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Laurent, Rational, and Hypergeometric Solutions of Linear *q*-Difference Systems of Arbitrary Order with Polynomial Coefficients

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Abstract—Systems of linear q-difference equations with polynomial coefficients are considered. Equations in the system may have arbitrary orders. For such systems, algorithms for searching polynomial, rational, and hypergeometric solutions, as well as solutions in the form of Laurent series, are suggested. Implementations of these algorithms are discussed.

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1. INTRODUCTION

Algorithms related to q-difference equations are of interest in many fields of mathematics [1], especially in combinatorics and the partition theory [2, 3]. In what follows, we consider systems of linear q-difference equations with coefficients from $\mathbb{K}[x]$, where $\mathbb{K} = K(q)$, K is a field of characteristic 0, q is transcendental over K, x denotes q^k, and k is a variable taking values in $\mathbb{Z}_{\geq 0}$. The system under consideration has the form

$$A_r(x)y(q^r x) + \ldots + A_1(x)y(qx) + A_0(x)y(x) = b(x), \quad (1)$$

where

- $A_0(x), A_1(x), \ldots, A_r(x)$ are $m \times m$ matrices with entries from $\mathbb{K}[x]$ (this is denoted as $A_0(x), A_1(x), \ldots, A_r(x) \in \mathrm{Mat}_m(\mathbb{K}[x])$); matrices $A_0(x)$ and $A_r(x)$ are assumed to be nonzero;
- $b(x) = (b_1(x), b_2(x), \ldots, b_m(x))^T \in \mathbb{K}[x]^m$ is the right-hand side of the system; and
- $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T$ is the column vector of the unknowns.

The number r is the *order* of the system. The homogeneous system corresponding to (1) is given by

$$A_r(x)y(q^r x) + \ldots + A_1(x)y(qx) + A_0(x)y(x) = 0. \quad (2)$$

Systems (1) and (2) can be written in the operator form as $L(y) = b(x)$ and $L(y) = 0$, where

$$L = A_r(x)\sigma_q^r + \ldots + A_1(x)\sigma_q + A_0(x), \quad (3)$$

σ_q is the operator of q-shift: $\sigma_q y(x) = y(qx)$. Matrices $A_r(x)$ and $A_0(x)$ are called *leading* and *trailing* matrices of systems (1), (2), and operator (3).

One of the generally accepted computer algebraic approaches to searching solutions of linear systems of equations is the cyclic vector method. This method transforms the system to a scalar equation that is equivalent, in some sense, to the original system. One of the difficulties associated with this transformation is the overgrowth of coefficients, due to which the method works only for small-order systems. The last circumstance stimulates development of direct (without constructing a scalar equation) methods and algorithms.

In this paper, we consider direct algorithms for searching solutions to systems of form (1), the components of which $y_1(x), \ldots, y_m(x)$ belong either to the field $\mathbb{K}((x))$ of formal Laurent series (in particular, the ring $\mathbb{K}[[x]]$ of formal power series) over $\mathbb{K}$, or to the ring $\mathbb{K}[x]$ of polynomials over $\mathbb{K}$, or to the field of rational functions $\mathbb{K}(x)$ over $\mathbb{K}$. The equations of the corresponding homogeneous system (2), as well as those of the original system, are assumed independent over $\mathbb{K}[x, \sigma_q, \sigma_q^{-1}]$ (such systems are referred to as systems of *full rank*).

A solution $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T \in \mathbb{K}((x))^m$ of the system is called a *Laurent* solution, and a solution $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T \in \mathbb{K}(x)^m$ is called a *rational* solution. If $y(x) \in \mathbb{K}[x]^m$, then this solution is called a *polynomial* solution (which is a particular case of the rational solution).

Remark 1. We could confine our consideration to the case of a homogeneous system, since a nonhomogeneous system with a polynomial right-hand side can easily be reduced to a homogeneous system by applying the expedient described, in particular, in [4]: given a nonhomogeneous system, one can construct an equivalent homogeneous system by adding one additional equation in a constant function and using components of the right-hand side of the original system as coefficients of this new function.

Search for Laurent solutions to differential systems was considered in [5, 6]; for difference systems, tropical calculations were used in [7]. Search for rational solutions of both scalar linear difference equations and systems of equations, together with relevant issues, was discussed, for example, in [8–17].

For the q-difference case, there exist algorithms (see, for example, [18–20]) for searching all rational solutions of scalar linear equations, as well as of normal linear systems of the first order, i.e., systems of the form

$$y(qx) = A(x)y(x), \tag{4}$$

where $A(x)$ is a nonsingular matrix in $\mathrm{Mat}_m(\mathbb{K}(x))$. Search for rational solutions of linear q-difference equations and systems is of interest by itself and as a part of various computer algebra algorithms (see, e.g., Section 6 of this paper).

Singularity of the leading or trailing matrices creates difficulties in searching solutions of the system (if we rewrite system (4) as $I_m y(qx) - A(x)y = 0$, where I_m is the identity $m \times m$ matrix, then the trailing matrix of this system is $-A(x)$). The same refers to the case of singular leading or trailing matrices of the so-called induced recurrence (difference) system: if the formal Laurent series $\sum a_n x^n$, $a_n \in \mathbb{K}^m$, satisfies the original q-difference system, then the sequence (a_n) of m-dimensional vectors satisfies this induced recurrence system. In Section 2, we present an algorithm of EG-eliminations, which transforms an original q-difference or induced recurrence system to that with a nonsingular trailing or leading matrices. Such a transformation of an induced recurrence system and subsequent finding of the determinant of the leading matrix allows one to easily find a lower bound of valuations of formal Laurent solutions of the original system. An upper bound for the degree of polynomial solutions is calculated by using the nonzero determinant of the trailing matrix of the induced recurrence system (see Section 2.2).

The search of rational solutions is implemented in two stages: (1) search of the so-called universal denominator and (2) search of the corresponding numerators of the solution components. The numerators are found as components of polynomial solutions of the system obtained from the original one by means of a special substitution involving the universal denominator. Construction of the universal denominator on the initial stage makes use of the leading and trailing matrices of the original q-difference system after application of the EG-eliminations. This stage allows one to obtain all factors of the universal denominator different from the factor x^k. After this, an appropriate exponent k can be found by treating the rational solution as a formal Laurent solution and determining a lower bound of valuations of such solutions. This is discussed in Section 5.

The very first algorithm and an example of constructing polynomial solutions of q-difference systems of an arbitrary order were presented in [21, Section 3.6]. As for universal denominators, it should be noted that, strictly speaking, only first-order systems are considered in paper [19]. However, this paper also formulates general principles allowing one to find rational solutions of higher-order systems. For instance, it is noted in [19] that, to construct a part of a universal denominator that contains only factors different from x, one can take advantage of a slightly modified version (with the substitution $x + i \to xq^i$) of the algorithm designed for the difference case. An algorithm for difference systems of an arbitrary order was proposed, for example, in [22]. The idea to consider rational solutions as Laurent solutions to deal with x^k in the denominator was also proposed in [19]. In the current paper, we follow this plan to obtain an algorithm for searching rational solutions to systems of form (1) (assuming that the system has full rank). For searching polynomial solutions, we use the algorithm from [23], which is also based on EG-eliminations (see Section 4).

Thus, the discussed algorithms for constructing rational solutions of linear q-difference systems systematically rely on EG-eliminations. Similar algorithms we suggested earlier for linear differential and difference systems of an arbitrary order [21, 22]. (For first-order linear difference systems $y(x+1) = A(x)y(x)$ with a nonsingular matrix $A(x)$, an algorithm was proposed in [9] that relied on super-irreducibility rather than on EG-eliminations.) J. Middeke [24] showed that bounds for the exponent k in factors of the form x^k and for the exponents of polynomial solutions can be found also with the help of the Popov normal form (see [25, 26]). Note that no comparison of various approaches was carried out in [24].

In Section 6, we consider search of hypergeometric solutions, i.e., solutions each component $y_i(x)$ of which is a finite sum of hypergeometric terms, i.e., terms $h(x)$ such that $h(qx) = r(x)h(x)$ for some rational function $r(x)$. For scalar linear q-difference equations, the problem was first considered in [27], where algorithm qHyper was suggested, which is a q-version of the algorithm by M. Petkovsek for the difference case [28]. (For the difference case, there are also algorithms described in [29, 30].) The proposed algorithm relies on the scalar algorithm, EG-eliminations, and resolving sequences of scalar operators proposed in [31]. The resolving sequences reduce the search of solutions of a system to the search of solutions of several scalar equations. Results of experiments described in [31] show that they are more efficient in terms of the time spent than the cyclic vector we mentioned earlier.

In Section 7, we discuss implementations of the proposed algorithms in Maple [32]. It is worth noting that detailed versions of the EG-eliminations designed for the differential and difference cases were proposed earlier, whereas the case of the q-difference system was noted without going into detail in [33, 34] as that where the EG-eliminations could be applied. (The above-mentioned example of constructing a polynomial solution to a q-difference system was obtained by applying EG-eliminations to the recurrent induced system.) In this paper, we fill this gap. Note that the results reported in this paper were announced in the extended abstracts [35, 36].

2. PRELIMINARIES

2.1. Embracing Systems

For any full-rank system S of form (1), one can construct an l-*embracing* system $\bar{S}$

$$\bar{A}_r(x)y(q^r x) + \ldots + \bar{A}_1(x)y(qx) + \bar{A}_0(x)y(x) = \bar{b}(x), \quad (5)$$

the leading matrix of which is invertible in $\mathrm{Mat}_m(\mathbb{K}(x))$ and the set of solutions contains all solutions of system S. Similarly, one can construct a t-*embracing* system $\bar{\bar{S}}$

$$\bar{\bar{A}}_r(x)y(q^r x) + \ldots + \bar{\bar{A}}_1(x)y(qx) + \bar{\bar{A}}_0(x)y(x) = \bar{\bar{b}}(x), \quad (6)$$

whose trailing matrix is invertible in $\mathrm{Mat}_m(\mathbb{K}(x))$ and the set of solutions contains all solutions of system S. In addition, if entries of matrices in (1), (2) and entries of the right-hand side of system (1) belong to $\mathbb{K}[x]$, then this property holds for systems (5), (6). The case where matrices $\bar{A}_0(x)$, $\bar{\bar{A}}_r(x)$ are zero (either one of them or both) is not excluded.

The embracing systems can be constructed by means of the EG-eliminations ([21, 33, 37]).

Remark 2. If $\bar{S}$ and $\bar{\bar{S}}$ are l- and t-embracing systems, respectively, constructed by EG-eliminations for system (1), then the l- and t-embracing systems constructed by EG-eliminations for system (2) coincide with the homogeneous systems corresponding to $\bar{S}$ and $\bar{\bar{S}}$.

The EG-elimination algorithm consists in the successive repetition of two—*reduction* and *shift*—stages, which continues until the rows of the leading (trailing) matrix remain linear dependent over $\mathbb{K}(x)$. On the reduction stage, coefficients $p_1, \ldots, p_m$ of the dependence are found; then, the equation corresponding to one of the dependent rows is replaced with the linear combination of the other equations, and the row of the leading (trailing) matrix is set zero. On the shift stage, operator σ_q (or, respectively, σ_q^{-1}) is applied to the new equation. The termination of the algorithm operation is guaranteed owing to using a certain simple rule when selecting equations to be replaced. The rule is as follows. Suppose that system S has form (1) and let the ith rows of matrices $A_0, \ldots, A_{k-1}$ be zero and that of matrix A_k be nonzero. The number k is called the *lower order* of the ith equation of system S. Accordingly, if the ith rows of matrices $A_r, A_{r-1}, \ldots, A_{l+1}$ are zero and the ith row of matrix A_l is nonzero, l is referred to as the *upper order* of the ith equation of system S. In the course of obtaining the system with a nonsingular leading matrix by means of the EG-eliminations, among the equations corresponding to the nonzero p_i, we select that with the least lower order (if there are several such equations, we take any of them). When obtaining the system with a nonsingular trailing matrix, the equation with the greatest upper order is selected.

This algorithm can also be applied to difference (recurrence) Section 2.2 systems. In this case, operators σ_q, σ_q^{-1} are replaced with σ, σ^{-1}: $\sigma f(x) = f(x+1)$, $\sigma^{-1} f(x) = f(x-1)$. The importance of this feature is discussed in Subsection 2.2. For such systems, we are first of all interested in *sequential* solutions, i.e., solutions in the form of sequences. Apart of constructing the l- or t-embracing systems, the EG-elimination algorithm can additionally construct a finite set of *linear constraints*, each of which involves a finite number of elements of a sequential solution and is a linear combination of these elements with constant coefficients. Any sequential solution of the original difference system satisfies the constructed linear constraints and, if the solution of the l- or t-embracing system satisfies all constructed linear constraints, is also a solution of the original difference system (i.e., it is not a "redundant" sequential solution of the corresponding embracing system).

Remark 3. One of the variants of this algorithm does not require special selection of the equation to be replaced. On the shift stage, σ_q is replaced with $\sigma_q - 1$ (in the difference case, σ is replaced with $\sigma - 1$). Each shift stage increases the dimension of the space of the solutions that have components in some special extension of the field $\mathbb{K}$ by one. This dimension cannot be greater than rm, which guarantees termination of the algorithm. The change of the dimension of the spaces of solutions the components of which belong to the so-called adequate extensions is discussed in [38].

Remark 4. In certain cases, for example, when constructing a resolving sequence of operators for system S, it is required to construct an lt-embracing system, i.e., an embracing system with nonsingular leading and trailing matrices. This can be done as follows: construct a t-embracing system $\bar{\bar{S}}$ by means of EG-eliminations, and, then, construct an l-embracing systems for $\bar{\bar{S}}$ by means of the EG-eliminations that use operator $\sigma_q - 1$ on the shift stage (see Remark 3).

2.2. Induced Recurrence Systems

If a formal Laurent series $\sum a_n x^n$, $a_n \in \mathbb{K}^m$, satisfies the original q-difference system (1), then the sequence (a_n) of m-dimensional vectors satisfies the *induced* recurrence system

$$P_l(n)a_{n+l} + \ldots + P_t(n)a_{n+t} = c_n, \tag{7}$$

where (c_n) is a sequence of the coefficients of the expansion of the right-hand side $b(x)$ of the original q-difference system into a series. The induced system can be obtained from the original one in the following three stages:

(i) rewrite the original system in the operator–matrix form $My = b$, where $M \in \operatorname{Mat}_m(\mathbb{K}[x, \sigma_q])$;

(ii) in the matrix M, make the substitution

$$\sigma_q \to q^n, \quad x \to \sigma^{-1},$$

where σ is the shift operator, $\sigma f_n = f_{n+1}$, for any two-sided sequence (f_n);

(iii) rewrite the system obtained in form (7).

In what follows, considering elements of the ring $\mathbb{K}[[x]]$ and field $\mathbb{K}((x))$, we will need the concept of series valuation: for a nonzero formal power or Lauren series $f(x) = \sum f_i x^i$, its *valuation* $\operatorname{val} f(x)$ is the integer

$$\min\{i \in \mathbb{Z} \mid f_i \neq 0\},$$

with $\operatorname{val} f(x) = \infty$ for the zero series $f(x)$. The valuation of a vector whose entries are series is the minimal valuation of the entries of this vector.

Accordingly, the degree of the vector whose entries are polynomials is the maximal degree of its entries. The degree of the zero polynomial, as well as the degree of the zero vector of polynomials, is equal to $-\infty$.

If the induced system (7) has a nonsingular leading matrix $P_l(n)$, i.e., $\det P_l(n)$ is a nonzero polynomial of q^n, then the valuation of all possible Laurent solutions of the original q-difference system can be lower estimated; if $\det P_t(n)$ is a nonzero polynomial of q^n, then the degrees of all possible polynomial solutions of the original q-difference system can be upper estimated. In some cases, we can immediately find out that Laurent solutions or, respectively, polynomial solutions are lacking

The following theorem is a combined variant of Theorem 1 and 2 from [19].

Theorem 1. *Let the recurrent system* (7) *satisfiy the condition that, if a formal Laurent series* $\sum a_n x^n$, $a_n \in \mathbb{K}^m$ (*in particular, a polynomial over* $\mathbb{K}^m$), *satisfies the original q-difference system* (1), *then the sequence* (a_n) *of m-dimensional vectors satisfies* (7). *Let* $p_l(n) = \det P_l(n)$ *and* $p_t(n) = \det P_t(n)$ (*thus*, $p_l(n)$, $p_t(n)$ *are polynomials in* q^n). *In this case*,

(i) *if the right-hand side* $b(x)$ *is Laurent and, additionally*,

- $p_l(n)$ *is a nonzero polynomial in* q^n,
- N_l *is a set* (*possibly, empty*) *of all integer roots of the equation* $p_l(n) = 0$,
- *the number* β *does not exceed the valuation of the right-hand side of the original q-difference system* ($\beta = \infty$ *when the right-hand side* $b(x)$ *is a zero column vector*),

then the valuation of any Laurent solution of system (1) *cannot be less than*

$$\min(N_l \cup \{\beta\}) + l, \tag{8}$$

(ii) *if the right-hand side* $b(x)$ *is polynomial and, additionally*,

- $p_t(n)$ *is a nonzero polynomial in* q^n,
- N_t *is a set* (*possibly, empty*) *of all integer roots of the equation* $p_t(n) = 0$,
- *the number* γ *is not less than the degree of the right-hand side of the original q-difference system* ($\gamma = -\infty$ *when the right-hand side* $b(x)$ *is a zero column vector*),

then the degree of any polynomial solution of system (1) *does not exceed*

$$\max(N_t \cup \{\gamma\}) + t. \tag{9}$$

If the leading or, respectively, trailing matrix of the induced system is singular, then we can first apply the required version of the EG-eliminations and, then, by means of Theorem 1, find the desired estimates of valuations and degrees and construct Laurent and polynomial solutions. This is discussed in Sections 3 and 4.

3. LAURENT SOLUTIONS

To search for Laurent solutions when the leading matrix of the induced system is singular, we construct an l-embracing recurrence system by means of EG-eliminations. In so doing, a finite set of linear constraints arises.

The expansion of a Laurent solution in a series is presented by initial terms, the number of which is selected such that, first, the subsequent terms can be calculated without regard to the linear constraints and, second, the determinant of the leading matrix of the l-embracing recurrence system does not vanish in the course of this calculation. Having found the lower bound of solution valuations by means of formula (8), we can write down the system of linear algebraic equations for these terms and to solve it. The subsequent terms can be obtained by means of the l-embracing recurrence system found.

The problem of convergence of the series is not discussed.

4. POLYNOMIAL SOLUTIONS

The upper bound for degrees of all polynomial solutions of system (1)can be obtained by formula (9). If the trailing matrix of the induced system (7) used in these calculations is singular, then EG-eliminations are first applied to construct a t-embracing system of the same form (7), and the calculations by formula (9) are carried out for this system

After the upper bound ρ of the degrees has been determined, to find the polynomial solutions themselves, one can apply not only the method of undetermined coefficients but also a more efficient method of constructing coefficients of the polynomial solutions with the help of the induced recurrence system [23].

The recurrence system allows one to calculate the coefficients either in the forward direction, from the lower-order coefficients to the leading ones, or in the backward direction, from the leading coefficients to the lower-order ones. In the calculation of a next coefficient in the case of the forward direction, the leading matrix plays the key role; in the backward direction, the key role is played by the trailing matrix. Since, in the general case, a t-embracing system is used for searching the bound of the degree of the polynomial solution, it is reasonable to use the same t-embracing system for searching the polynomial solution as well in view of the fact that it is convenient to use a nonsingular matrix for calculating the coefficients

Let us rewrite the constructed t-embracing system (7) in the form

$$P_t(n)a_{n+t} = c_n - P_{t+1}(n)a_{n+t+1} - \ldots - P_l(n)a_{n+l}. \quad (10)$$

Taking into account that $a_n = 0$ for $n > \rho$, we successively find

$$a_\rho,\ a_{\rho-1},\ \ldots,\ a_0,\ a_{-1},\ \ldots,\ a_{-l+t}.$$

To this end, for a fixed n, we consider (10) as a system of linear algebraic equations in a_{n+t}. For $n = \rho - t$, $\rho - t - 1, \ldots$, solutions of such systems include a set of constants that will change when n changes, if the matrix on the left hand side of the current system (10) is singular. On the one hand, the system is to be consistent, and this may yield relations (linear algebraic equations) for earlier introduced variables. On the other hand, new constants appear. The number of such constants is equal to the difference of m and the rank of matrix on the left-hand side. To the linear algebraic equations obtained, we add, first, the equations

$$a_{-1} = 0, \quad a_{-2} = 0, \quad \ldots, \quad a_{-l+t} = 0 \quad (11)$$

in the constants and, second, the linear constraints, in which unknowns a_η for $\eta < 0$ and $\eta > \rho$ are replaced by zeros. The resulting set of expressions $a_\rho x^\rho + a_{\rho-1}x^{\rho-1} + \ldots + a_0$ is the set of all polynomial solutions of the original system (the constants are linear terms in $a_\rho, a_{\rho-1}, \ldots, a_0$).

Remark 5. If, for example, owing to the approach mentioned in Remark1, the algorithm is applied to a homogeneous system, then $c_n = 0$ on the left-hand side of (10). In this case, like in the algorithms from [9, 11], there is no need to search successively all degrees of the coefficients, which is an important advantage in the case of sparse solutions. For this purpose, the number of zero coefficients of the solution in succession is counted, and, if this number exceeds the degree of the left-hand side of system (10), the calculations continue starting from n corresponding to the next integer root of the determinant of the trailing matrix $P_t(n)$.

5. RATIONAL SOLUTIONS

When constructing rational solutions, we first determine $U(x) \in \mathbb{K}(x)$ $(U(0) \neq 0)$ and $k \in \mathbb{Z}$ such that any rational solution $y(x)$ of the system can be written as

$$y(x) = \frac{x^k}{U(x)} z(x), \quad (12)$$

where $z(x)$ is a polynomial vector. Then, substitution (12) replacing $y(x)$ with $z(x)$ is made, and, after simplification, polynomial solutions of the new system are sought (how to search them, was discussed in Section 4). The difference between x and irreducible factors of the polynomial $U(x)$ is as follows: if the polynomial $p(x) \in \mathbb{K}[x]$ is irreducible and $p(0) \neq 0$, then $p(q^h x)$ is an irreducible polynomial that is coprime with $p(x)$ for any $h \in \mathbb{Z}$. Note that different h result in different irreducible polynomials. However, this is not true for the polynomial x that is not coprime with qx. This imposes a special status on the irreducible polynomial x, as opposed to, for example, the difference case, where any irreducible $p(x)$ (including the case of $p(x) = x$) is coprime with $p(x+1)$.

As has already been noted, each rational solution may be viewed as a Laurent solution. Therefore, Theorem 1 (ii) allows one to determine the exponent k in the factor x^k. As for the polynomial $U(x)$, in accordance with the scheme outlined in Section 1, we find it by the "difference" algorithm replacing shifts σ^i

with the corresponding q-shifts σ_q^i and excluding from the consideration the factors x. Below, we give some definitions and, then, present the algorithm.

For a rational function $F(x)$, the notation $\operatorname{den} F(x)$ will denote its *denominator*, i.e., a polynomial with the leading coefficient equal to 1 such that $F(x) = \frac{f(x)}{\operatorname{den} F(x)}$ for some polynomial $f(x)$ coprime with $\operatorname{den} F(x)$. If $F(x)$ is a polynomial (in particular, a zero polynomial), then $\operatorname{den} F(x) = 1$. If $F(x)$ is a vector the components of which are rational functions $F_1(x)$, ..., $F_m(x)$, then $\operatorname{den} F(x)$ is the least common multiple (l cm) of the polynomials $\operatorname{den} F_1(x)$, ..., $\operatorname{den} F_m(x)$.

Given $f(x), g(x) \in \mathbb{K}[x]$, we write $f(x) \perp g(x)$ if these polynomials are coprime and $f(x) \not\perp g(x)$, if they have a common divisor of positive degree.

Any polynomial $f(x) \in \mathbb{K}(x)\backslash\{0\}$ can be represented in the form $f(x) = x^v s(x)$, where $v \in \mathbb{Z}_{\geq 0}$ and the polynomial $s(x)$ is not divisible by x; i.e., $s(0) \neq 0$. In this case, $s(x)$ is said to be a *base* of $f(x)$, and v is denoted as $\nu(f(x))$. If $\nu(f(x)) = \nu(g(x)) = 0$, we can consider the *q-dispersion set* of polynomials $f(x)$ and $g(x)$,

$$\operatorname{qds}(f(x), g(x)) = \{h \in \mathbb{Z}_{\geq 0} \mid f(x) \not\perp g(q^h x)\}, \qquad (13)$$

and their *q-dispersion*,

$$\operatorname{qdis}(f(x), g(x)) = \max(\operatorname{qds}(f(x), g(x)) \cup \{-\infty\}). \quad (14)$$

Like in the difference case, q-dispersion is either a nonnegative integer or $-\infty$, with the latter taking place if and only if $f(x) \perp g(q^h x)$ for all nonnegative integers h.

As has already been noted, if a polynomial $p(x) \in \mathbb{K}[x]$ is irreducible and $\nu(p(x)) = 0$, then polynomial $p(q^h x)$, $h \in \mathbb{Z}_{\geq 0}$, is also irreducible and, for different values of h, such polynomials are coprime. From this and from uniqueness of the factorization of an arbitrary polynomial into irreducible factors, we find that, if $\nu(f(x)) = \nu(g(x)) = 0$, then the set $\operatorname{qds}(f(x), g(x))$ is finite. This set can be found either by calculating all roots of the equation $R(\lambda) = 0$ of the form $\lambda = q^h$, $h \in \mathbb{Z}_{\geq 0}$, where $R(\lambda) = \operatorname{Res}_x(f(x), g(\lambda x))$, or by applying an analogue of the difference algorithm by Man and Wright [39], i.e., by considering irreducible multipliers of the polynomials $f(x)$ and $g(x)$ and using the fact that, if an irreducible $p(x)$ has the form $p(x) = x^l + a_{l-1}x^{l-1} + \ldots$, then $p(q^h x) = q^{lh}(x^l + q^{-h}a_{l-1}x^{l-1} + \ldots)$; it is essential that $\deg g(x) = \deg g(q^h x)$ for any $h \in \mathbb{Z}_{\geq 0}$. (In [40], an algorithm is suggested that is applicable to the case where q is an algebraic number not equal to the root of one.)

After the k has been found, it is required to construct a polynomial $U(x)$ such that

- $\nu(U(x)) = 0$;
- if the original system has a rational solution with the denominator $u(x)$, then $U(x)$ is divided by the base of the polynomial $u(x)$.

When k and $U(x)$ are found, we can take

$$x^k \cdot \frac{1}{U(x)} \qquad (15)$$

for substitution (12).

Finding of $U(x)$ is similar to the finding of a universal denominator in the difference case [22]. In the algorithm, we use notation $\gcd(f(x), g(x))$ for the greatest common divisor of the polynomials $f(x), g(x)$.

Set

$$A(x) = (\operatorname{den} \bar{A}_r^{-1}(q^{-r}x))/x^{a_r},$$

$$B(x) = (\operatorname{den} \bar{\bar{A}}_0^{-1}(x))/x^{a_0},$$

where $a_r = \nu(\operatorname{den} \bar{A}_r^{-1}((q^{-r}x))$, $a_0 = \nu(\operatorname{den} \bar{\bar{A}}_0^{-1}(x))$.

Find $H = \operatorname{qds}(A(x), B(x))$. *If* $H = \varnothing$, *then terminate the algorithm execution with the result* $U(x) = 1$ (*further, we assume that* $H = \{h_1, h_2, \ldots, h_s\}$ *and* $h_1 > h_2 > \ldots > h_s$, $s \geq 1$). *Set* $U(x) = 1$ *and, for all* h_i *in the order of their decreasing starting from* h_1, *perform the following assignments*:

$$N(x) = \gcd(A(x), B(q^{h_i}x)),$$

$$A(x) = A(x)/N(x),$$

$$B(x) = B(x)/N(q^{-h_i}x),$$

$$U(x) = U(x)\prod_{j=0}^{h_i} N(q^{-j}x).$$

The resulting value $U(x)$ *is the polynomial that can be used in* (15).

The following assertion can be proved in the same way as the similar assertion for the difference case in [18, 22].

Theorem 2. *Let any rational solution of the original q-difference system* (1) *be also a solution to systems* (5), (6), *with the determinants* $\det \bar{A}_r(x)$ *and* $\det \bar{\bar{A}}_0(x)$ *being not equal to zero. Then, polynomial* $U(x)$ *obtained by the above algorithm satisfies the condition* $\nu(U(x)) = 0$ *and, if the original system has a rational solution with the denominator* $u(x)$, *is divided by the base of the polynomial* $u(x)$.

The basic idea of the proof is similar to that used in [12, 14, 22] for the difference case. Valuation of a polynomial $f(x)$ with respect to an irreducible polynomial $p(x)$ (which is denoted as $\operatorname{val}_{p(x)} f(x)$) is defined to be the greatest integer n such that $f(x)$ is divided by $p(x)^n$;

if $f(x)$ is the zero polynomial, we set $\mathrm{val}_{p(x)} f(x) = \infty$. Valuation of a rational function $\frac{f(x)}{g(x)}$ is the difference $\mathrm{val}_{p(x)} f(x) - \mathrm{val}_{p(x)} g(x)$. Valuation of a vector consisting of polynomials or rational functions is the maximal valuation of the components of the vector. Then, if the assumptions of Theorem 2 are fulfilled, the following inequality holds for any rational solution $y(x)$ and any irreducible $p(x)$:

$$\mathrm{val}_{p(x)}\, y(x) \geq \max\left\{-\sum_{n\in\mathbb{Z}_{\geq 0}} \mathrm{val}_{p(q^n x)}\, A(x),\; -\sum_{n\in\mathbb{Z}_{\geq 0}} \mathrm{val}_{p(q^{-n} x)}\, B(x)\right\}. \tag{16}$$

It can be proved that $\mathrm{val}_{p(x)} U(x)$ does not exceed valuation of the right-hand side of (16).

Note that inequality (16) can underlie other algorithms for finding polynomial $U(x)$ similar to that was done in [14]. However, the algorithm presented in this section is considerably simpler and is more convenient from the point of view of implementation.

Remark 6. The suggested algorithm for constructing $U(x)$ remains correct after the substitution of $\mathrm{den}\, \bar{A}_r^{-1}(q^{-r}x)$, $\mathrm{den}\, \bar{\bar{A}}_0^{-1}(x)$ for $\det \bar{A}_r(q^{-r}x)$, $\det \bar{\bar{A}}_0(x)$, respectively, since $\det \bar{A}_r(q^{-r}x)$ is divided by $\mathrm{den} \bar{A}_r^{-1}(q^{-r}x)$ and $\det \bar{\bar{A}}_0(x)$ is divided by $\mathrm{den}\, \bar{\bar{A}}_0^{-1}(x)$. Inequality (16) also holds after the substitution. This substitution simplifies the algorithm but can increase the degree of the desired polynomial $U(x)$.

6. HYPERGEOMETRIC SOLUTIONS

6.1. Hypergeometric Terms and Certificates

According to [27, 41], a function $h(x)$ is called a *hypergeometric term* over $\mathbb{K}$ if the ratio $h(qx)/h(x)$ is a rational function of x with coefficients from $\mathbb{K}$; this rational function is referred to as *certificate* of $h(x)$. Let $\mathcal{H}_{\mathbb{K}(x)}$ denote the set of all hypergeometric terms and $\mathcal{L}(\mathcal{H}_{\mathbb{K}(x)})$ denote the set of all finite sums of elements from $\mathcal{H}_{\mathbb{K}(x)}$; the latter is a linear space over $\mathbb{K}$.

Let $\mathbb{M} = \overline{\mathbb{K}}$ be an algebraic closure of field $\mathbb{K}$. For a scalar q-difference equation of an arbitrary order

$$a_r y(q^r x) + \ldots + a_1 y(qx) + a_0 y(x) = 0,$$

$a_r(x), \ldots, a_0(x) \in \mathbb{K}[x]$, algorithm qHyper was suggested in [27]. It constructs a set of certificates $r_1(x)$, ..., $r_s(x) \in \mathbb{M}(x)$ such that the corresponding $h_1(x)$, ..., $h_s(x) \in \mathcal{H}_{\mathbb{M}(x)}$ are a basis of the space of solutions of this equation from $\mathcal{L}(\mathcal{H}_{\mathbb{M}(x)})$. Let us represent the certificate $r(x)$ in the normal form (see [27, Theorem 1]):

$$r(x) = z\frac{a(x)}{b(x)}\frac{c(qx)}{c(x)}\frac{d(x)}{d(qx)} = zU(x)\frac{V(qx)}{V(x)}, \tag{17}$$

where $z \in \mathbb{K}$; $a(x)$, $b(x)$, $c(x)$, $d(x) \in \mathbb{K}[x]$ are monic polynomials (their leading coefficients are ones); $a(x) \perp b(q^n x)$ for $n \in \mathbb{Z}$; $a(x) \perp c(x)d(qx)$; $b(x) \perp c(qx)d(x)$; $c(0) \neq 0$, and $d(0) \neq 0$; $U(x) = a(x)/b(x)$, $V(x) = c(x)/d(x)$. Such a representation of a rational function is unique.

Then, the corresponding hypergeometric term $h(x) = h(q^k)$, where k is a variable taking values from $\mathbb{Z}_{\geq 0}$, can be written as

$$h(q^k) = Cz^k V(q^k)\prod_{j=k_0}^{k-1} U(q^j),$$

where $C \in \mathbb{K}$ and q^j is not a pole of $U(q^k)$ for $j \in \mathbb{Z}_{\geq k_0}$. If $U(x) = 1$ and $z = q^j$, where $j \in \mathbb{Z}$, $h(x)$ is a rational function of $x = q^k$.

Let $h_1(x)$, $h_2(x) \in \mathcal{H}_{\mathbb{K}(x)}$. Then, $h_1(x)$ and $h_2(x)$ are said to be *similar* if their ratio is a rational function of x: $h_1(x)/h_2(x) \in \mathbb{K}(x)$. Let certificates of $h_1(x)$, $h_2(x)$ have normal forms $r_1(x) = z_1 U_1(x) V_1(qx)/V_1(x)$ and $r_2(x) = z_2 U_2(x) V_2(qx)/V_2(x)$, respectively. Then, $h_1(x)$ and $h_2(x)$ are similar if and only if

$$U_1(x) = U_2(x) \quad \text{and} \quad \frac{z_1}{z_2} = q^j, \quad \text{where} \quad j \in \mathbb{Z}. \tag{18}$$

6.2. Resolving Sequences of Operators

Let us reformulate the definition and proposition from [31] for the q-difference case.

Let the leading and trailing matrices of system (2) be nonsingular and $l_1, \ldots, l_p$ be pairwise different positive integers not exceeding m. Let scalar operators L_1, ..., $L_p \in \mathbb{K}[\sigma_q, x]$ satisfy the following condition: if $y_{l_1}(x) = \ldots = y_{l_j}(x) = 0$ for $j \leq p$ for some solution $y(x)$ of system (2), then,

- for $j = p$, all components of this solution are equal to zero,

$$y_1(x) = y_2(x) = \ldots = y_m(x) = 0,$$

- for $j < p$,

$$L_{j+1}(y_{l_{j+1}}(x)) = 0.$$

In this case, a finite sequence

$$L_1, \ldots, L_p \tag{19}$$

is called a *resolving sequence of operators* for this system [31].

A resolving sequence of operators for system (2) with singular leading or trailing matrix is defined to be a resolving sequence of operators of an lt-embracing system for system (2) (see Remark 4).

Proposition 1. Let $y(x) = h(x)R(x)$ be a nonzero solution of system (2), where $h(x) \in \mathcal{H}_{\mathbb{M}(x)}$, $R(x)$ is a column vector m of rational functions from $\mathbb{M}(x)$. Let also (19) be a resolving sequence of operators for (2). Then, there exists s, $1 \le s \le p$, such that the scalar equation $L_s z(x) = 0$ has a solution $z(x)$ that is similar to $h(x)$: $z(x) = h(x)f(x)$, where $f(x) \in \mathbb{M}(x)$.

Proof. According to the definition of the resolving sequence, there is an index j such that $j < p$, $y_{l_1}(x), \dots, y_{l_j}(x)$ are zero components of the solution, $y_{l_{j+1}}(x) = h(x)R_{l_{j+1}}(x)$ is a nonzero component, and, hence, equation $L_{j+1}(h(x)R_{l_{j+1}}(x)) = 0$ holds. □

6.3. Construction of Hypergeometric Solutions for Systems

We propose an algorithm for constructing hypergeometric solutions to the homogeneous system of q-difference equations (2), i.e., solutions from $\mathcal{L}(\mathcal{H}_{\mathbb{M}(x)})^m$. Like the algorithm from [31] for systems of difference equations, this algorithm uses the algorithm for constructing sequences of resolving operators [31], algorithm qHyper, and the algorithm for constructing rational solutions for systems of q-difference equations (Section 5).

• Construct a resolving sequence of operators (19) for (2).

• Set $\ell = \varnothing$.

• For $s = 1, \dots, p$ construct a set of certificates b_s for the equation $L_s z(x) = 0$ by means of algorithm qHyper. Place into ℓ only those elements b_s that do not satisfy condition (18) with any element from ℓ. Thus, ℓ contains certificates of non-similar hypergeometric terms.

• For each $r_j(x)$ from $\ell = \{r_1(x), \dots, r_d(x)\}$ represented in the normal form $z_j U_j(x) V_j(qx)/V_j(x)$, make the substitution

$$y(x) = \frac{h_j(x)R(x)}{V_j(x)}$$

into the original system (2), where $R(x)$ is the column vector of new unknowns and $h_j(x)$ has certificate $r_j(x)$. Having divided all equations by $h_j(x)/V_j(x)$, we get the following system with the coefficients from $\mathbb{M}(x)$:

$$B_r R(q^r x) + \dots + B_1 R(qx) + B_0 R(x) = 0, \qquad (20)$$

where, for $i = 0, 1, \dots, r$,

$$B_i = z^i U(q^{i-1}x)\dots U(qx)U(x)A_i.$$

• By means of the algorithm from Section 5, find all rational solutions of system (20). Let $R_{j,1}(x), \dots, R_{j,s_j}(x) \in \mathbb{M}(x)^m$ be a basis of the space of rational solutions. Then, $h_j(x)R_{j,1}(x)/V_j(x), \dots, h_j(x)R_{j,s_j}(x)/V_j(x)$ are linearly independent solutions of the original system (2). In accordance with Proposition 1, the set of all solutions $h_j(x)R_{j,i}$ $(1 \le j \le d$, $1 \le i \le s_j)$ is a basis of the space of all hypergeometric solutions of system (2).

7. IMPLEMENTATION

The algorithms presented in the paper were implemented in Maple 2017 (see [32]) as procedures of package `LqRS` (Linear q-Recurrence Systems). The package includes procedures for constructing

• embracing systems for q-difference systems;
• Laurent solutions for (1);
• polynomial solutions for (1);
• universal denominator (in the given case, rational function of the form $\frac{x^k}{U(x)}$), rational solutions for (1);
• resolving sequences of operators for (2);
• hypergeometric solutions for (2).

When implementing algorithms for nonhomogeneous systems, we used the approach described in Remark 1, and the implementation of the algorithm for constructing embracing systems for a q-difference system uses the shift step described in Remark 3 (the implementation of the algorithm for difference systems from package Maple `LinearFunctionalSystems` applied to solving induced recurrence systems uses on the shift step the approach based on selection of a system equation).

The implementation is designed for systems whose coefficients are rational functions of one variable, for example, x, over $\mathbb{K} = \mathbb{Q}(q)$, where $\mathbb{Q}$ is a field of rational numbers, and q is a name.

Each system of form (1) and (2) is specified in the input parameters of the procedures as a linear q-difference equation with matrix coefficients or in the traditional for Maple way, as a set of r equations in r unknown functions.

Figure 1 shows a homogeneous system with a singular leading matrix. An l-embracing system for it can be constructed by means of procedure `EG`, the first argument of which is the keyword `'lead'`, the second argument is a system, and the third argument, a name of a vector of unknowns. The procedure returns an embracing system and `true`, which means that the system has the full rank. The call of the procedure and the result of its operation are shown in Fig. 2.

By means of procedure `LaurentSolution`, Laurent solutions of the system are constructed. If only

$$> S := \begin{bmatrix} -\dfrac{q^2x^2}{qx+1} & -qx \\ -\dfrac{x(1+q^2x^2+q(q^2+1)x)}{qx+1} & -q^2x-qx^2 \end{bmatrix} \cdot y(x) + \begin{bmatrix} \dfrac{q^2x^2}{q^2x+1} & 1 \\ \dfrac{-1+q^2(q^2+1)x^2+q^3x}{q^2x+1} & x+q \end{bmatrix} \cdot y(qx) + \begin{bmatrix} 0 & 0 \\ q^2 & 0 \end{bmatrix} \cdot y(q^2x)$$

[>

Fig. 1. System of equations *S*.

$> LqRS:\text{-}EG('lead', S, y(x));$

$$\begin{bmatrix} \dfrac{q^2x^2}{qx+1} & qx \\ -\dfrac{x(q^3x+q^2x^2+qx+1)}{qx+1} & -q^2x-qx^2 \end{bmatrix} \cdot y(x) + \begin{bmatrix} -\dfrac{q^4x^2}{q^2x+1} - \dfrac{q^2x}{q^2x+1} & -q^2x-1 \\ \dfrac{q^4x^2+q^3x+q^2x^2-1}{q^2x+1} & x+q \end{bmatrix} \cdot y(qx) + \begin{bmatrix} \dfrac{q^3x}{q^3x+1} & 1 \\ q^2 & 0 \end{bmatrix} \cdot y(q^2x),$$

true

[>

Fig. 2. 1-embracing system for *S*.

two arguments—the system and the name of the vector of unknowns—are given, the procedure calculates as many initial terms of the series as required for determining the dimension of the solution space. The names $\_c_1$, $\_c_2$,… denote arbitrary constants:

```
>LqRS:-LaurentSolution(S, y(x));
```

$$\begin{bmatrix} \dfrac{\_c_1}{q^4} x^{-2} + O(x^{-1}) \\ O(x^{-1}) \end{bmatrix}.$$

To get more terms, it is required to specify their number in the third parameter:

```
>LqRS:-LaurentSolution(S, y(x), 3);
```

$$\begin{bmatrix} \dfrac{\_c_1}{q^4} x^{-2} + O(x^4) \\ -\dfrac{\_c_1}{q^3} x^{-1} + \dfrac{\_c_1}{q^2} + O(x) \end{bmatrix}.$$

System *S* in Fig. 1 has no polynomial solutions:

```
>LqRS:-PolynomialSolution(S, y(x));
```

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The universal denominator is given by:

```
>LqRS:-UniversalDenominator(S, y(x));
```

$$\frac{q}{x^2(qx+1)}.$$

The rational solutions are as follows:

```
>LqRS:-RationalSolution(S, y(x));
```

$$\begin{bmatrix} -\dfrac{\_c_1}{x^2} \\ \dfrac{\_c_1 q}{x(qx+1)} \end{bmatrix}.$$

The resolving sequence of equations for this system is shown if Fig. 3. When constructing hypergeometric solutions, it is required to specify in the third parameter the name of variable k such that $x = q^k$:

```
>LqRS:-HypergeometrisSolution(S, y(x), k);
```

$$\begin{bmatrix} -\dfrac{\_c_1}{(q^k)^2} \\ \dfrac{\_c_1 q}{q^k(qq^k+1)} + q^{\binom{k}{2}} q^k \_c_2 \end{bmatrix}.$$

As an example of a system with polynomial solutions, we consider system *S*1 (Fig. 4).

```
>LqRS:-PolynomialSolution(S1, z(x));
```

$$\begin{bmatrix} -\dfrac{\_c_1}{q} - \_c_1 x \\ \_c_1 x \end{bmatrix}.$$

The package, its description, and examples of using its procedures are available at the address http://www.ccas.ru/ca/lqrs.

$$> L := LqRS\text{:-}ResolvingSequence(S, y(x));$$
$$L := \left[(qx^2+qx)y_1(x)+(-q^3x^2-q^3x-q^2x-qx^2+qx+q)y_1(qx)+(q^4x+q^3x^2-q^3x-q^3-q-x)y_1(q^2x)\right.$$
$$\left.+(q^3+q^2x)y_1(q^3x), y_2(qx)-qxy_2(x)\right]$$
$$>$$

Fig. 3. Resolving sequence for S.

$$> S1 := \begin{bmatrix} -\frac{q^3}{(qx+1)^2} & -\frac{q^2}{x(qx+1)} \\ -\frac{q(q^2x^2+1+q(q^2+1)x)}{x(qx+1)^2} & \frac{q(-q^2x-qx^2)}{x^2(qx+1)} \end{bmatrix} \cdot z(x) + \begin{bmatrix} \frac{q}{x(q^2x+1)^2} & \frac{1}{qx^2(q^2x+1)} \\ \frac{-1+q^2(q^2+1)x^2+q^3x}{qx^2(q^2x+1)} & \frac{x+q}{qx^2(q^2x+1)} \end{bmatrix} \cdot z(qx)$$
$$+ \begin{bmatrix} 0 & 0 \\ \frac{1}{qx^2(q^3x+1)} & 0 \end{bmatrix} \cdot z(q^2x):$$
$$>$$

Fig. 4. System of equations $S1$.

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