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Solution Spaces of *H*-Systems and the Ore–Sato Theorem (Extended Abstract)[1]

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Linear homogeneous recurrence equations with polynomial coefficients and systems of such equations play a significant role in combinatorics and in the theory of hypergeometric functions; the question of the dimension of the space of solutions of such systems is of great importance for many problems.

Let E_{n_i} be the corresponding shift operators acting on functions (sequences) of $n_1, \ldots, n_d$ by $E_{n_i} f(n_1, \ldots, n_i) = f(n_1, \ldots, n_i + 1, \ldots, n_d)$, $i = 1, \ldots, d$.

Definition 1. An *H*-system is *a* system of equations

$$\begin{gathered} f_i(n_1, \ldots, n_d)T(n_1, \ldots, n_i + 1, \ldots, n_d) \\ = g_i(n_1, \ldots, n_d)T(n_1, \ldots, n_i, \ldots, n_d), \\ i = 1, 2, \ldots, d, \end{gathered} \tag{1}$$

where $f_i, g_i \in \mathbb{C}[n_1, \ldots, n_d]\backslash\{0\}$ and f_i, g_i are coprime. We say that a d-variate sequence T (i.e., a complex function defined on a subset of $\mathbb{Z}^d$) is a *solution* of (1) if (1) is satisfied at all those $(n_1, \ldots, n_i, \ldots, n_d)$ in the domain of T for which $(n_1, \ldots, n_i + 1, \ldots, n_d)$ belongs to the domain of T as well. We call a *hypergeometric term* any solution T of an *H*-system.

The prefix "*H*" in the name "*H*-system" refers to "hypergeometric" and to Jakob Horn, who, factually, defined a d-variate hypergeometric series as a d-variate Laurent series, whose coefficients form a hypergeometric term.

The notion of singular points (singularities) of such systems can be defined in the usual way. Such singularities make obstacles (sometimes, insuperable) for continuation of partial solutions of the system on all of $\mathbb{Z}^d$.

Definition 2. Let H be an *H*-system of the form (1).

A d-tuple $(n_1, \ldots, n_d) \in \mathbb{Z}^d$ is a *singularity* of H if there exists i, $1 \le i \le d$, such that $f_i(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_d)g_i(n_1, \ldots, n_d) = 0$.

Let $S(H)$ denote the set of all integer singularities of H. Denote by $V_1(H)$ the $\mathbb{C}$-linear space of all solutions of H which are defined at all elements of $\mathbb{Z}^d$, and by $V_2(H)$ the $\mathbb{C}$-linear space of all solutions of H which are defined at all elements of $\mathbb{Z}^d\backslash S(H)$.

Sometimes, we will drop the name of the *H*-system and will write V_1, V_2 instead of $V_1(H)$, $V_2(H)$.

We investigate the dimensions of the spaces V_1, V_2. It is well known [1] that, if (in the case $d = 1$) one considers the germs of sequences at infinity (i.e., classes of sequences which agree from some point on), then the dimension of the solution space is 1. However, the situation is different with $\dim V_1$ and $\dim V_2$.

When $d = 1$, the system (1) is of the form

$$f(n)T(n+1) = g(n)T(n), \tag{2}$$

where $f(n), g(n) \in \mathbb{C}[n]\backslash\{0\}$ and $f(n), g(n)$ are coprime. We prove for the case $d = 1$ the following.

Theorem 1. *Let S denote the set of singularities of Eq.* (2).

(a) *If $S = \varnothing$, then* $\dim V_1 = \dim V_2 = 1$.

(b) *If $S \ne \varnothing$, then* $1 \le \dim V_1 < \dim V_2 < \infty$, *and, for any integers s, t such that* $1 \le s < t$, *there exists an equation of the form* (2) *such that* $\dim V_1 = s$ *and* $\dim V_2 = t$.

For example, for the recurrence $T(n+1) = \prod_{i=0}^{k-2}(n - 2i + 1)T(n)$, $k \ge 1$, we have $\dim V_1 = 1$, $\dim V_2 = k$ (we use the convention that a product is 1 if its lower limit exceeds its upper limit). For $q_k(n+1)T(n+1) = q_k(n)T(n)$, where $k \ge 1$ and $q_k(n) = \prod_{i=0}^{k-2}(n + 2i + 1)$, we have $\dim V_1 = k$, $\dim V_2 = k + 1$.

We show that, in the case $d > 1$, the possibilities are even richer.

Theorem 2. *For any $d > 1$ and $1 \le s, t \le \infty$, there exists an H-system such that* $\dim V_1 = s$ *and* $\dim V_2 = t$.

For example, $\dim V_1 = \infty$, $\dim V_2 = 1$ for the system

$$(n_1 - 4n_2 + 1)T(n_1 + 1, n_2) = (n_1 - 4n_2)T(n_1, n_2),$$

$$(n_1 - 4n_2 - 4)T(n_1, n_2 + 1) = (n_1 - 4n_2)T(n_1, n_2),$$

while $\dim V_1 = 1$, $\dim V_2 = 1$ for

[1] The text was submitted by the authors in English.

$$(n_1 - 4n_2)T(n_1 + 1, n_2) = (n_1 - 4n_2 + 1)T(n_1, n_2),$$
$$(n_1 - 4n_2)T(n_1, n_2 + 1) = (n_1 - 4n_2 - 4)T(n_1, n_2),$$

and $\dim V_1 = \dim V_2 = \infty$ for

$$(n_1 - n_2 - 1)(n_1 - n_2 + 1)T(n_1 + 1, n_2)$$
$$= (n_1 - n_2)(n_1 - n_2 + 2)T(n_1, n_2),$$
$$(n_1 - n_2 - 1)(n_1 - n_2 + 1)T(n_1, n_2 + 1)$$
$$= (n_1 - n_2)(n_1 - n_2 - 2)T(n_1, n_2).$$

Definition 3. Rational functions $F_1, \ldots, F_d \in \mathbb{C}(n_1, \ldots, n_d)$ are *consistent* if

$$(E_{n_j} F_i) F_j = F_i (E_{n_i} F_j)$$

for all $1 \le i \le j \le d$.

We revisit the Sato–Ore theorem [2–4] which describes the structure of consistent rational functions. We show that, contrary to some interpretations in the literature (e.g., [5, 6]), this theorem does not imply that every solution of an H-system (even when the solution is defined everywhere on $\mathbb{Z}^d$) is of the form

$$R(n_1, \ldots, n_d) = \frac{\prod_{i=1}^{p} \Gamma(a_{i1} n_1 + \ldots + a_{id} n_d + \alpha_i)}{\prod_{j=1}^{q} \Gamma(b_{j1} n_1 + \ldots + b_{jd} n_d + \beta_j)} \times u_1^{n_1} \ldots n_d^{n_d}, \quad (3)$$

where $R \in \mathbb{C}(x_1, \ldots, x_d)$, a_{ik}, $b_{jk} \in \mathbb{Z}$, and α_i, $\beta_j \in \mathbb{C}$.

Let $A(n_1, n_2) = |n_1 - n_2|$. Then,

$$(n_1 - n_2)A(n_1 + 1, n_2) = (n_1 - n_2 + 1)A(n_1, n_2),$$
$$(n_1 - n_2)A(n_1, n_2 + 1) = (n_1 - n_2 - 1)A(n_1, n_2)$$

for all $n_1, n_2 \in \mathbb{Z}$, so $A(n_1, n_2)$ is a hypergeometric term. However, as we prove, $A(n_1, n_2)$ is not of the form (3). Notice that generating function of $A(n_1, n_2)$ is rational

$$\sum_{n_1, n_2 \ge 0} |n_1 - n_2| z_1^{n_1} z_2^{n_2}$$
$$= \left(\frac{z_1}{(1 - z_1)^2} + \frac{z_2}{(1 - z_2)^2} \right) \frac{1}{1 - z_1 z_2}.$$

It may be the case that, in the literature referred to above, a hypergeometric term $T(n_1, n_2)$ is implicitly assumed to be nonzero for all $n_1, n_2 \in \mathbb{Z}$. This possibility is supported by the fact that, e.g., in [5], the corresponding H-system is given in terms of the two quotients (rational functions) $T(n_1 + 1, n_2)/T(n_1, n_2)$ and $T(n_1, n_2 + 1)/T(n_1, n_2)$ (Horn, factually, used the same way). But such a severe restriction would preclude many important functions from being hypergeometric, such as the binomial coefficient $T(n_1, n_2) = \binom{n_2}{n_2}$, and all polynomials with integer roots. However, if we do not adopt this restriction, then there are hypergeometric terms which cannot be written in the form (3), as illustrated by the hypergeometric term $A(n_1, n_2)$.

Finally, we give an appropriate corollary of the Ore–Sato theorem on possible forms of solutions of systems under consideration.

Definition 4. Call the two d-tuples $(n_1, \ldots, n_d)$, $(n'_1, \ldots, n'_d) \in \mathbb{Z}^d$ *adjacent* if $\sum_{i=1}^{d} |n_i - n'_i| = 1$. Call a finite sequence $t_1, \ldots, t_k \in \mathbb{Z}^d$ a *path* from t_1 to t_k if t_i is adjacent to t_{i+1} for all $i = 1, \ldots, k - 1$.

Given an H-system H, we define *components* induced by H on $\mathbb{Z}^d$ as the equivalence classes of the following equivalence relation ~ in $\mathbb{Z}^d$: $t' \sim t''$ iff there exists a path from t' to t'' which contains no singularity of H. If T is a solution of an H-system H, then its *constituent* is the sequence that is the restriction of T on a component induced by H.

We call a set $M \subset \mathbb{Z}^d$ *algebraic* if there is a nonzero polynomial $p \in \mathbb{C}[x_1, x_2, \ldots, x_d]$ such that $p(n_1, n_2, \ldots, n_d) = 0$ for all $(n_1, n_2, \ldots, n_d) \in M$.

Corollary. *Any constituent of a hypergeometric term $T(n_1, \ldots, n_d)$ whose domain is a nonalgebraic component of $\mathbb{Z}^d$ induced by the original system is of the form* (3).

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REFERENCES

1. Petkov¡ek, M., Wilf, H.S., and Zeilberger, D., *A = B*, Wellesley, Mass.: A.K. Peters, 1996.
2. Ore, O., Sur les fonctions hypergéométriques de plusieurs variables, *Comp. Rend.*, 1929, vol. 189, p. 1238.
3. Ore, O., Sur la forme des fonctions hypergéométriques de plusieurs variables, *J. Math. Pure Appl.*, 1930, vol. 9, pp. 311–326.
4. Sato, M., Shintani, T., and Muro, M., Theory of Prehomogeneous Vector Spaces (Algebraic Part), *Nagoya Math. J.*, 1990, vol. 120, pp. 1–34.
5. Bateman, H. and Erdelyi, A., New York: McGraw-Hill, 1953, vol. 1.
6. Gel'fand, I.M., Graev, M.I., and Retakh, V.S., General Hypergeometric Systems of Equations and Series of Hypergeometric Type, 1992, vol. 47, pp. 3–82 [(Engl. Translation), 1992, vol. 47, pp. 1–88].