Rational solutions of linear difference and q-difference equations with polynomial coefficients *

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Abstract

We propose a simple algorithm to construct a polynomial divisible by the denominator of any rational solution of a linear difference equation

$$a_n(x)y(x+n)+\ldots+a_0(x)y(x)=b(x)$$

with polynomial coefficients and a polynomial right-hand side. Then we solve the same problem for q-difference equations.

Nonhomogeneous equations with hypergeometric right-hand sides are considered as well.

§1.**Difference equations**

Consider the problem of finding all rational solutions of linear difference equations of the form

$$a_n(x)y(x+n)+\ldots+a_0(x)y(x)=b(x) \tag{1}$$

or in operator form, $Ly(x)=b(x)$, where

$$L=a_n(x)E^n+\ldots+a_1(x)E+a_0(x). \tag{2}$$

Here $a_0(x)$,..., $a_n(x), b(x)$ are polynomials over a field K of characteristic zero. In [Abr89b] an algorithm to find a polynomial $u(x)$ such that $u(x)$ is divisible by the denominator of any (reduced) rational solution of Eq. (1) has been proposed. After constructing $u(x)$ one can substitute $z(x)/u(x)$ in Eq. (1) for $y(x)$, where $z(x)$ is an unknown polynomial. This results in an equation for $z(x)$ with polynomial coefficients and a polynomial right-hand side. The search for polynomial solutions has been considered in [Abr89a], [ABP95].

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The algorithm of [Abr89b] is quite complicated. Here, we can demonstrate how the same goal can be achieved more directly. If one takes as input

$$A(x) = a_n(x-n),\ \ B(x) = a_0(x)$$

then the following algorithm gives a polynomial $u(x)$ which can be used as the denominator of any rational solution of Eq. (1):

`input:` nonzero polynomials $A(x), B(x)$
`output:` a polynomial $u(x)$

> $u(x) := 1;$
> $R(m) := Res_x(A(x), B(x+m));$
> `if` $R(m)$ has some nonnegative integer root `then`
> $N :=$ the largest nonnegative integer root of $R(m)$;
>   `for` $i = N, N-1, ..., 0$ `do`
> $d(x) := gcd(A(x), B(x+i));$
> $A(x) := A(x)/d(x);$
> $B(x) := B(x)/d(x-i);$
> $u(x) := u(x)d(x)d(x-1)\dots d(x-i)$
>   `od`
> `fi.`

Example 1.

$$(2x^3 + 13x^2 + 22x + 8)E^3y - (2x^3 + 11x^2 + 18x + 9)E^2y +$$

$$+(2x^3 + x^2 - 6x)Ey - (2x^3 - x^2 - 2x + 1)y = 0.$$

It is easy to see that $u(x) = x^3 - x$ in this example. The substitution $y(x) = z(x)/(x^3 - x)$ yields the equation

$$(2x^4 + 7x^3 + 7x^2 + 2x)E^3z + (-2x^4 - 11x^3 - 18x^2 - 9x)E^2z +$$

$$+(2x^4 + 7x^3 - 3x^2 - 18x)Ez + (-2x^4 - 11x^3 - 16x^2 - x + 6)z = 0.$$

The general polynomial solution of this equation is $C(2x^2 - 3x)$. Therefore the general rational solution of the initial equation is $C(2x-3)/(x^2-1)$.

Below we prove correctness of the algorithm, but first we introduce some terminology.

We will call a polynomial *special* if its full factorization over K has the form

$$p^{\gamma_0}(x)p^{\gamma_1}(x+1)\dots p^{\gamma_h}(x+h) \tag{3}$$

where $p(x)$ is irreducible, $h, \gamma_0, \dots, \gamma_h$ are nonnegative integers. We will call two special polynomials *related* if their product is special again.

Let $g(x)$ be a special polynomial of the form (3). We will call a divisor

$$p^{\sigma}(x+l) \tag{4}$$

of $g(x)$ *critical of the first kind* if the relationship

$$p^{\sigma_1}(x+l_1)|g(x) \tag{5}$$

along with $l_1 > l$ implies $\sigma_1 < \sigma$ and along with $\sigma_1 > \sigma$ implies $l_1 < l$ (i.e., it is impossible to increase σ without decreasing l, and to increase l without decreasing σ). We will call a divisor of the form (4) of $g(x)$ *critical of the second kind* if the relationship (5) along with $l_1 < l$ implies $\sigma_1 < \sigma$ and along with $\sigma_1 > \sigma$ implies $l_1 > l$ (i.e., it is impossible to increase σ without increasing l, and to decrease l without decreasing σ).

Let $p^{\alpha_1}(x+M_1), \dots, p^{\alpha_s}(x+M_s)$ be all critical divisors of the first kind, and $p^{\beta_1}(x+m_1), \dots, p^{\beta_t}(x+m_t)$ be all critical divisors of the second kind. Let $M_1 > \dots > M_s$ and $m_1 < \dots < m_t$. Then $\alpha_1 < \dots < \alpha_s$ and $\beta_1 < \dots < \beta_t$. Let $\alpha_0 = \beta_0 = 0$ additionally.

Let $A(x), B(x) \in K[x]$. We will call a special polynomial $g(x)$ of the form (3) *bounded* by the pair $(A(x), B(x))$ if

$$p^{\alpha_i - \alpha_{i-1}}(x+M_i)|A(x), i = 1, \dots, s, \tag{6}$$

$$p^{\beta_j - \beta_{j-1}}(x+m_j)|B(x), j = 1, \dots, t. \tag{7}$$

Theorem 1. Let the result of applying the operator (2) to a rational function $S(x)$ with special denominator be a polynomial. Then the denominator of $S(x)$ is bounded by $(a_n(x-n), a_0(x))$.

Proof. Let the denominator of $S(x)$ be a polynomial $g(x)$ of the form (3). Let us prove, for example, (6). Let

$$a_n(x-n) = p^{\delta_1}(x+M_1) \dots p^{\delta_s}(x+M_s) f(x),$$

where $\delta_1, \dots, \delta_s$ are nonnegative integers, and $f(x)$ is not divisible by $p(x+M_i), i = 1, \dots, s$. Note that $S(x)$ can be decomposed as the sum of a polynomial and of fractions of the form

$$\frac{w(x)}{p^\gamma(x+l)}, \deg w(x) < \deg p^\gamma(x+l),$$

where values of l are pairwise different. This decomposition includes fractions with denominators

$$p^{\alpha_1}(x+M_1), \dots, p^{\alpha_s}(x+M_s).$$

Apply the operator L to each element of the decomposition and sum the elements with equivalent denominators. This gives us the decomposition of the function $LS(x)$. Therefore if $\alpha_i - \delta_i > \alpha_{i-1}$ for some i then the decomposition of $LS(x)$ includes an element with the denominator $p^{\alpha_i - \alpha_{i-1}}(x+M_i)$ and $LS(x)$ is not a polynomial. Contradiction.

Theorem 2. Let a special polynomial $g(x)$ be bounded by $(A(x), B(x))$. Let

$$R(m) = Res_x(A(x), B(x+m)).$$

Let $R(m)$ have nonnegative integer roots and let v be the largest of them. Let

$$d(x) = gcd(A(x), B(x+v)), c(x) = d(x)d(x-1)\dots d(x-v).$$

Let

$$\tilde{g}(x) = g(x)/gcd(g(x), c(x)),$$
$$\tilde{A}(x) = A(x)/d(x),$$
$$\tilde{B}(x) = B(x)/d(x - v).$$

Then $\tilde{g}(x)$ is bounded by $(\tilde{A}(x), \tilde{B}(x))$ and $g(x)|c(x)\tilde{g}(x)$.

Proof. The second part of the statement is obvious. To prove the first part, we note that critical divisors of $\tilde{g}(x)$ are also critical divisors of $g(x)$. Back to (6),(7). If $g(x)$ has been transformed to $\tilde{g}(x)$ then the differences $\alpha_i - \alpha_{i-1}, i = 1, \ldots, s$ and $\beta_j - \beta_{j-1}, j = 1, \ldots, t$ have been transformed without increasing. Distinction between $A(x)$ and $\tilde{A}(x)$, and between $B(x)$ and $\tilde{B}(x)$, respectively, either does not concern the factors $p(x+M_i), i = 1, \ldots, s, p(x+m_j), j = 1, \ldots, t$, or concerns only the exponents of $p(x + M_1), p(x + m_1)$ (or even only one of them). But in the latter case the exponents of $p(x+M_1), p(x+m_1)$ in $g(x)$ either undergo the same change as the exponents in $A(x)$ and $B(x)$, or vanish.

Finally, note that if $R(m)$ has no nonnegative integer root then 1 is the only polynomial bounded by $(A(x), B(x))$.

The last theorem shows that the algorithm proposed in the beginning of the paper allows one to compute a polynomial $u(x)$ divisible by any special polynomial bounded by

$$(A(x), B(x)).$$

But any rational nonpolynomial function $S(x)$ can be presented in the form $S_1(x)+\ldots+S_k(x)$ where $S_1(x), \ldots, S_k(x)$ are rational functions with nonrelated special denominators. The product of the denominators is equal to the denominator of $S(x)$. Applying operator (2) to $S_i(x), 1 \le i \le k$, gives either a polynomial, or a sum of a polynomial and a rational function with a denominator which is special and related to the denominator of $S_i(x)$. Therefore if $S(x)$ is a solution of Eq. (1) then every $S_i(x), i = 1, \ldots, k$, has the denominator which is bounded by $(a_n(x-n), a_0(x))$. And we obtain the desired "universal denominator" by using $A(x) = a_n(x - n), B(x) = a_0(x)$ as input for this algorithm.

In conclusion of the paragraph we have to refine our suppositions on the field K. Apparently we must know how to find integer roots of an algebraic equation $R(m)$ over K. Our coefficient field is (as in [Abr89b]) so-called *suitable* field in the sense of the following definition:

1) $\mathbf{Q} \subseteq K$;

2) there is an algorithm for finding integer roots of algebraic equations over K in one unknown.

The field $\mathbf{Q}$ is obviously suitable. It is easy to see that a simple extension (algebraic or transcendental) of a suitable field K is itself suitable.

The algorithm presented in this paragraph is a version of the algorithm which was given in [Abr 94]. Unfortunately, there was a mistake in the text of the published paper which has been corrected in Errata. The paper [Abr95] has a mistake also. The new version of the algorithm and its verification have not been published before.

The algorithm proposed in [Abr89b] is more complicated. But it takes into account all $a_0(x), \ldots, a_n(x), b(x)$, not only $a_0(x), a_n(x)$. Conceivably such an algorithm could give $u(x)$

of smaller degree than the algorithm proposed in this paragraph. But this question has not been investigated yet.

Undoubtedly the algorithm described above is quite convenient for implementation (much more so than the one proposed in [Abr89b]). Additionally, it can be adapted for the case of q-difference equations.

§2. q-Difference equations

We consider now the problem of looking for denominators of rational solutions of linear q-difference equations of the form

$$a_n(x)y(q^n x) + \ldots + a_1(x)y(qx) + a_0(x)y(x) = b(x), \tag{8}$$

or in operator form, $Ly(x) = b(x)$, where

$$L = a_n(x)Q^n + \ldots + a_1(x)Q + a_0(x).$$

Here $a_0(x), \ldots, a_n(x), b(x) \in K[x]$, q is an indeterminate parameter. Our coefficient field K is a so-called *q-suitable* field in the sense of the following definition:

1) $\mathbf{Q}(q) \subseteq K$;

2) there is an algorithm for finding the roots of the form q^m, where m is a nonnegative integer, of algebraic equations over K in one unknown.

The field $\mathbf{Q}(q)$ is obviously q-suitable. It is easy to see that a simple extension (algebraic or transcendental) of a q-suitable field K is itself q-suitable.

Let us investigate the polynomial $u(x)$ which arises as the result of the q-analog of the algorithm that has been proposed in §1 (consider $B(q^i)$ instead of $B(x+i)$, $d(q^{-i}x)$ instead of $d(x-i)$ and so on). The algorithm we describe as the function

```
function P(A(x), B(x))
    u(x) := 1;
    R(m) := Res_x(A(x), B(q^m x));
    if R(m) has some nonnegative integer root then
        N := the largest nonnegative integer root of R(m);
        for i = N, N-1, ..., 0  do
            d(x) := gcd(A(x), B(q^i x));
            A(x) := A(x)/d(x);
            B(x) := B(x)/d(q^{-i} x);
            u(x) := u(x)d(x)d(q^{-1} x) ... d(q^{-i} x)
        od
    fi;
    return(u(x)).
```

Note that the equation $R(m) = 0$ has the form

$$f_t q^{tm} + f_{t-1} q^{(t-1)m} + \ldots + f_1 q^m + f_0 = 0,$$

$f_0, \ldots, f_t \in K$. Searching for its nonnegative roots is equivalent to searching for roots of the form q^m of the algebraic equation

$$f_t X^t + f_{t-1} X^{(t-1)} + \ldots + f_1 X + f_0 = 0$$

over K. Since K is a q-suitable field, the roots can be found.

Theorem 3. Let Eq. (8) have a solution $v(x)/w(x)$ s.t.

$$v(x), w(x) \in K[x], gcd(v(x), w(x)) = 1,$$

$$w(x) = x^{\alpha} w^*(x), w^*(0) \neq 0.$$

Let

$$a_0(x) = x^{\beta} a_0^*(x), a_n(x) = x^{\gamma} a_n^*(x), a_0^*(0) \neq 0, a_n^*(0) \neq 0.$$

Let

$$A(x) = a_n^*(q^{-n}x), B(x) = a_0^*(x), u(x) = P(A(x), B(x)).$$

Then

$$w^*(x) | u(x). \tag{9}$$

Proof. First of all we remark that for any irreducible polynomials $p_1(x), p_2(x) \in K[x]$ s.t. $p_1(0) \neq 0, p_2(0) \neq 0$, we can find at most one nonnegative integer l s.t. $p_1(q^l x)$ and $p_2(q^l x)$ are equal up to a factor from K. Thus, all arguments that we gave in §1 will hold if we replace shifts

$$x \to x + h, x \to x - h,$$

where h is a nonnegative integer, by

$$x \to q^h x, x \to q^{-h} x,$$

and if we ignore the factor x when considering irreducible factors of polynomials.

To consider $a_0(x), a_n(x)$ instead of $a_0^*(x), a_n^*(x)$, we can remark that any rational solution of Eq. (8) can be presented in the form

$$\frac{f(x)}{g(x)} + \frac{l_1}{x} + \ldots + \frac{l_m}{x^m}, \tag{10}$$

where $f(x), g(x) \in K[x]$; $g(0) \neq 0$; $l_1, \ldots, l_m \in K$; m is a nonnegative integer. The results of substituting $f(x)/g(x)$ and

$$\frac{l_1}{x} + \ldots + \frac{l_m}{x^m} \tag{11}$$

for $y(x)$ in the left-hand side of Eq. (8) are rational functions with relatively prime denominators. Therefore the results must be some polynomials. We already know how to find a polynomial $u(x)$ s.t. $g(x)|u(x)$. It is necessary now to find an upper bound M for m. Then $U(x) = x^M u(x)$ can be taken as a universal denominator of rational solutions of Eq. (8).

To obtain a bound on m one can use the technique of indicial equations. (This technique is well known in the theory of linear ordinary differential equations.) We write all the $a_i(x)$ involved in (1) in the form

$$a_i(x) = x^{\alpha_i} a_i^*(x) \tag{12}$$

where $a_i^*(x) \in K[x], a_i^*(0) \neq 0, \alpha_i$ is a nonnegative integer. If $a_i(x)$ is the zero polynomial, then $a_i^*(x) = a_i(x), \alpha_i = \infty$. Let

$$\alpha = \min\{\alpha_0, \ldots, \alpha_n\}.$$

Assume that $\alpha = \alpha_i$ for $i = i_1, \dots, i_s$. If $m < \alpha$ then substituting (11) for $y(x)$ in the left-hand side of (8) trivially gives a polynomial. If $m > \alpha$ then the monomial $x^{\alpha - m}$ in the result of such substitution must have coefficient zero (the degree $\alpha - m$ of x is formally minimal in the result of substitution). The condition allows to write down the indicial equation

$$a^*_{i_1}(0)(q^{-m})^{i_1} + \ldots + a^*_{i_s}(0)(q^{-m})^{i_s} = 0. \tag{13}$$

Therefore, if $q^{m_1}, \dots, q^{m_t}$ are all the roots of the form q^l, l is a nonnegative integer, of the algebraic equation

$$a^*_{i_s}(0)X^{i_1 - i_s} + a^*_{i_{s-1}}(0)X^{i_1 - i_{s-1}} + \ldots + a^*_{i_1}(0) = 0, \tag{14}$$

then we can take $M = \max\{m_1, \dots, m_t, \alpha\}$ as an upper bound for m in (10).

The full algorithm to compute a universal denominator $U(x)$ of rational solutions of Eq. (8) can be given as follows:

`input:` Eq. (8)
`output:` a universal denominator $U(x)$

compute $a^*_i(x), \alpha_i (i = 0, \dots, n)$ as in (12);
$\alpha := \min\{\alpha_0, \dots, \alpha_n\}$;
construct Eq. (14);
$N := \max\{m | q^m$ is a root of Eq. (14)$\}$, or
-1 if the set is empty;
$M := \max\{N, \alpha\}$;
$U(x) := x^M P(a^*_n(q^{-n}x), a^*_0(x))$.

After constructing $U(x)$ one can substitute $z(x)/U(x)$ in Eq. (8) for $y(x)$, where $z(x)$ is an unknown polynomial. This results in an equation for $z(x)$ with polynomial coefficients and a polynomial right-hand side. The search for polynomial solutions has been considered in [ABP95].

Example 2. Consider the equation

$$q^3(qx+1)Q^2y - 2q^2(x+1)Qy + (x+q)y =$$
$$= (q^5 - 2q^3 + 1)x^2 + (q^4 - 2q^3 + 1)x.$$

Compute the universal denominator $U(x)$. Here $\alpha = 0$ and the algebraic equation (14) has the form

$$qX^2 - 2q^2X + q^3 = 0,$$

q is its unique solution of the wanted form. Hence $M = 1$. We can additionally find the factor $x + q$ of $U(x)$ with the help of the function P. So, $U(x) = x(x + q)$. Replacing $y = z(x)/x(x + q)$ in the original equation we get that

$$Q^2z - 2Qz + z = (q^5 - 2q^3 + 1)x^2 + (q^4 - 2q^3 + q)x.$$

The last equation has the polynomial solutions

$$C + qx^2 + x^3,$$

where C is an arbitrary constant. Therefore

$$\frac{C+qx^2+x^3}{x(qx+1)}$$

is the general rational solution of the original equation.

§3.**The case when q is a number**

In the previous paragraph we discussed the case when q is an undetermined parameter. But q-difference equations can be considered with a fixed number in the role of q. Let q be a number, $q \neq 1$. Let K be the field $\mathbf{Q}$ if q is a rational number, and $\mathbf{Q}(q)$ otherwise. If we want to retain the main steps of the previous algorithm we have to place restrictions on q.

First, we cannot use a root of unity in the role of q, because otherwise our remark at the begining of the proof of Theorem 3 is not valid. Next, we have to inspect closely the problem of finding roots of the form q^m of algebraic equations over K. If q is rational or transcendental this is easy. If q is algebraic over $\mathbf{Q}$ and $|q| \neq 1$ then we can construct an algebraic equation over $\mathbf{Q}$ s.t. all the roots of the old equation satisfy the new equation, and we can find upper bounds for the moduli of the nonzero roots of the new equation. Then we find an upper bound for m using the existing information about $|q|$.

But if q is an algebraic number and $|q| = 1$, then our problem becomes too difficult. The author knows a full solution of the problem of solving algebraic equations mentioned above only for the case when q is a quadratic irrationality and the given equation is quadratic over $\mathbf{Q}$ ([Abr87]).

All other steps of the algorithm described in §2 remain the same.

§4.**Hypergeometric right-hand sides**

So, one has a key to find rational solutions of linear difference and q-difference equations with polynomial coefficients and polynomial right-hand sides. A fast algorithm to solve the analogous problem connected with differential equations has been proposed in [Abr89b,AbrKva91]. The problem of computing rational solutions of Eq. (1), Eq. (8) or of analogous differential equations is quite important in computer algebra because some interesting problems can be reduced to it.

Consider the problem of finding hypergeometric term solution y of the nonhomogeneous difference equation $Ly = b$ where b is a nonzero hypergeometric term. Thus

$$Eb(x) = R(x)b(x) \tag{15}$$

for some $R(x) \in K(x)$. Let the coefficients of L be polynomials. In [Pet92] it has been shown that if a hypergeometric term $y(x)$ satisfies Eq. (1), then it is of the form $F(x)b(x)$ with a rational $F(x)$. One can substitute $F(x)b(x)$ for $y(x)$ in Eq. (1) with an unknown $F(x)$. This yields an equation with polynomial coefficients and a polynomial right-hand side. Thus to search for solutions in the form of hypergeometric terms the algorithm which finds rational solutions is quite useful.

Example 3. The equation

$$y(x+1) - y(x) = b(x)$$

with $b(x)$ in the form of hypergeometric term will be transformed, after substituting $F(x)b(x)$ for $y(x)$, to the following equation for $F(x)$:

$$R(x)F(x+1) - F(x) = 1$$

where $R(x) = b(x+1)/b(x)$. The last equation can easily be transformed into an equation with polynomial coefficients and a polynomial right-hand side:

$$a_1(x)F(x+1) + a_0(x)F(x) = c(x).$$

We can apply our algorithm to the last equation.

Let $A(x) = a_1(x-1), B(x) = a_0(x)$, where $a_1(x), a_0(x)$ are the coefficients of the last equation for $F(x)$. If we use the well-known Gosper's algorithm ([Gos78]), designed specially for indefinite hypergeometric summation, then the denominator $u(x)$ of $F(x)$ will be computed as follows

$u(x) := 1;$
$R(m) := Res_x(A(x), B(x+m));$
`if` $R(m)$ has some nonnegative integer root `then`
    $N :=$ the largest nonnegative integer root of $R(m)$;
    `for` $i = 0, 1, ..., N$ `do`
        $d(x) := gcd(A(x), B(x+i));$
        $A(x) := A(x)/d(x);$
        $B(x) := B(x)/d(x-i);$
        $u(x) := u(x)d(x)d(x-1)\dots d(x-i)$
    `od`
`fi.`

It gives in general $u(x)$ of smaller degree. But the last algorithm works well only in the case of indefinite hypergeometric summation. In the general case of Eq. (1) we can not use this way.

Example 4. The equation

$$(x+5)(x+8)^2y(x+4) - (x^3 + 11x^2 + 38x + 40)y(x+2) + x(x+2)y(x) = 0$$

has the rational solution

$$\frac{1}{x(x+2)^2(x+4)^2},$$

but the last algorithm gives

$$u(x) = x(x+1)(x+2)(x+3)(x+4).$$

Our algorithm gives

$$u(x) = x(x+1)(x+2)^2(x+3)^2(x+4)^2.$$

The idea of this example was communicate me by prof. V. Strehl.

The result of Petkovšek mentioned above can be generalized easily for the case of an arbitrary ring of Ore polynomials (see [BroPet94] for definitions). Let k be a field of coefficients. If $\theta f/f \in k$, then $\theta f = af$ for some $a \in k$. An induction on m allows us to show that $\theta^m f/f \in k$ for any $m > 0$. Indeed, let $\theta^{m-1} f = cf, c \in k$. Then $\theta^m f = \theta(cf) = \sigma(c)\theta f + \delta(c)f = (a\sigma(c) + \delta(c))f$, but $\sigma(c), \delta(c) \in k$. It is clear now that $L *_\theta f = df, d \in k$ for any $L \in k[x; \sigma, \delta]$.

Thus we can use this result not only in the difference case, but, for example, in the q-difference (see [AbrPet95]) and in the differential cases, i.e. one can use Q or, resp. d/dx in (15) instead of E and so on.

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