. For the q-difference operator L in Example 2 in the problem of constructing solutions in $K((x))$, the indicial equation $(q^\lambda - q^7)(q^\lambda + q^7) = 0$ has a single integer root $\lambda = 7$. Let us write out the initial terms of the Laurent solution for $L(y) = 0$ up to x^{13}:

$$C\left(x^7 - \frac{q^{20}x^{10}}{q^{20} - q^{14}} + \frac{q^{46}x^{13}}{(q^{26} - q^{14})(q^{20} - q^{14})} + \ldots\right),$$

where C is an arbitrary constant.

In the problem of constructing solutions in $K((x^{-1}))$, the indicial equation $q^6 q^{2\lambda} = 0$ has no integer roots. Therefore, the equation $L(y) = 0$ has no nonzero solutions in $K((x^{-1}))$ and has no polynomial solutions.

Example 6. For the difference operator L in Example 3 and the basis $\mathcal{B} = \langle x^{\underline{n}} \rangle_{n\in\mathbb{Z}}$, the indicial polynomial $I_{L,+}(\lambda) = (\lambda + 5)\lambda$ has two integer roots $\{-5, 0\}$. The initial terms of the solutions in $K((\mathcal{B}))$ for $L(y) = 0$ are

$$C_1 x^{\underline{-5}} + C_2 x^{\underline{0}} - \frac{5}{6}C_2 x^{\underline{1}} + \ldots,$$

where C_1, C_2 are arbitrary constants.

In the problem of constructing solutions in $K((\langle x^{\underline{n}} \rangle_{n\in\mathbb{Z}}))^-$, the indicial polynomial $I_{L,-}(\lambda) = (\lambda + 5)$ has a single root, and it is associated with the solutions

$$C_1 x^{\underline{-5}} + \ldots$$

Here the ellipsis denotes the terms of the series beginning with $x^{\underline{n}}$ for $n < -5$. It is easy to verify that all coefficients in these terms are zero: $c_n = 0$, $n < -5$. That is, the equation $L(y) = 0$ has solutions in the form of the Laurent polynomial

$$C_1 x^{\underline{-5}} = \frac{C_1}{(x+5)(x+4)(x+3)(x+2)(x+1)}.$$

Remark 1. The indicial equation obtained using the induced operator in the differential case coincides with the equation given by the Frobenius method (Section 2.1), which can be confirmed by the straightforward verification. In this sense, the induced operators help establish a relationship between the indicial equations for the difference and q-difference cases with the classical differential case.

5. INDICIAL POLYNOMIALS OF THE PRODUCT OF OPERATORS

Theorem 2 (On the multiplicative property of the indicial polynomial). *Let $\mathcal{B}$ be a two-sided basis compatible with the operator ξ. Furthermore, let L_1, $L_2 \in K[x,\xi]$. Then, for ω_*, $I_*(\lambda)$, where $* \in \{+, -\}$, it holds that*

(*i*) $\omega_{L_1L_2,*} = \omega_{L_1,*} + \omega_{L_2,*}$;

(*ii*) $I_{L_1L_2,*}(\lambda) = I_{L_1,*}(\lambda + \omega_{L_2,*}) I_{L_2,*}(n)$.

Proof. Let the induced operators for L_1 and L_2 be

$$L_1^{①} = b_{1,l_1}(n)E^{l_1} + \ldots + b_{1,t_1}(n)E^{t_1},$$

$$L_2^{①} = b_{2,l_2}(n)E^{l_2} + \ldots + b_{2,t_2}(n)E^{t_2}.$$

The proof of Theorem 1 implies that

$$l_i = -\omega_{L_i,+} \quad \text{and} \quad b_{i,l_i}(n) = E^{-\omega_{L_i,+}} I_{L_i,+}(n),$$

$$t_i = -\omega_{L_i,-} \quad \text{and} \quad b_{i,t_i}(n) = E^{-\omega_{L_i,-}} I_{L_i,-}(n),$$

$i = 1, 2$. It is known from [8] that

$$(L_1L_2)^{\textcircled{1}} = L_1^{\textcircled{1}}L_2^{\textcircled{1}},$$

Expand the product on the right-hand side of the last equality to obtain

$$(E^{-\omega_{L1,+}}I_{L_1,+}(n) + \ldots + E^{-\omega_{L1,-}}I_{L_1,-}(n))$$
$$\times(E^{-\omega_{L2,+}}I_{L_2,+}(n) + \ldots + E^{-\omega_{L2,-}}I_{L_2,-}(n))$$
$$= \sigma^{-\omega_{L1,+}-\omega_{L2,+}}I_{L_1,+}(n + \omega_{L_2,+})I_{L_2,+}(n) + \ldots$$
$$+ \sigma^{-\omega_{L1,-}-\omega_{L2,-}}I_{L_1,-}(n + \omega_{L_2,-})I_{L_2,-}(n).$$

This implies (i) and (ii).

It follows from this proof of Theorem 2 that the indicial polynomial has the multiplicative property formulated in (ii).

Corollary 1. Let $L_1,\ldots,L_m \in K[x,\xi]$ and $\mathcal{B}$ be a basis compatible with ξ. Then for $* \in \{+,-\}$, we have

(i) $\omega_{L_1\ldots L_m,*} = \omega_{L_1,*} + \ldots + \omega_{L_m,*}$;

(ii) $I_{L_1\cdots L_m,*}(\lambda)\prod_{i=1}^{m} I_{L_i,*}(\lambda + \sum_{j=i+1}^{m}\omega_{L_j,*})$.

Example 7. Let us construct for the differential operator L defined in Examples 1 and 4 the indicial polynomials for L^5. To this end, there is no need to perform exponentiation of the operator and construct the induced operator for L^5. Using Corollary 1, we obtain

$$\omega_{L^5,+} = 5\omega_{L,+} = -10;$$

$$I_{L^5,+}(\lambda) = \prod_{i=1}^{5} I_{L,+}(\lambda + (5-j)\omega_{L,+})$$
$$= \prod_{i=1}^{5}(\lambda - 2(5-j))(\lambda - 1 - 2(5-j))$$
$$= \prod_{i=0}^{9}(\lambda - i) = \lambda^{\underline{10}};$$

$$\omega_{L^5,-} = 5\omega_{L,-} = 0;$$

$$I_{L^5,-}(\lambda) = \prod_{i=1}^{5} I_{L,-}(\lambda + (5-j)\omega_{L,-})$$
$$= \prod_{i=1}^{5}(3-\lambda)(\lambda+4) = (3-\lambda)^5(\lambda+4)^5.$$

Example 8. Let

$$L_1 = (q^6x^3 + 1)Q^2 - q^{14},$$

$$L_2 = (x^2 + q)Q - (q^2x^2 + q).$$

The indicial equations for L_1 are given in Example 5. According to the rules formulated in Example 2, the induced operator for L_2 is

$$L_2^{\textcircled{1}} = (-q + qq^n) + \left(-q^2 + \frac{q^n}{q^2}\right)E^{-2}.$$

Then we obtain

$$\omega_{L_1,+} = 0, \quad I_{L_1,+}(\lambda) = (q^\lambda - q^7)(q^\lambda + q^7),$$
$$\omega_{L_1,-} = 3, \quad I_{L_1,-}(\lambda) = q^6q^{2\lambda},$$
$$\omega_{L_2,+} = 0, \quad I_{L_2,+}(\lambda) = q(q^\lambda - 1),$$
$$\omega_{L_2,-} = 2, \quad I_{L_2,-}(\lambda) = q^\lambda - q^2.$$

Using Theorem 2, we obtain for the product of operators L_1L_2 the expression

$$\omega_{L_1L_2,+} = 0,$$
$$I_{L_1L_2,+}(\lambda) = q(q^\lambda - 1)(q^\lambda - q^7)(q^\lambda + q^7),$$
$$\omega_{L_1L_2,-} = 5,$$
$$I_{L_1L_2,-}(\lambda) = q^{10}q^{2\lambda}(q^\lambda - q^2).$$

Example 9. Let

$$L_1 = (x+6)\sigma - (x+1),$$

$$L_2 = x(x-1)\sigma^2 + 1.$$

The indicial equations for L_1 are given in Example 6. According to the rules formulated in Example 3, the induced operator for L_2 is

$$L_2^{\textcircled{1}} = n(-1+n)(n+2)(n+1)E^2$$
$$+ 4n(n-1)(n+1)E + (6n^2 - 6n + 1)$$
$$+ 4(n-1)E^{-1} + E^{-2}.$$

Then we obtain

$$\omega_{L_1,+} = -1, \quad I_{L_1,+}(\lambda) = (\lambda+5)\lambda,$$
$$\omega_{L_1,-} = 0, \quad I_{L_1,-}(\lambda) = (\lambda+5),$$
$$\omega_{L_2,+} = -2, \quad I_{L_2,+}(\lambda) = (\lambda-3)(\lambda-2)(\lambda-1)\lambda,$$
$$\omega_{L_2,-} = 2, \quad I_{L_2,-}(\lambda) = 1.$$

Using Theorem 2, we obtain for the product of operators L_1L_2 the expression

$$\omega_{L_1L_2,+} = -3,$$
$$I_{L_1L_2,+}(\lambda) = (\lambda-3)(\lambda-2)^2(\lambda-1)\lambda(\lambda+3),$$
$$\omega_{L_1L_2,-} = 2,$$
$$I_{L_1L_2,-}(\lambda) = (\lambda+7).$$

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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