Indicial Rational Functions of Linear Ordinary Differential Equations with Polynomial Coefficients*

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Abstract

The notion of indicial rational function is introduced for ordinary differential equations with polynomial coefficients and polynomial right-hand sides, and the algorithms for its construction are proposed.

1 Introduction

It is known that if an analytic solution of a differential equation

$$a_d(x)y^{(d)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0, \tag{1}$$

where $a_0(x), a_1(x), \ldots, a_d(x)$ are polynomials over $\mathbb{C}$, has a singularity (in particular, a pole) in a point α, then $a_d(\alpha) = 0$ (we assume that $a_d(x)$ is a non-zero polynomial). One can compute a lower bound of the order of the pole using the least integer root of the indicial equation. This equation is an algebraic equation of degree not exceeding d, and it corresponds to equation (1) and to the point α [5, 6]. If the indicial equation has no integer root, then equation (1) has no non-zero solutions that either are regular, or have a pole at α. Assume that every indicial equation corresponding to the roots of polynomial $a_d(x)$ has integer roots. Let $\alpha_0, \alpha_1, \ldots, \alpha_k$ be all complex roots of the polynomial $a_d(x)$, and $l_0, l_1, \ldots, l_k$ be the least integer roots of the corresponding indicial equations (they may be arbitrary, not necessarily negative, integers). Then any solution of (1) that is meromorphic in the whole complex plane $\mathbb{C}$ can be represented as a product of the rational function

$$V(x) = (x - \alpha_0)^{l_0}(x - \alpha_1)^{l_1} \ldots (x - \alpha_k)^{l_k} \tag{2}$$

by some entire function. In this paper we call $V(x)$ the indicial rational function of equation (1).

The problem of recognizing the existence of an indicial function for (1) and its construction if it exists can be also considered in the case when $a_0(x), a_1(x), \ldots, a_d(x)$ are polynomials over an arbitrary field K of characteristic zero. In this case $\alpha_0, \alpha_1, \ldots, \alpha_k$ belong to the splitting field K' of the coefficient $a_d(x)$. The substitution

$$y(x) = u(x)V(x), \tag{3}$$

where $V(x)$ is the indicial rational function, and $u(x)$ is a new unknown function, reduces the problem of finding rational solutions of (1) to finding polynomial solutions.

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In Section 3 we extend the notion of the indicial rational function to the case of inhomogeneous linear equations with coefficients and right-hand sides belonging to $K[x]$.

In Proposition 6 of Section 4 we show that an indicial rational function is in a sense the optimal choice of rational factor for substitution that reduces finding rational solutions to finding polynomial ones (the algorithm for constructing a rational factor of that kind for the difference case was given in [8]; rougher, but also simpler algorithms for the difference case were proposed in [2, 3]).

In [11] it is described how a rational function of the form

$$\tilde{V}(x) = (x - \alpha_0)^{m_0}(x - \alpha_1)^{m_1} \dots (x - \alpha_k)^{m_k}, \tag{4}$$

where $m_0 \leq l_0$, $m_1 \leq l_1, \dots, m_k \leq l_k$, can be constructed without a computation in algebraic extensions of K by means of the factorization of $a_d(x)$ into factors irreducible over K, and p-adic decompositions of rational functions. The resulting function has the form $\frac{1}{v(x)}$, where $v(x)$ is a polynomial. This algorithm does not use the fact, useful for further computation, that some of irreducible factors of the polynomial $a_d(x)$ can be included into rational function $\tilde{V}(x)$ with positive degrees. In Section 5 of the present paper we describe in elementary terms the algorithm for constructing the indicial rational function $V(x)$ based on the complete factorization of $a_d(x)$, which is a slightly modified version of the algorithm from [11].

In [2] an algorithm which, similarly to the algorithm from [11], constructs a rational function of the form (4) has been proposed. This algorithm is based on a computation of greatest common divisors and resultants of polynomials over K, without computing roots of $a_d(x)$ and the complete factorization of polynomials. It was noticed in [4] that the algorithm from [2] can be improved such that its result will be the indicial function (2). The substitution (3) with the indicial function $V(x)$ reduces finding rational solutions to finding polynomial ones, which have in the general case smaller degrees in comparison with $\tilde{V}(x)$. In Section 6 of the present paper we give a detailed description of the improved version of the algorithm from [2].

It is worth to note that algorithms from [11, 2] as well as new algorithms use finding integer roots of polynomials in $K[x]$.

In Section 7 we discuss an implementation of algorithms described in Sections 5 and 6 in the computer algebra system Maple, and also demonstrate results of some experiments.

The article is a kind of a review in a sense, and a part of the text is a summary of the special course lectured by the first author to the students of the faculty of Computational Mathematics and Cybernetics of Lomonosov Moscow State University. At the same time, Proposition 6, the sub-partition procedure from Section 6 and the implementation of some algorithms described in Section 7 are new.

2 Indicial equations

It is convenient to use formal series to introduce the notions of the indicial equation and the indicial rational function. If K is a field then, as usual, $K[[x]]$ denotes the ring of formal power (Taylor) series over K, i.e., the series of the form

$$c_0 + c_1 x + c_2 x^2 + \dots,$$

$c_0, c_1, c_2, \dots \in K$. The quotient field of this ring $K((x))$ is the field of (Laurent) series of the form

$$c_m x^m + c_{m+1} x^{n+1} + c_{m+2} x^{m+2} + \dots, \tag{5}$$

$c_m, c_{m+1}, c_{m+2}, \dots \in K$. The number m is an arbitrary integer, not necessarily non-negative. The least m such that the coefficient at x^m in $s(x)$ is non-zero is called the *order* of the series w.r.t. x (or simply order) and is denoted by $\nu(s)$; for the zero series we assume $\nu(0) = \infty$. By $\mathrm{tc}(s)$ we denote the coefficient at $x^{\nu(s)}$ assuming $\mathrm{tc}(0) = 0$.

The derivative of the series $s(x) = \sum_{i=-\infty}^{\infty} c_i x^i$ (in our case only finitely many coefficients with negative indices may be non-zero) is defined as $D(s(x)) = s'(x) = \sum_{i=-\infty}^{\infty} d_i x^i$, where $d_i = (i+1)c_{i+1}$ for all i. It follows that the coefficient d_{-1} is always zero.

Let K be a field of characteristic 0 and L be a differential operator

$$a_d(x)D^d + \cdots + a_1(x)D + a_0(x), \tag{6}$$

where $a_0(x), a_1(x), \ldots, a_{d-1}(x) \in K[[x]]$, $a_d(x) \in K[[x]] \setminus \{0\}$. We consider the equations of the form

$$L(y) = f(x), \tag{7}$$

where $f(x) \in K[[x]]$. The main question we are interested in concerns a bound for the orders of the series $s(x) \in K'((x))$ such that $L(s(x)) = f(x)$.

Given the operator L and equation (7), we consider the integer

$$b = \min_{0 \le j \le d} (\nu(a_j) - j) \tag{8}$$

and the algebraic equation

$$I(t) = 0,$$

where

$$I(t) = \sum_{\substack{0 \le j \le d \\ \nu(a_j) - j = b}} \mathrm{tc}(a_j) t^{\underline{j}}, \tag{9}$$

$t^{\underline{j}} = t(t-1)\cdots(t-j+1)$. The equation $I(t) = 0$ is called the *indicial* equation, corresponding to the operator L and the equation $L(y) = f(x)$.

Let N be the set of all integer roots of the indicial equation. We put

$$\lambda = \begin{cases} \min N, & \text{if } N \neq \emptyset, \\ \infty, & \text{if } N = \emptyset. \end{cases} \tag{10}$$

It is easy to check that $s(x) \in K[[x]]$, $\nu(s) = m$, imply $\nu(L(s(x))) \ge m + b$, and the coefficient at x^{m+b} of the series $L(s(x))$ is equal to $s_m I(m)$, where $s_m = \mathrm{tc}(s)$. Therefore, this coefficient is equal to zero iff $I(m) = 0$. Thus, we have

Proposition 1. *Let L has the form (6), $f(x) \in K[[x]]$. Let $s(x) \in K((x))$ and $L(s) = f(x)$. Then we have*

$$\nu(s) \ge \min(\nu(f) - b, \ \ \lambda), \tag{11}$$

where b and λ are defined by (8) and (10).

(The detailed proof can be performed by considering the cases $m+b = \nu(f)$ and $m+b < \nu(f)$; the inequality $m + b > \nu(f)$ is impossible when $L(s(x)) = f(x)$.)

As a consequence, we get that a homogeneous differential equation $L(y) = 0$ has no non-zero solution in the field $K((x))$ if the indicial equation $I(t) = 0$ corresponding to the operator L has no integer roots.

We denote by l the value of the right-hand side of inequality (11):

$$l = \min(\nu(f) - b, \ \ \lambda). \tag{12}$$

If $f(x)$ is zero series, i.e., equation (7) is homogeneous, then $l = \lambda$. The values λ and l do not depend on the way the solutions of the differential equation (7) are considered: as series over K or over some its extension.

If $\nu(a_d) = 0$ in equation (6), and $L(s(x)) = f(x)$, then $\nu(s) \geq 0$. This follows from the following fact: if $\nu(a_d) = 0$ then the series $a_d(x)$ is inverible in $K[[x]]$. We add that for $\nu(a_d) = 0$ we have $b = -d$ and $I(t) = t(t-1)\cdots(t-d+1)$.

Consider the accuracy of the estimation (11). In the following proposition, speaking about solutions of the equations of the form (7) with coefficients and right-hand sides in $K[[x]]$, we mean solutions in $K((x))$.

Proposition 2. *Let an equation $L(y) = f(x)$, where L is the operator of the form (6) and $f(x) \in K[[x]]$, have a partial solution, and additionally the equation $L(y) = 0$ have d linearly independent solutions. Then the equation $L(y) = f(x)$ has a solution of order l.*

Proof. At first, we show that the indicial equation has d different integer roots, and that for each root there exists a solution of $L(y) = 0$ whose order is equal to this root. Indeed, performing Gaussian elimination on d linearly independent solutions of $L(y) = 0$, one can construct d new solutions with pairwise different orders. These orders must be the roots of the indicial equation, and the degree of this equation cannot exceed d. Thus, the indicial equation cannot have "extraneous" roots.

Now we turn to the proof of the proposition. We shall prove the stronger statement: the equation $L(y) = f(x)$ has a solution of the order $\nu(f) - b$, and if $\lambda < \nu(f) - b$ then it also has a solution of the order λ. Let $v(x)$ be a solution of the equation $L(y) = f(x)$. Using d solutions of $L(y) = 0$ described above and performing Gaussian elimination we can find a solution $\tilde{v}(x)$ of $L(y) = f(x)$ of the order different from all roots of the indicial equation. For $\tilde{v}(x)$ we have $\nu(\tilde{v}) = \nu(f) - b$. If $\lambda < \nu(f) - b$ then we take $\tilde{v}(x) + w(x)$, where $w(x)$ is a solution of the homogeneous equation such that $\nu(w) = \lambda$. $\square$

3 Indicial rational functions

We turn to the case when the coefficients of our operator L and the right-hand side of the equation $L(y) = f(x)$ are polynomials. >From now on we assume that in this equation

$$L = a_d(x)D^d + \cdots + a_1(x)D + a_0(x), \tag{13}$$

and $a_0(x), a_1(x), \ldots, a_{d-1}(x) \in K[x]$, $a_d(x) \in K[x] \setminus \{0\}$. We also assume that $f(x) \in K[x]$.

Turning from the field K to some its extension K' containing all roots of the polynomial $a_d(x)$, for each root α we can construct the equation

$$L_{x+\alpha}(y(x)) = f(x+\alpha), \tag{14}$$

where

$$L_{x+\alpha} = a_d(x+\alpha)D^d + \cdots + a_1(x+\alpha)D + a_0(x+\alpha). \tag{15}$$

Polynomials can be viewed as (Taylor) series, that is why we can determine the value of l using (12). For convenience, we denote it by l_α (similarly, one may write λ_α). We call the rational function

$$\prod_{a_d(\alpha)=0} (x-\alpha)^{l_\alpha} \tag{16}$$

the *indicial rational function* (indicial function for brevity) of the equation $L(y) = f(x)$. If any exponent l_α is infinity then the indicial function (16) does not exist.

We show that one can find the values l_α without necessarily using shifted equations of the form (14). First of all, for $f(x), p(x) \in K[x]$, where $p(x)$ is irreducible, we define the value $\nu_{p(x)}(f)$ as the maximal $k \in \mathbb{N}$ such that $p^k(x) \mid f(x)$, for a non-zero $f(x)$, and ∞ for $f(x) = 0$. Let α be fixed such that $a_d(\alpha) = 0$. In our case $f(x)$ and $a_j(x)$, $j = 0, 1, \ldots, d$, are polynomials.

We have $\nu_x(f(x+\alpha)) = \nu_{x-\alpha}(f(x))$, and the value $\mathrm{tc}(f(x+\alpha))$ can be found using the Taylor formula, which is true for polynomials over any field of characteristic 0:

$$\mathrm{tc}(f(x+\alpha)) = \frac{f^{(m)}(\alpha)}{m!},$$

where the order of the derivative m is equal to $\nu_{x-\alpha}(f(x))$. One can obtain similar relations for $a_j(x)$, $j = 0, 1, \ldots, d$. Using the notation

$$m_{\alpha,j} = \nu_{x-\alpha}(a_j(x)), \quad j = 0, 1, \ldots, d,$$

we can rewrite (8) as

$$b_\alpha = \min_{0 \le j \le d} (m_{\alpha,j} - j), \tag{17}$$

Formula (9) becomes

$$I_\alpha(t) = \sum_{\substack{0 \le j \le d \\ m_{\alpha,j} - j = b_\alpha}} \frac{a_j^{(m_{\alpha,j})}(\alpha)}{m_{\alpha,j}!} t^{\underline{j}}; \tag{18}$$

and, eventually, $\nu(f(x))$ can be rewritten as $\nu_{x-\alpha}(f(x))$ in (11). Thus, Proposition (1) yields

Proposition 3. *For each root α of the polynomial $a_d(x)$ the exponent l_α in (16) is*

$$\min(\nu_{x-\alpha}(f(x)) - b_\alpha, \ \lambda_\alpha), \tag{19}$$

where λ_α is the least integer root of the indicial equation $I_\alpha(t) = 0$ (if there are no integer roots then $\lambda_\alpha = \infty$).

Proposition 4. *Let the indicial function $V(x)$ exist for the equation $L(y) = f(x)$. Then $V(x) \in K(x)$.*

Proof. Let an irreducible over K polynomial $p(x)$ be a divisor of $a_d(x)$, and K' be an extension of the field K that contains all roots of the polynomial $a_d(x)$. For a fixed j, $0 \le j \le d$, the values $\nu_{x-\alpha}(a_j(x))$ coincide for all α such that $p(\alpha) = 0$: over the field K' we have $\nu_{x-\alpha}(a_j(x)) = \nu_{p(x)}(a_j(x))$. >From this it follows that the values b_α defined via (17) also coincide for these α. Similarly, $\nu_{x-\alpha}(f(x)) = \nu_{p(x)}(f(x))$. Obviously, the values λ_α also coincide, since the equations $I_\alpha(t) = 0$ have the same sets of integer roots. Thus the values l_α defined via (19) coincide too for all α such that $p(\alpha) = 0$. Denote these values l_α by $l_{p(x)}$, then the expression (16) for the indicial function can be rewritten as

$$\prod_{\substack{p(x) \in \mathrm{Irr}(K) \\ p(x) | a_d(x)}} p^{l_{p(x)}}(x), \tag{20}$$

where $\mathrm{Irr}(K)$ is the set of normalized irreducible polynomials over K. The proof follows from this. $\square$

4 Rational solutions

Proposition 5. *Let our equation $L(y) = f(x)$ have a solution $F(x) \in K(x)$. Then*

(i) the indicial function does exist for the equation $L(y) = f(x)$;

(ii) $F(x) = q(x)V(x)$, where $q(x) \in K[x]$ and $V(x)$ is the indicial function of the equation.

Proof. Let K' be an extension of the field K that contains all the roots of the polynomial $a_d(x)$. For any rational function we put $G(x) = \frac{g(x)}{h(x)}$,

$$\nu_{p(x)}(G(x)) = \nu_{p(x)}(g(x)) - \nu_{p(x)}(h(x)).$$

The last definition is correct (it does not depend on the choice of $f(x), g(x)$).

Since every polynomial in x over K can be viewed as a (Taylor) series, we can write $K[x] \subset K[[x]]$. A rational function represented as a quotient of two polynomials $g(x)$ and $h(x)$, can be viewed as a Laurent series obtained by multiplying $g(x)$ by $h^{-1}(x)$ in $K((x))$. This series doesn't depend on a specific representation of the original rational function. In terms of this correspondence the field $K(x)$ can be isomorphically embedded into the field $K((x))$. If the series $\hat{G}(x)$ corresponds to the rational function $G(x)$ then $\nu_x(G(x)) = \nu(\hat{G}(x))$, and the derivative of the series $\hat{G}(x)$ corresponds to the derivative of the rational function $G(x)$. It follows that a rational solution can have a pole only in a such point α that $a_d(\alpha) = 0$. The last statement also holds true when considering rational functions over arbitrary extension of the field K'.

Consider an arbitrary root α of the polynomial $a_d(x)$. The differential equation $L_{x+\alpha}(y) = f(x+\alpha)$ (see (14), (15)) has the rational solution $G(x) = F(x+\alpha)$ and, respectively, the solution in the form of the series $\hat{G}(x)$, whence the indicial equation $I_\alpha(t) = 0$ has integer roots. Thus we conclude (i). The least integer root does not exceed $\nu(\hat{G}(x))$, i.e., does not exceed $\nu_{x-\alpha}(F(x))$, therefore we have (ii). □

Proposition 5 reduces the problem of finding rational solutions of equations of the type being concerned to finding polynomial solutions of the same type. It is well known (e.g., [1, 10]) that for a given differential equation $L(y) = f(x)$ of this type one can a priori give an upper bound to the degrees of its polynomial solutions. Let's find

$$c = \max_{0 \le j \le d} (\deg a_j(x) - j)$$

and construct the algebraic equation $I_\infty(t) = 0$, where

$$I_\infty(t) = \sum_{\substack{0 \le j \le d \\ \deg a_j(x) - j = c}} \operatorname{lc}(a_j) t^{\underline{j}}$$

(as usual, lc denotes the leading coefficient of a polynomial). If $q(x) \in K[x]$ and $L(q(x)) = f(x)$ then

$$\deg q(x) \le \max(\deg f(x) - c,\ \mu), \tag{21}$$

where μ is the greatest integer root of the equation $I_\infty(t) = 0$, and if there are no integer roots then $\mu = -\infty$.

This proof of Proposition 2 can be modified to the case of polynomial solutions: if a differential equation has "many" polynomial solutions then the estimate (21) is precise, i.e., among polynomial solutions there is a solution such that its degree is equal to the degree of the right-hand side of (21).

After setting the upper bound for finding polynomial solutions, the method of undetermined coefficients can be applied. It reduces the problem to solving a system of linear algebraic equations with coefficients in K.

A rational function $U(x) \in K(x)$ such that any rational solution of the original equation can be written as $u(x)U(x)$, $u(x) \in K[x]$ will be called a *universal factor* of the equation being considered. >From Proposition 5(ii) it follows that the indicial function is a universal factor.

The following proposition shows that the indicial rational function in some sense is the optimal variant of a universal factor.

Proposition 6. *Let the indicial function $V(x)$ exist for an equation $L(y) = f(x)$ and have the form (20). Let $p_1(x), p_2(x), \dots, p_k(x)$ be all different irreducible factors of the leading coefficient $a_d(x)$ of the operator L. Let the equation $L(y) = 0$ have d linearly independent solutions in $K(x)$ and let $U(x) \in K(x)$ be its universal factor. Then $U(x) = p_1^{s_1}(x)p_2^{s_2}(x)\dots p_k^{s_k}(x)r^{-1}(x)$, where $r(x) \in K[x]$, $r(x)$ is not divisible by $p_1(x), p_2(x), \dots, p_k(x)$, and*

$$s_1 \le l_{p_1(x)},\ s_2 \le l_{p_2(x)},\ \dots\ , s_k \le l_{p_k(x)}.$$

As a consequence, if $F(x) \in K(x)$, $L(F(x)) = f(x)$ and $F(x) = u(x)U(x) = v(x)V(x)$, $u(x), v(x) \in K[x]$, then $\deg v(x) \le \deg u(x)$.

Proof. It is possible to represent a rational solution of the original equation as a series. Using Proposition 2 we get $\nu_{p_i(x)}(U(x)) \ge l_{p_i(x)}$, $i = 1, 2, \dots, k$, and also $\nu_{q(x)}(U(x)) \le 0$ for all irreducible $q(x)$ that does not divide $a_d(x)$ (in this case $l_{q(x)} = 0$). □

Thus, using the substitution with a universal factor, we get the problem of finding polynomial solutions. We may find an upper bound for the degrees of all polynomial solutions and use, for example, the method of undetermined coefficients. In the case considered in Proposition 6 we get an equation that has "many" polynomial solutions, and the estimate of the form (21) is precise in the sense of our discussion. That is why the order of the system of linear algebraic equations we need to solve using the method of undetermined coefficients will reach its minimum when the indicial function is used as a universal factor.

5 Constructing the indicial rational function using the complete factorization

Suppose that all different irreducible factors $p_1(x), p_2(x), \dots, p_k(x)$ of the leading coefficient $a_d(x)$ of operator L are known. Let $p(x)$ be one of these factors. We can find

$$m_{p(x),j} = \nu_{p(x)}(a_j(x)),\quad j = 0, 1, \dots, d,$$

and

$$b_{p(x)} = \min_{0 \le j \le d} (m_{p(x),j} - j),$$

and construct a polynomial in two variables

$$J_{p(x)}(t, x) = \sum_{\substack{0 \le j \le d \\ m_{p(x),j} - j = b_{p(x)}}} \frac{a_j^{(m_{p(x),j})}(x)}{m_{p(x),j}!} t^{\underline{j}}. \tag{22}$$

By construction, this polynomial is such that the substitution of any root α of the polynomial $p(x)$ for x gives $I_\alpha(t)$. We noticed above in the proof of Proposition 4 that for all α such that $p(\alpha) = 0$ the sets of integer roots of equations $I_\alpha(t) = 0$ are the same. We denote the set of these roots by $N_{p(x)}$ and show two ways of computing it.

The first method. Rewrite the equation $J_{p(x)}(t, x) = 0$ in the form

$$u_v(x)t^v + u_{v-1}(x)t^{v-1} + \dots + u_0(x) = 0, \tag{23}$$

where $u_0(x), \dots, u_{v-1}(x), u_v(x)$ are polynomials in x of degree smaller than $\deg p(x)$ (each polynomial in x can be replaced by its remainder of division of this polynomial by $p(x)$), and $u_v(x)$ is a non-zero polynomial. Expand this equation by powers of x:

$$w_k(t)x^k + w_{k-1}(t)x^{k-1} + \dots + w_0(t) = 0, \tag{24}$$

where $k \le \deg p(x) - 1$, $w_i(t) \in K[t]$, $i = 0, 1, \dots, k$, $w_k(t) \in K[t] \setminus \{0\}$. After substitution of some root α of the polynomial $p(x)$ for x, the equation obtains an integer root n_0 iff all $w_i(t)$ occurring in (24) becomes zero when $t = n_0$ (because an element of the field $K(\alpha), p(\alpha) = 0$, written as a polynomial in α of degree smaller than $\deg p(\alpha)$ is zero iff all its coefficients are zero). The set of common integer roots of polynomials $w_i \in K[t]$, $i = 0, 1, \dots, k$, is finite, because $w_k(t) \in K[t] \setminus \{0\}$. This set is just $N_{p(x)}$.

The second method is based on the fact that $N_{p(x)}$ is the set of integer roots of the equation

$$\mathrm{Res}_x(J_{p(x)}(t, x), p(x)) = 0$$

(the resultant in the left-hand side is a polynomial in t).

After finding $N_{p(x)}$ in some way, we put $\lambda_{p(x)}$ equal to the minimal element of this set if it is not empty, and equal to ∞ in the other case. Further we easily find the value $l_{p(x)}$ of degree of polynomial $p(x)$ in (20):

$$l_{p(x)} = \min(\nu_{p(x)}(f(x)) - b_{p(x)},\ \lambda_{p(x)}).$$

If the exponent $l_{p(x)}$ is equal to infinity for some irreducible factor $p(x)$ of the polynomial $a_d(x)$, the indicial function does not exist.

Example 1. Let $p(x), q(x)$ be irreducible over $\mathbb{Q}$, $m, n \in \mathbb{N}^+$. Consider the differential equation

$$p(x)q(x)y' - (mp(x)q'(x) - nq(x)p'(x))y = 0. \tag{25}$$

Taking the factor $q(x)$ of the leading coefficient, we get $b_{q(x)} = 0$ and $J_{q(x)}(t, x) = p(x)q'(x)t - mp(x)q'(x)$. Hence, $l_{q(x)} = \lambda_{q(x)} = m$. Similarly, taking $p(x)$ we get $l_{p(x)} = \lambda_{p(x)} = -n$. Therefore $V(x) = p^{-n}(x)q^m(x)$. The substitution $y(x) = u(x)V(x)$ in (25) leads to the equation $u' = 0$, therefore the general rational solution of the original equation is $C\frac{q^m(x)}{p^n(x)}$, where C is an arbitrary constant.

Now we consider the inhomogeneous equation

$$\begin{aligned} p(x)q(x)y' - (mp(x)q'(x) - nq(x)p'(x))y = \\ p(x)q(x)q'(x) - mp(x)q(x)q'(x) + nq^2(x)p'(x). \end{aligned} \tag{26}$$

The left-hand side of the equation is not changed, hence $b_{p(x)}$, $b_{q(x)}$, $J_{p(x)}(t, x)$, $J_{q(x)}(t, x)$, $\lambda_{p(x)}$ and $\lambda_{q(x)}$ remain the same. Denote the right-hand side of equation (26) by $f(x)$; we get $\nu_{q(x)}(f(x)) = 1$, $\nu_{p(x)}(f(x)) = 0$. Thus we have $l_{p(x)} = -n$, $l_{q(x)} = 1$ and $V(x) = p^{-n}(x)q(x)$. After substitution $y(x) = u(x)V(x)$ we obtain an equation having the polynomial solution $Cq^{m-1}(x) + p^n(x)$. This corresponds to the fact that the general rational solution of (26) is $\frac{Cq^m(x)+q(x)p(x)}{p^n(x)}$.

6 Constructing the indicial rational function using a balanced factorization and the sub-partitioning

In the algorithm from Section 5 one needs to find all irreducible factsors $p_1(x), p_2(x), \dots, p_s(x)$ of the polynomial $a_d(x)$. If the complete factorization is undesirable by some reason, it is possible to use the other variant of this algorithm based on a *balanced factorization* [2]. We give necessary definitions.

Let $f(x), g(x) \in K[x]$, $\deg f(x) > 0$. The polynomial $f(x)$ is called balanced w.r.t. $g(x)$ if either $g(x)$ is zero, or $g(x) = f^l(x)\hat{g}(x)$, $\hat{g}(x) \in K[x]$, $l \ge 0$, and polynomials $f(x)$, $\hat{g}(x)$ are relatively prime. A factorization

$$f(x) = u_1(x)u_2(x)\dots u_k(x),$$

$\deg u_i(x) > 0, i = 1, 2, \ldots, k$, is called a balanced factorization of $f(x)$ w.r.t. $g(x)$ if every polynomial $u_i(x)$ is balanced w.r.t. $g(x)$. Let S be a finite subset of polynomials in $K[x]$ and $f(x) \in K[x]$, $\deg f(x) > 0$. Then $f(x)$ is called balanced w.r.t. S if it is balanced w.r.t. every element of S. A representation $f(x)$ in the form of product of factors balanced w.r.t. S is called a balanced factorization of the polynomial $f(x)$ w.r.t. S.

The algorithm for constructing a balanced factorization using base operations on polynomials and computation of the greatest common divisors of polynomials (gcd-technique) is given in [2]. A more formal definition (a pseudocode) of this algorithm can be found in [7].

We can make the polynomial $a_d(x)$ square-free taking the quotient of $a_d(x)$ by $\gcd(a_d(x), a'_d(x))$. Denote the result by $A(x)$. Let a balanced factorization of $A(x)$ w.r.t. the set of polynomials

$$f(x), a_0(x), a_1(x), \ldots, a_d(x) \tag{27}$$

has the form

$$h_1(x)h_2(x)\cdots h_k(x). \tag{28}$$

Let $g(x)$ be one of polynomials (27), and $h(x)$ be one of the factors of the product (28). We denote by

$$\nu_{h(x)}(g(x))$$

the greatest exponent such that the power $h(x)$ divides $g(x)$. It follows from the balanced factorization definition that for any irreducible factor $p(x)$ of $h(x)$ we have

$$\nu_{p(x)}(g(x)) = \nu_{h(x)}(g(x)).$$

Thus the approach described in Section 5 can be used for factors that may not be irreducible but are balanced in this sense. However, we need to introduce clarity into finding integer roots of equation (24), since we cannot assume that all $w_i(t)$ are be zero if the left-hand side of the equation is zero. Thus the first method of computing integer roots described in Section 5 does not work in this case. But the formal application of the second method creates no difficulties. By analogy with (22), find the polynomial $J_{h(x)}(t, x)$ using $h(x)$ instead of $p(x)$ and consider the set of integer roots of the equation

$$\mathrm{Res}_x(J_{h(x)}(t, x), h(x)) = 0.$$

We can define $\lambda_{h(x)}$ similarly to $\lambda_{p(x)}$, and then define the exponent for $h(x)$ as

$$l_{h(x)} = \min(\nu_{h(x)}(f(x)) - b_{h(x)},\ \lambda_{h(x)}).$$

Finding the product of all balanced factors with such exponents we obtain the rational function $\tilde{V}(x)$.

The substitution $y(x) = u(x)\tilde{V}(x)$ allows, for example, to turn from problem of finding rational solutions of original differential equation to the problem of finding polynomial solutions of the new equation of the same order [2], but $\tilde{V}(x)$ in the general case is not the indicial function, and the substitution described above may be more crude in the sense of Proposition 6 than the substitution $y(x) = u(x)V(x)$ with the indicial function $V(x)$.

However, $\tilde{V}(x)$ will be the indicial function if every balanced factor $h(x)$ is *flat* with a finite exponent, i.e., for all roots $\alpha, \beta, \ldots$ of the polynomial $h(x)$ the indicial equations

$$I_\alpha(t) = 0,\ I_\beta(t) = 0,\ \ldots \tag{29}$$

have integer roots and the least integer roots of these equations equal the same number n. Then n is the exponent of the factor $h(x)$ in $V(x)$. If all equations (29) have no integer roots then we say that the balanced factor $h(x)$ is flat with the exponent ∞. The existence of a flat factor with the exponent ∞ implies that the indicial function does not exist.

Let $N_{h(x)}$ be the set of integer roots of the equation $\mathrm{Res}_x(J_{h(x)}(t,x), h(x) = 0$. Starting with

$$h(x),\ J_{h(x)}(t,x),\ N_{h(x)}$$

a factorization of $h(x)$ into flat factors can be done by a simple procedure based on the gcd-technique. This procedure is called the *sub-partitioning*. We describe this procedure using for simplicity the notation $J(t,x)$ instead of $J_{h(x)}(t,x)$.

Let $N = \{n_0, n_1, \ldots, n_\delta\}$ and $n_0 < n_1 < \cdots < n_\delta$. Then for n_0 we find $h_{[n_0]}(x) = \gcd(J(n_0,x), h(x))$ and change $h(x)$ replacing it by the quotient of $h(x)$ by $h_{[n_0]}(x)$. Then we do the same with $J(t,x)$; using changed $h(x)$ and n_1 we obtain the polynomial $h_{[n_1]}(x)$ and so on. As a result we decompose $h(x)$ into the factors

$$h_{[n_0]}(x),\ h_{[n_1]}(x),\ \ldots\ , h_{[n_\delta]}(x). \tag{30}$$

If after computing $h_{[n_\delta]}(x)$ and changing $h(x)$ we get $\deg h(x) > 0$ then the indicial function does not exist. If it is not the case then the polynomials from (30) that is equal to 1 can be excluded from the further consideration; the remained polynomials of the form $h_{[n_i]}(x)$, $0 \le i \le \delta$, are flat with the exponent n_i.

Example 2. Let us turn to equation (25). The leading coefficient (denote it by $a_1(x)$) is square-free, and its possible balanced factorization consists of the single factor $h(x)$ equal $a_1(x)$. We get $b_{h(x)} = 0$ and

$$J_{h(x)}(t,x) = (p(x)q(x))'t - (mp(x)q'(x) - nq(x)p'(x)).$$

The set of integer roots of the equation $\mathrm{Res}_x(J_{h(x)}(t,x), h(x))$ is $N = \{-n, m\}$. If the sub-partitioning is not used, we obtain $\tilde{V}(x) = (p(x)q(x))^{-n}$. Let us apply the sub-partitioning. We have $\gcd(J_{h(x)}(-n,x), h(x)) = p(x)$. Changing $h(x)$, we get $h(x) = q(x)$ and $\gcd(J_{h(x)}(m,x), q(x)) = q(x)$. We obtain the indicial function $V(x) = p^{-n}(x)q(x)^m$.

The substitution $y(x) = u(x)\tilde{V}(x)$ leads to the equation

$$q(x)u'(x) - (n+m)q'(x)u(x) = 0$$

having polynomial solution $Cq^{n+m}(x)$. The substitution $y(x) = u(x)V(x)$ results in the equation $u'(x) = 0$, its polynomial solutions are constants.

Treating in the similar way equation (26) we get $(p(x)q(x))^{-n}$ before the sub-partitioning, and $p^{-n}(x)q(x)$ after it.

In this example $p(x)$ and $q(x)$ are not necessary irreducible. A balanced factorization, as well as a balanced factorization with the sub-partitioning, can be performed in the same way under the weaker assumption that $p(x), q(x)$, and also $p(x), p'(x)$ and $q(x), q'(x)$ are relatively prime.

7 Implementation and experiments

The algorithm for constructing a universal factor has already been implemented in computer algebra system Maple [9] as an auxiliary procedure used for finding rational solutions of a differential equation with polynomial coefficients in procedure **DEtools[ratsols]**. There exist even two such procedures. One of them uses the complete factorization and constructs the indicial function for a homogeneous equation. The other uses a balanced factorization (without the sub-partitioning) and constructs a universal denominator (i.e., $U(x)$ from Proposition 6 with $s_1 \leq 0, \ldots, s_k \leq 0$).

To compare the effectiveness of algorithms described in Sections 5, 6, we had to modify the existing code, to implement the sub-partitioning and to design all as a separate procedure that is available to users.

Thus, the package **IndicialFunction** has been implemented in Maple 11. We demonstrate the work of the main procedure of the package using the equation from Example 1. The equation (25) is written as usual in Maple

```
> ode := p(x)*q(x)*diff(y(x),x)-m*p(x)*diff(q(x),x)-
>         n*q(x)*diff(p(x), x))*y(x):
```

We put $p(x) = x^5 + 2, m = 5, q(x) = x^3 + x - 3, n = 7$:

```
> ode1 := eval(ode=0,  {p(x) = x^5+2, m = 5,
>          q(x) = x^3+x-3, n = 7}):
```

Compute the indicial function:

```
> IndicialFunction(ode1, y(x));
```

$$\frac{(x^3 + x - 3)^5}{(x^5 + 2)^7}$$

Similarly, for the inhomogeneous equation (26)

```
> f := p(x)*q(x)*diff(q(x),x)-(m*p(x)*diff(q(x),x)-
>         n*q(x)*diff(p(x), x))*q(x):
> ode2 := eval(ode=f,  {p(x) = x^5+2, m = 5,
>          q(x) = x^3+x-3, n = 7}):
```

we get

```
> IndicialFunction(ode2, y(x));
```

$$\frac{x^3 + x - 3}{(x^5 + 2)^7}$$

The package contains three procedures for constructing the indicial function:

- **IndicialFunction:-ByFactors**: uses complete factorization of the leading coefficient; calls standard Maple procedure **factors**;
- **IndicialFunction:-ByFactorsAndResultant**: uses complete factorization and computation of $N_{p(x)}$ by a resultant;
- **IndicialFunction:-BySubpartition**: uses a balanced factorization followed by the sub-partitioning.

The Maple procedure for a balanced factorization **'DEtools/balancedfacts'**[1] has also been modified and included into the package **IndicialFunction**. The procedure **'DEtools/balancedfacts'** returns a balanced factorization of the polynomial $f(x) = p_0 p_1^{s_1}(x) \cdots p_k^{s_k}(x)$ w.r.t. $g(x)$ in the form of the list $[p_0, [p_1(x), s_1], \ldots, [p_k(x), s_k]]$. For example, for:

[1]Such double (and also triple) names with apostrophes were given to auxiliary procedures in older versions of Maple until the module structure had been introduced. The procedure **'DEtools/balancedfacts'** has not its own help page but is available for use.

```
> f := x^13+x^11-3*x^10+4*x^8+4*x^6-12*x^5+4*x^3+4*x-12:
> g := x^4+x^2-2*x+x^3-3:
```

we get

```
> `DEtools/balancedfacts`(f, [g], x);
```

$$[1, [[x^5 + 2, 2], [x^3 + x - 3, 1]]]$$

This means that

$$f(x) = (x^5 + 2)^2(x^3 + x - 3).$$

During the factorization process we obtain the representation

$$g(x) = p_i(x)^{\nu_i}\,\hat{g}_i(x).$$

Since this information is necessary for constructing the indicial function, we programmed the procedure **IndicialFunction:-BalancedFactorization** to return it in the second list of the form $[g_0(x), \nu_{p_1(x)}(g(x)), \dots, \nu_{p_k(x)}(g(x))]$:

```
> IndicialFunction:-BalancedFactorization(f, [g], x);
```

$$[1, [[x^5 + 2, 2], [x^3 + x - 3, 1]]], [[x + 1, 0, 1]]$$

This means that

$$g(x) = (x + 1)(x^5 + 2)^0(x^3 + x - 3)^1.$$

Several experiments were made in order to compare the effectiveness of algorithms for constructing the indicial function. In the following way we get the working time of each algorithm (in seconds):

```
> st := time();
> IndicialFunction:-BySubpartition(ode1, y(x)):
> time()-st;
```

$$0.012$$

```
> st := time();
> IndicialFunction:-ByFactors(ode1, y(x)):
> time()-st;
```

$$0.016$$

```
> st := time();
> IndicialFunction:-ByFactorsAndResultant(ode1, y(x)):
> time()-st;
```

$$0.014$$

The working times of all procedures are rather small and are practically the same. Let's increase the degree of polynomial $p(x)$ in equation (25). We use the standard Maple procedure **randpoly** to generate a random polynomial of desired degree:

```
> randpoly(x, degree = 10);
```

$$-56x^{10} - 62x^7 + 97x^6 - 73x^3 - 4x^2$$

We tested the equation for $50 \le \deg p(x) \le 100$. The working times for the degrees 50, 60, 70, 80, 90, 100 are given below:

$\deg p(x)$	50	60	70	80	90	100
ByFactors	0.036	0.032	0.040	0.072	0.104	0.080
ByFactorsAndResultant	0.052	0.044	0.052	0.096	0.148	0.100
BySubpartition	0.328	0.380	0.732	2.360	2.464	3.021

These experiments showed that, as a rule, **ByFactors** and **ByFactorsAndResultant** have the same speed and work essentially faster than **BySubpartition**. Probably, the reason is that polynomial factorization in Maple is implemented more thoroughly than the search for the greatest common divisor.

After experiments we chose **ByFactors** as a default procedure for constructing the indicial function. Our package has been used for finding rational solutions of a differential equation. For the inhomogeneous equation **ode2** we have already obtained the indicial function:

```
> V := IndicialFunction(ode2, y(x));
```

$$V := \frac{x^3 + x - 3}{(x^5 + 2)^7}$$

Let us make a substitution $y(x) = V(x)u(x)$:

```
> ode3 := eval(ode2, y(x) = V*u(x));
```

$$ode3 := (x^5+2)(x^3+x-3)\left(-\frac{35(x^3+x-3)\,u(x)\,x^4}{(x^5+2)^8} + \frac{(3x^2+1)\,u(x)}{(x^5+2)^7} + \frac{(x^3+x-3)\,\frac{d}{dx}u(x)}{(x^5+2)^7}\right) - \frac{(5\,(x^5+2)\,(3x^2+1) - 35\,(x^3+x-3)\,x^4)\,(x^3+x-3)\,u(x)}{(x^5+2)^7} = (x^5+2)\,(x^3+x-3)\,(3x^2+1) - (5\,(x^5+2)\,(3x^2+1) - 35\,(x^3+x-3)\,x^4)\,(x^3+x-3)$$

and construct a polynomial solution of the new equation using **DEtools[polysols]**:

```
> Psol := DEtools[polysols](ode3, u(x));
```

$$Psol := [[81 - 108x + 54x^2 - 120x^3 + 109x^4 - 36x^5 + 58x^6 - 36x^7 + 6x^8 - 12x^9 + 4x^{10} + x^{12}], 128 + 448x^5 + 560x^{15} + 280x^{20} + 84x^{25} + 14x^{30} + x^{35} + 672x^{10}]$$

This answer means that the general polynomial solution of the equation **ode3** is:

```
> Psol[1][1]*C+Psol[2];
```

$$(81 - 108x + 54x^2 - 120x^3 + 109x^4 - 36x^5 + 58x^6 - 36x^7 + 6x^8 - 12x^9 + 4x^{10} + x^{12}\ )\,C + 128 + 448x^5 + 560x^{15} + 280x^{20} + 84x^{25} + 14x^{30} + x^{35} + 672x^{10}$$

where C is an arbitrary constant. Multiplying it by $V(x)$ we obtain the general rational solution of the equation **ode2**:

```
> V*Psol[1][1]*C+V*Psol[2];
```

$$\frac{x^3+x-3}{(x^5+2)^7}\left(81-108x+54x^2-120x^3+109x^4-36x^5+\right.$$
$$\left.58x^6-36x^7+6x^8-12x^9+4x^{10}+x^{12}\right)C+$$
$$\frac{x^3+x-3}{(x^5+2)^7}\left(128+448x^5+560x^{15}+280x^{20}+84x^{25}+14x^{30}+x^{35}+672x^{10}\right)$$

Using the procedure **DEtools[ratsols]** we can also get the general rational solution (in another form):

```
> DEtools[ratsols](ode2, y(x));
```

$$[[\frac{(x^3+x-3)^5}{(x^5+2)^7}, \frac{1}{(x^5+2)^7}(-71339352+118898408+79557752x^5+$$
$$145320248x^3-79265520x^2-162934680x^4+2936432x^{11}+1468552x^{13}+$$
$$560x^{16}+560x^{18}+280x^{21}+280x^{23}+84x^{26}+84x^{28}+14x^{31}+$$
$$14x^{33}+x^{36}+x^{38}-26421392x^8-96879632x^6+80733400x^7-17616576x^{10}+$$
$$291896x^{15}-4403640x^{12}+29357600x^9-840x^{20}-252x^{25}-42x^{30}-3x^{35})]$$

Now compare the running times of the new procedure:

```
> st := time();
> V := IndicialFunction(ode2, y(x)):
> ode3 := eval(ode2, y(x) = V*u(x)):
> Psol := DEtools[polysols](ode3, u(x)):
> [Psol[1][1]*V, Psol[2]*V]:
> time()-st;
```

$$0.068$$

and the procedure **DEtools[ratsols]**:

```
> st := time():
> DEtools[ratsols](ode2, y(x)):
> time()-st;
```

$$0.372$$

The new program is faster.

In the package **IndicialFunction** for a differential equation with the leading coefficient $a_d(x)$ the program first gets its factorization (complete or balanced)

$$a_d(x) = p_1^{s_1}(x)\dots p_k^{s_k}(x),$$

then it successively constructs indicial equations for all factors, finds its integer roots and computes the exponents l_{p_i}. The factors $p_i(x)$ with finite exponents are included into the product

$$V = p_{i_1}^{l_{p_{i_1}}}(x)\cdots p_{i_t}^{l_{p_{i_t}}}(x),$$

that is not always the indicial function. The factors with $l_{p_i}=\infty$ are saved in the list

$$B = [p_{j_1}(x),\dots,p_{j_r}(x)].$$

The procedure returns the pair (V,B). E.g., for the equation

```
> ode4 := (2+2*x^3+2 x)*y(x)+(x^3+x^4)*diff(y(x), x):
```

we get

```
> IndicialFunction(ode4, y(x));
```

$$\frac{1}{(1+x)^2},\ [x]$$

This answer means that this equation cannot have rational solutions, at the point $x = -1$ it has solutions in the form of a formal (Laurent) series, and at the point $x = 0$ such solutions do not exist. Applying the substitution $y(x) = u(x)/(x+1)^2$ to **ode4** and multiplying by common denominator, we get the equation with coefficients of lower degree:

```
> ode5 := numer(normal(eval(ode4, y(x) = u(x)/(1+x)^2)));
```

$$ode5 := x^3 \frac{d}{dx} u(x) + 2u(x)$$

for which $V = 1$:

```
> IndicialFunction(ode5, u(x));
```

$$1,\ [x]$$

If only rational solutions are needed then one can stop the execution of the program as soon as it gets $l_{p_i} = \infty$ to reduce the working time. To do this, use an additional argument **'ratsols'**:

```
> IndicialFunction(ode4, y(x), 'ratsols');
```

Here the program returns **NULL**, and this means that the indicial function does not exist.

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