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Finite Decimal Fractions As Entries of Nonsingular Matrices

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Abstract—How can one check, for a given nonsingular real number matrix the entries of which have only a finite number of decimal digits, whether this matrix will remain nonsingular after some decimal digits are arbitrarily added to some (explicitly specified in advance) of its entries? It turns out that this problem is algorithmically solvable. A computer implementation of the proposed algorithmic solution is discussed.

Keywords: truncated number matrices, nonsingularity of number matrices, Tarski's algorithm, cylindrical decomposition algorithm, computer algebra

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1. INTRODUCTION

In the problem under discussion, it is assumed that some real matrix M that is of interest from some point of view is not completely known. Only a matrix P, the entries of which are "truncated" entries of M—the initial digits of the original entries of M—is known. The entries of P are rational numbers the notation of which has only a finite number of digits after the decimal point (if an entry is integer, a finite number of zeros may be added after the decimal point). Let the given matrix P be nonsingular. Is it possible to say whether the matrix M, which we do not know exactly, will also be nonsingular? The answer is not always unambiguous, and in this paper we discuss an algorithm that, based on P, answers the question of nonsingularity of the matrix with digits added to the entries of P. If no addition of digits can "damage" the nonsingularity of P, then this matrix is said to be strongly nonsingular. In this case, it can be definitely stated that the matrix M is nonsingular.

A similar problem was considered earlier in [1] under the assumption that the entries of P are polynomials in x, which, in turn, are truncated formal power series. The truncation is performed by discarding in the entries of M all terms the degree of which exceeds a given nonnegative integer d. The numerical version leads to a somewhat more difficult problem, due, e.g., to the fact that arithmetic operations on numbers in the decimal system prescribe, e.g., "carries to the higher order," which is not the case with operations with polynomials and series. The latter circumstance (the absence of carries) additionally made possible in [1] an algorithmic solution to the problem of calculating the first few terms of the entries of the matrix M^{-1} from the first terms of the entries of P^{-1} (these matrix entries themselves are formal Laurent series and, generally speaking, contain terms with negative powers of x). In Section 7.2, we show that such calculations are not always possible for strongly nonsingular number matrices.

The proposed algorithms for checking the strong nonsingularity of numerical matrices are based on Tarski's theorem [2] on checking the truth in a class of logical formulas containing quantifiers for all variables taking values in the set of real numbers. In [3], the discussed result of Tarski ("theorem") is presented both in the title and as an algorithm. In the title of Tarski's own work [2], the first words are *a decision method*. It is Tarski's theorem that states the existence of a method (algorithm) for calculating the value of a given formula.

A later algorithm [4], called the cylindrical decomposition algorithm, provides a transition to a quantifier-free formula belonging to a class broader than Tarski's quantifier-free formulas. Nevertheless, this transition simplifies the problem of truth checking and often allows one to check the truth of formulas with quantifiers covered by Tarski's theorem in reasonable time. (However, the complexity of the cylindrical decomposition algorithm is twice exponential in the length of the original formula [5], [6, Section 3.2.3.]) The cylindrical decomposition algorithm is implemented, in particular, in Maple. We use this implementation in our program (Section 6).

A preliminary version of this paper was presented as a talk at the conference Differential Equations and Related Problems of Mathematics (Kolomna, 2024); for an extended abstract of this report, see [7]. The impossibility of obtaining a truncated representation of a number matrix M^{-1} (Subsection 7.2) was not discussed in [7].

2. ADDING DIGITS TO ENTRIES OF A NUMBER MATRIX

Let a finite number of digits after the decimal point be known for each entry of a real number $n \times n$ matrix P. The number of known digits may be different for different entries. Some of the matrix entries may be known exactly, i.e., all significant digits after the decimal point in them may be given.

However, for some entries, there is no information initially whether all the digits after the decimal point are written out. For example, it may happen that the full decimal notation of an entry requires an infinite number of digits. Or, perhaps, in a given notation of an entry, only a finite number of digits are missing. We assume that in both cases such entries are somehow marked in advance (further we speak of *marked entries*).

Suppose, the calculation of the determinant of the matrix P carried out based on the given finite representation of the entries yielded a nonzero value. Is it true that the determinant remains nonzero and, accordingly, the matrix remains nonsingular with any addition of digits to the least significant digits of the marked entries of P?

Consider two examples each of which includes a nonsingular 2×2 matrix $P = (p_{ij})$.

Let

$$P = \begin{pmatrix} 1.0 & -3.333 \\ -0.3 & 0.99 \end{pmatrix}, \qquad (1)$$

the only marked entry be $p_{11} = 1.0$, only one digit after the decimal point be known, and there is no guarantee that we know all the digits. It is easy to verify that, if we add 1 at the end of p_{11} thus obtaining $p_{11} = 1.01$, then the matrix becomes singular. If we assume that two zero digits after the decimal point in p_{11} are known ($p_{11} = 1.00$), then the matrix becomes singular if we add an infinite number of nines at the end of p_{11}, which gives $p_{11} = 1.00999... = 1.01$.

Here is an example of a matrix that remains nonsingular no matter how many digits are added to each of its entries:

$$\begin{pmatrix} 1.40 & -7.8 \\ 9.954 & 0.46 \end{pmatrix}. \qquad (2)$$

We will return to these two examples in Section 6.

Let the entry p_{ij} of the original matrix have d digits after the decimal point (several last digits may be zero). We can add new digits to this entry as follows: take a decimal fraction β, $0 \le \beta \le 1$, which we represent in the form

$$0.\beta_1\beta_2...,$$

and add $10^{-d}\beta$ (with the negative sign if $p_{ij} < 0$) to p_{ij}. As a result, the digits $\beta_1, \beta_2, \ldots$ of the fraction β will be added to the end of p_{ij}. In fact, β is an arbitrary real number in the interval [0, 1], and $\beta = 1$ is represented by the fraction 0.999.... This representation was used above when we considered matrix (1) after replacing 1.0 with 1.00.

3. THE CASE OF A POLYNOMIAL MATRIX

A problem of this kind was considered in [1] for matrices whose entries are polynomials in x over a field K of characteristic 0. Is it possible to obtain a nonsingular matrix by transforming the entries of the original nonsingular polynomial matrix P into formal power series by adding new terms whose degrees are greater than the maximum degree d of the matrix entries? If this is impossible, then the matrix P was called a strongly nonsingular (polynomial) matrix in [1]. Largely due to the fact that a single lower bound $d + 1$ for the powers of the added terms is fixed for all entries of P and also due to the fact that, in contrast to numbers, carries do not occur when operating with series, the criterion for the strong nonsingularity of a polynomial matrix turned out to be quite simple. Before formulating it, we introduce some notation and concepts. The ring of *formal power series* over a field K is denoted by $K[[x]]$, and the field of *formal Laurent series,* which is the field of fractions of the ring $K[[x]]$, is denoted by $K((x))$. For a nonzero entry $s = \sum s_i x^i \in K((x))$, the notation $\mathrm{val}\, s$ is used for its *valuation* defined as $\min\{i | s_i \neq 0\}$. It is assumed that $\mathrm{val}\, 0 = \infty$. The valuation $\mathrm{val}\, M$ of a matrix M over a field $K((x))$ is the smallest of the valuations of the entries of this matrix. For a polynomial matrix P, its degree $\deg P$ is the largest degree of the entries of this matrix (for polynomials, it is assumed that $\deg 0 = -\infty$).

It was proved in [1] that, a polynomial square matrix P is strongly nonsingular if and only if

$$\deg P + \mathrm{val}\, P^{-1} \ge 0. \qquad (3)$$

4. QUANTIFIER ELIMINATION

For the class of statements about real numbers, Tarski's theorem gives an algorithm for checking the truth of these statements.

Let $x_1,\dots,x_\nu$ be variables taking real values (real variables). We consider polynomials in $x_1,\dots,x_\nu$ with rational coefficients and the corresponding polynomial equations and inequalities for $x_1,\dots,x_\nu$. Using logical operations $\vee,\wedge,\neg,\Rightarrow$, we can construct more complex relations from these equations and inequalities. In such relations, existential and universal quantifiers are imposed on the variables $x_1,\dots,x_\nu$; and as a result, all these real variables must be bound by quantifiers. Each quantifier is related to the entire set of real numbers. Tarski's algorithm finds out whether a given logical formula of this kind is true or false.

The cylindrical decomposition algorithm allows one to replace the considered type of logical formulas by quantifier-free formulas of a certain class, which facilitates the verification of the truth of formulas.

5. CHECKING STRONG NONSINGULARITY OF A NUMBER MATRIX

Definition 1. Let each entry of a nonsingular number real matrix P contain only a finite number of digits after the decimal point. We call this matrix *strongly nonsingular* with respect to a specified set of its marked entries U if it remains nonsingular after arbitrary addition of digits to the least significant digits of the entries in U. If U contains all entries of P, then we call the matrix P strongly nonsingular without referring to any set of its marked entries.

It turns out that the decision problem (i.e., the problem in which the answer is *yes* or *no*) of checking the strong nonsingularity of the given matrix is algorithmically solvable.

Theorem 1. *There is an algorithm that that recognizes strongly nonsingular matrices (strong nonsingularity can be considered both with respect to a given set of marked entries and with respect to all entries).*

Proof. This algorithm is based on the method of cylindrical decomposition with quantifier elimination [4, 5, 8, 9], which is a version of Tarski's algorithm [2]. The input of this algorithm is a nonsingular real number $n \times n$ matrix P the entries of which contain only a finite number of digits after the decimal point. The set of marked entries $U = \{p_{i_1j_1},\dots,p_{i_kj_k}\}$ and the set $D = \{d_{i_1j_1},\dots,d_{i_kj_k}\}$ of the number of known digits after the decimal point in the marked entries are also specified. The algorithm itself additionally introduces a set of real variables $S = \{T_{i_1j_1},\dots,T_{i_kj_k}\}$.

Algorithm.

Construct the matrix $P' = (p'_{ij})$ in which $p'_{ij} = p_{ij}$ if $p_{ij} \notin U$. If $p_{ij} \in U$, then

$$p'_{ij} = \begin{cases} p_{ij} + T_{ij}\cdot 10^{-d_{ij}}, & \text{if} \quad p_{ij} > 0, \\ p_{ij} - T_{ij}\cdot 10^{-d_{ij}}, & \text{if} \quad p_{ij} < 0, \\ p_{ij} + \sigma T_{ij}\cdot 10^{-d_{ij}}, & \text{if} \quad p_{ij} = 0, \end{cases}$$

where σ equals 1 or -1, depending on whether $p_{ij} = 0$ is a truncation of a positive or negative number.

The next steps are as follows:

1. Calculate the determinant $\Delta = \det P'$, which gives $\Delta \in \mathbb{Q}[S]$, where $\deg_{T_{ij}}\Delta \in \{0,1\}$ for all $T_{ij} \in S$.

2. If Δ is independent of variables, then terminate the algorithm execution with the conclusion that P is a strongly nonsingular matrix.

3. If Δ depends on a single variable $T \in S$, then find the root of the equation $\Delta(T) = 0$. If this root does not belong to the interval [0,1], then P is a strongly nonsingular matrix. Otherwise, P is not strongly nonsingular.

4. Using the variables $T_{ij} \in S$ such that $\deg_{T_{ij}}\Delta = 1$, write the formula with the quantifier $\exists$: there exist values of variables such that $\Delta = 0$ and these values of the variables belong to the interval [0,1] on the number line.

5. Using the cylindrical decomposition algorithm decide whether the formula constructed at the preceding step is true. If it is true, then P is not strongly nonsingular; otherwise, P is strongly nonsingular. □

6. IMPLEMENTATION IN MAPLE

The algorithm described above was implemented in Maple 2024 [10]. The Maple-package *Quantifier-Elimination* [11], which makes it possible to apply the cylindrical decomposition method, was used.

Suppose we want to find out if matrix (1) is strongly nonsingular with respect to the entries p_{12}, p_{21}, p_{22}. To avoid rounding, we write the matrix entries as rational numbers in Maple. Mark the entries as described in the preceding section and add $T_{ij} \times 10^{-d_{ij}}$ to each marked entry p_{ij} or distract it from this entry. Thus, in the Maple session we carry out the assignment

$> P :=$

$$\begin{bmatrix} 1 & -3333\cdot 10^{-3} - T_{1,2}\cdot 10^{-3} \\ -3\cdot 10^{-1} - T_{2,1}\cdot 10^{-1} & 99\cdot 10^{-2} + T_{2,2}\cdot 10^{-2} \end{bmatrix}:$$

(In Maple, the first and the second indices are separated by a comma.) Next, we find the determinant. This polynomial in all T_{ij} with rational coefficients is denoted by *Dt*:

$> Dt := LinearAlgebra{:}-Determinant(P);$

$$Dt := -\frac{99}{10000} + \frac{1}{100}T_{2,2} - \frac{3333}{100000}T_{2,1}$$

$$-\frac{3}{10000}T_{1,2}-\frac{1}{100000}T_{1,2}T_{2,1}$$

Next, we obtain the list S of variables on which the determinant Dt depends:

$> S := [indets(Dt)[\ ]]:$

$$S := [T_{1,2}, T_{2,1}, T_{2,2}]$$

Construct the corresponding expression (formula in terms of mathematical logic) with a quantifier: there exist values of the variables $T_{1,2}, T_{2,1}, T_{2,2}$ such that $Dt = 0$ and these values belong to the interval [0,1]. In Maple, this is done as follows:

$> exists(S, \mathrm{And}(Dt = 0, seq([i \ge 0, i \le 1][], i = S)));$

$$\exists([T_{1,2}, T_{2,1}, T_{2,2}], -\frac{99}{10000}+\frac{1}{100}T_{2,2}-\frac{3333}{100000}T_{2,1} -\frac{3}{10000}T_{1,2}-\frac{1}{100000}T_{1,2}T_{2,1} = 0 \wedge 0 \le T_{1,2} \wedge T_{1,2} \le 1 \wedge 0 \le T_{2,1} \wedge T_{2,1} \le 1 \wedge 0 \le T_{2,2} \wedge T_{2,2} \le 1)$$

The procedure *QuantifierEliminate* available in the package *QuantifierElimination* finds out if this formula is true:

$> QuantifierElimination:-QuantifierEliminate(\%);$

true

The result *true* means that the addition of some digits to the marked entries can make the determinant equal to zero, i.e., matrix (1) is not strongly nonsingular with respect to p_{12}, p_{21}, p_{22}.

The sequence of commands is designed as the *StronglyNonSingular* procedure. The procedure argument is a square number matrix with entries marked in the manner specified above (i.e. each entry (i, j) of the matrix is a polynomial in the variable T_{ij} of degree no greater than 1 with rational coefficients). In this case, unlike the sequence of commands considered above, the procedure returns true if the matrix is strongly nonsingular, and returns false otherwise.

Let us write matrix (1) in Maple, and mark only one entry $p_{11} = 1.0$ (only one digit after the decimal point is known):

$$> P := \begin{bmatrix} 1+T_{1,1}\cdot 10^{-1} & -3333\cdot 10^{-3} \\ -3\cdot 10^{-1} & 99\cdot 10^{-2} \end{bmatrix}:$$

Apply the procedure

$> StronglyNonSingular(P);$

false

The result *false* means that the matrix is not strongly nonsingular with respect to p_{11}. The same result is obtained if $p_{11} = 1.00$ (two digits after the decimal point are known):

$$> P := \begin{bmatrix} 1+T_{1,1}\cdot 10^{-2} & -3333\cdot 10^{-3} \\ -3\cdot 10^{-1} & 99\cdot 10^{-2} \end{bmatrix}:$$

$> StronglyNonSingular(P);$

false

Let us write matrix (2) in Maple, and mark all its entries:

$> P :=$

$$\begin{bmatrix} 14\cdot 10^{-2}+T_{1,1}\cdot 10^{-2} & -78\cdot 10^{-1}-T_{1,2}\cdot 10^{-1} \\ 9954\cdot 10^{-3}+T_{2,1}\cdot 10^{-3} & 46\cdot 10^{-2}+T_{2,2}\cdot 10^{-2} \end{bmatrix}:$$

The result is that this matrix is strongly nonsingular:

$> StronglyNonSingular(P);$

true

We experimented with a 5 × 5 matrix with random entries for which one or two digits after the decimal point are known. Two entries were marked in this matrix, and then three and four entries were marked. The matrix turned out to be strongly nonsingular with respect to these entries. The execution time[1] of the procedure *QuantifierElimination:-QuantifierEliminate* was 0.182 for two marked entries, 7.091 for three marked entries, and 640.624 for four marked entries.

Experiments were also carried out with a not strongly nonsingular 5 × 5 matrix for the entries of which the values of one or two digits after the decimal point are known. Two entries were marked, then three and four entries with respect to which the matrix is not strongly nonsingular. The execution time of the procedure *QuantifierElimination:-QuantifierEliminate* was 0.451 for two marked entries, 0.362 for three marked entries, and 0.457 for four marked entries.

The Maple session with the procedure *StronglyNonSingular* and examples is available in [12] and a pdf version of this session at [13].

7. ON THE INVERSE MATRIX

7.1. The Case of a Polynomial Original Matrix

Let us continue the topic of Section 3. Let M be an $n \times n$-matrix over $K[[x]]$, where the field K has characteristic 0, and the matrix P be obtained from M by discarding all terms of degree higher than d ($d = \deg P$) in its entries (series). Let P be strongly nonsingular. Then, as shown in [1, Section 4], due to the strong nonsingularity of P, we have for M $\mathrm{val}\, M^{-1} = \mathrm{val}\, P^{-1}$.

[1] In seconds. The computations were carried out in Maple 2024 under Ubuntu 8.04.4 LTS on a computer with an AMD Athlon(tm) 64 Processor 3700+, 3GB RAM.

In addition, the matrices composed of the coefficients of x^{ν}, $\nu = \operatorname{val} P^{-1}$ in the entries of P^{-1}, M^{-1} coincide. Furthermore, a larger number of initial terms in the series that are entries of P^{-1} and M^{-1} may also coincide. An upper bound on the degree of x up to which coincidence occurs can be found in [1, Proposition 3].

7.2. The Case of a Number Original Matrix

The next proposition shows that analogs of the properties listed in Subsection 7.1 do not hold for number matrices.

Proposition 1. *For any positive integers n, N, one can specify nonsingular number $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ such that the entries a_{ij} and b_{ij} for any i and j coincide up to the N th digit after the decimal point; however, for some i, j, some first significant digits of the entries $\bar{a}_{ij}$ and $\bar{b}_{ij}$ in the matrices $A^{-1} = (\bar{a}_{ij})$ and $B^{-1} = (\bar{b}_{ij})$ as well as the decimal positions of these digits do not coincide. This also holds for strongly nonsingular A and B.*

Proof. First, consider the case of a single number (or, equivalently, the case of a 1×1 matrix). For $u = 10^{-N}$, adding a digit 9 on the right gives $v = 19 \times 10^{-N-1}$. We have $u^{-1} = 10^{N}$, i.e., the first digit of the $(N+1)$-digit number is 1, and $v^{-1} = 0.0526\ldots \times 10^{N+1}$—the first digit of the N-digit part before the decimal point—is 5. (Adding the digit 2 to u on the right would give $v^{-1} = 0.0833\ldots \times 10^{N+1}$, i.e., the first digit of the N-digit number is 8; and if we add an infinite sequence of nines, we will have $v = 2 \times 10^{-N}$ and $v^{-1} = 0.05\ldots \times 10^{N+1}$, i.e., the same effect as adding a single nine, etc.)

Now take diagonal $n \times n$ matrices A and B for which $a_{ii} = b_{ii} = 1$ for $i = 2, \ldots, n$ and $a_{11} = u$, $b_{11} = v$. The entries of the matrices A^{-1} and B^{-1} with the indices $i = j = 1$ are equal, respectively, to u^{-1} and v^{-1}.

The matrices A and B are strongly nonsingular.

8. FACTORIZATION OF THE DETERMINANT

If the matrix P has a large number of exact zeros (i.e., zero entries that do not belong to the set of marked entries), then it makes sense to check whether the determinant under study can be represented as a product of determinants of matrices of smaller sizes: equality to zero of the original determinant is equivalent to the fact that at least one of the factors equals zero. Such factorization could reduce the cost of checking strong nonsingularity. Checking the possibility of such a factorization is based on the theorem on the determinant of a 2×2 block matrix one of the blocks of which is zero [14, Chapter 3, Subsection 2], item 4. The best effect is obtained if the original matrix can be reduced to a block-diagonal or block-triangular matrix on the diagonal of which there are small blocks.

It is assumed that this reduction is performed by rearranging rows and columns.

For example, arrange rows in the matrix

$$P = \begin{pmatrix} p_{11} & 0 & p_{13} & p_{14} & 0 \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} \\ p_{31} & 0 & 0 & p_{34} & 0 \\ p_{41} & 0 & 0 & p_{44} & 0 \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} \end{pmatrix} \qquad (4)$$

in the order 2, 5, 1, 3, 4 and the columns in the order 5, 2, 3, 4, 1; then we obtain the block-triangular matrix

$$P' = \begin{pmatrix} \boxed{\begin{matrix} p_{25} & p_{22} \\ p_{55} & p_{52} \end{matrix}} & \begin{matrix} p_{23} & p_{24} & p_{21} \\ p_{53} & p_{54} & p_{51} \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} \boxed{p_{13}} & p_{14} & p_{11} \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \boxed{\begin{matrix} p_{34} & p_{31} \\ p_{44} & p_{41} \end{matrix}} \end{pmatrix}.$$

The original matrix P is strongly nonsingular if each matrix block on the diagonal of the matrix P' is strongly nonsingular. Blocks consisting of a single entry (like the second block on the diagonal of P'—the entry p_{13}) are strongly nonsingular, which follows from the problem statement—the original number matrix is nonsingular (therefore, $p_{13} \neq 0$). Even if the entry is classified as marked, no addition of digits to a nonzero number can turn it into zero.

So, instead of applying the algorithm to the matrix P or P', we apply it to the diagonal blocks of size greater than one. Thus, we apply the algorithm to the matrix

$$A = \begin{pmatrix} p_{25} & p_{22} \\ p_{55} & p_{52} \end{pmatrix}.$$

If it turns out that A is not strongly nonsingular, then the matrix P is not strongly nonsingular either. Otherwise, we apply the algorithm to

$$B = \begin{pmatrix} p_{34} & p_{31} \\ p_{44} & p_{41} \end{pmatrix}.$$

The matrix P is strongly nonsingular if and only if B is strongly nonsingular.

In Section 6, we mentioned experiments with a 5×5-matrix with random entries. The execution time of the procedure *QuantifierEliminate* for several marked entries with respect to which the matrix turned out to be strongly nonsingular was given. In these experiments, there were no zero entries in the matrix. Next, we set some of the matrix entries equal to 0 as shown in (4) and marked all its nonzero entries. The running time of *QuantifierEliminate* was 2250.033. The matrix turned out to be strongly nonsingular with respect to all nonzero entries. The running time for A and B was 17.946 and 27.517, respectively.

Compaction transformation. This transformation is applied to some matrix P and positive integers m, l not exceeding the number of its rows and, respectively, columns. Let P have size $s \times t$ and $m \le s, l \le t$. Of all the rows of P with the indices $1, \ldots, m$ in which the number of zero entries at the intersection with the columns with the indices $1, \ldots, l$ takes the greatest value (let this value be equal to k), we take one row that has, e.g., the greatest index and exchange it with the mth row. Then the zero entries of the mth row located in columns with the indices not exceeding l are moved to the beginning of the row by permutations of columns $1, \ldots, l$; thus, they become entries $p_{m1}, \ldots, p_{mk}$ of the transformed matrix P. The values m, l are transformed into $m-1, k$ (these indices select the noncompacted part of the matrix).

This transformation may be applied to P several times with the step-by-step update of m and l. If after the sequence of w compactions of the $n \times n$ matrix P with the initial $m = l = n$ it holds that $l \ge n - w$, then the lowest w rows of the transformed matrix P admit block notation UV with the zero $w \times l$ matrix U and the square matrix V of size $w \times w$. The further compactions are applied to P with $m = n - w$ and l following the previous steps (let it be l_0). These compactions are continued up to the time when $l \ge n - w_1$ after the total number of w_1 compaction transformations. This gives the new block size $(w_1 - w) \times (w_1 - w)$, etc.

The execution time of the Maple implementation of the compaction transformation of matrix (4) is 0.033 s.

Remark 1. The choice of the row with the greatest number of zeros is not uniquely defined. For simplicity, it was suggested above to take the row with the greatest index. However, other (e.g., heuristic) selection strategies are possible. For example, from the rows with the greatest number of zeros, it is possible to choose a row such that, after swapping it with the row with the index m, the total number of zeros in the subcolumns lying above the zero entries of the new row with the index m is the greatest possible.

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CONFLICT OF INTEREST

The of this work declare that they have no conflicts of interest.

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