Eventually rational points and eventually m-points of linear ordinary differential operators

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Abstract

Let $L(y) = 0$ be a linear homogeneous ordinary differential equation with polynomial coefficients. One of the general problems connected with such an equation is to find all points a (ordinary or singular) and all formal power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ which satisfy $L(y) = 0$ and whose coefficient c_n – considered as a function of n – has some "nice" properties: for example, c_n has an explicit representation in terms of n, or the sequence $(c_0, c_1, \ldots)$ has many zero elements, and so on. It is possible that such properties appear only eventually (i.e., only for large enough n).

We consider two particular cases:

1. $(c_0, c_1, \ldots)$ is an eventually rational sequence, i.e., $c_n = R(n)$ for all large enough n, where $R(n)$ is a rational function of n;

2. $(c_0, c_1, \ldots)$ is an eventually m-sparse sequence, where $m \geq 2$, i.e., there exists an integer N such that

$$(c_n \neq 0) \Rightarrow (n \equiv N \pmod{m})$$

for all large enough n.

Note that those two problems were previously solved only "for all n" rather than "for n large enough", although similar problems connected with polynomial and hypergeometric sequences of coefficients have been solved completely.

Résumé

Soit $L(y) = 0$ une équation différentielle linéaire ordinaire homogène et à coefficients polynomiaux. Un problème général en liaison avec une telle équation est la recherche de tous les points a (ordinaires ou singuliers) et de toutes les séries formelles $\sum_{n=0}^{\infty} c_n(x-a)^n$ qui vérifient $L(y) = 0$ et dont les coefficient c_n – considérés comme une fonction de n – vérifient de "bonnes" propriétés, comme par exemple, que c_n admette une représentation explicite en termes de n, ou que la suite $(c_0, c_1, \ldots)$ comprend de nombreux termes nuls. Un autre cas intéressant est par ailleurs celui où de telles propriétés n'apparaissent qu'asymptotiquement (ex: pour des n assez grands).

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Dans cet article, nous considérons les deux cas particuliers suivants :

1. $(c_0, c_1, \ldots)$ est une suite ultimement rationnelle, i.e., $c_n = R(n)$ pour tout n assez grand, où $R(n)$ est une fraction rationnelle en n ;

2. $(c_0, c_1, \ldots)$ est une suite ultimement m-creuse, où $m \geq 2$, i.e., il existe un entier N tel que

$$(c_n \neq 0) \Rightarrow (n \equiv N \pmod m)$$

pour tout n assez grand.

Remarquons que ces deux problèmes n'avaient été jusqu'ici résolus que "pour tout n", et non "pour des n assez grands", bien que des problèmes similaires en connection avec des suites de coefficients polynomiaux ou hypergéométriques aient été résolus de façon complète.

Keywords: Linear differential equations, Formal solutions, Recurrences for coefficients, m-Sparse power series, Eventually rational points, Eventually m-points.

1 Introduction

Algorithms for solving ordinary differential equations by means of power series date back to Newton. It is of interest in the context of modern computer algebra and theory of generating functions to consider the problem of the search for formal power series solutions

$$\sum_{n=0}^{\infty} c_n (x-a)^n \tag{1}$$

whose coefficients c_n have some "nice" properties, for example, c_n as a function of n has an explicit representation in terms of n, or there are many zeros among $c_0, c_1, \ldots$, and so on. In the general case a fixed class $\mathcal{M}$ of sequences $c = (c_0, c_1, \ldots) \in \mathbb{C}^\infty$ is given. For a given differential equation, one of the problems is connected with the search for such solutions which have the form (1) with $(c_0, c_1, \ldots) \in \mathcal{M}$. The choice of the point a is of fundamental importance in such a problem, because it is possible that such a solution exists at one point and does not exist at another.

We will consider also the following more general problem: to find all points a (ordinary or singular) and all formal power series solutions (1) of the given equation such that elements of $(c_0, c_1, \ldots)$ coincide with the corresponding elements of some sequence of the class $\mathcal{M}$ for all large enough n (i.e., eventually). In particular, we can discuss solutions in the form of series whose coefficient sequence is eventually polynomial (i.e., there exists a polynomial $p(n)$ such that $c_n = p(n)$ for all large enough n) or eventually rational (i.e., there exists a rational function $r(n)$ such that $c_n = r(n)$ for all large enough n) and so on. Such a formulation of the problem is quite natural because, for example, a rational function can be undefined for some nonnegative integer numbers.

We will call any solution of the form (1) of a differential equation *local* at the point a. Local solutions at a fixed point a form a linear space over $\mathbb{C}$ that we will denote by $\mathcal{O}_a(L)$.

The problem of the search for the local solutions that have the coefficient sequence $(c_0, c_1, \ldots)$ belonging to one or another class was considered in a few papers. The basis of each of the approaches was the following: if a point a is fixed then the coefficients of any local solution at a of the equation

$$p_r(x)y^{(r)} + \cdots + p_1(x)y' + p_0(x)y = 0, \tag{2}$$

$p_0(x), \ldots, p_r(x) \in \mathbb{C}[x]$, satisfy the linear recurrence

$$q_l(n)c_{n+l} + q_{l-1}(n)c_{n+l-1} + \cdots + q_t(n)c_{n+t} = 0, \tag{3}$$

$q_l(n), q_{l-1}(n), \ldots, q_t(n) \in \mathbb{C}[n]$. The last recurrence can be easily constructed. In [14] the search for local solutions of (2) at a fixed point a with hypergeometric sequences (i.e., sequences which satisfy first order linear homogeneous recurrences with polynomial coefficients) of coefficients has been considered. It was shown that if the corresponding solutions of recurrence (3) are found, then constructing the desired local solutions of (2) is a simple linear algebra problem. Algorithm Hyper [13] can be used to search for all hypergeometric solutions of recurrence (3). Additionally in [14] an algorithm to search for primitive m-hypergeometric sequences satisfying a recurrence of the form (3) is given. This allows one to find all local solutions with primitive m-hypergeometric sequences of coefficients (for all n or eventually). We remark that a sequence $(c_k, c_{k+1}, \ldots)$ is *m-hypergeometric* if $a(n)c_{n+m} + b(n)c_n = 0$, $n = k, k+1, \ldots$, for some polynomials $a(n), b(n)$; an m-hypergeometric sequence $(c_k, c_{k+1}, ...)$ is *primitive* if it satisfies no linear homogeneous recurrence with polynomial coefficients of order $< m$.

But in [14] only the case of a fixed point a was discussed, and the search for suitable points a was not considered. In [6] the problem was considered for polynomial, rational and hypergeometric sequences of coefficients. Looking for suitable points was the principal moment of the investigation. It was shown that if (2) has a local solution with a polynomial sequence of coefficients (for all n or eventually) then $a+1$ is a singularity of equation (2), i.e., $p_r(a+1) = 0$. It was shown also that if (2) has a local solution with a rational sequence of coefficients (for all n) then a is a singularity of equation (2), i.e., $p_r(a) = 0$. It was shown that if (2) has a local solution with a hypergeometric sequence of coefficients at an ordinary point a then such solutions exist at any ordinary point, i.e., an ordinary point a can be chosen arbitrarily and then investigated. All singular points have to be investigated one after another (there is a finite set of them).

In [2, 3] the case of m-sparse sequences of coefficients was considered. The sequence $(c_0, c_1, \ldots)$ is *m-sparse*, where $m \geq 2$, if there exists an integer N such that

$$(c_n \neq 0) \Rightarrow (n \equiv N \pmod{m}). \tag{4}$$

The problem of the search for corresponding points a was solved for the case where the sequence of coefficients is m-sparse for all n. An upper bound for m was found and it was shown that for any fixed m either there exist only finitely many suitable points a (they are called *m-points* of the given equation) and they can be found explicitly, or all points $a \in \mathbb{C}$ are m-points of the given equation and the operator

$$L = p_r(x)D^r + \cdots + p_1(x)D + p_0(x) \tag{5}$$

can be factored as

$$L = \tilde{L} \circ C \tag{6}$$

where C is an operator of the special m-sparse form with constant coefficients.

The solutions in the form of power series with m-sparse coefficients are of interest by themselves and especially in connection with the search for m-hypergeometric m-sparse power series solutions like power series for $\sin(x), \cos(x)$ (2-hypergeometric 2-sparse), Airy functions (3-hypergeometric 3-sparse), etc. The sum of any of these power series and a polynomial is an eventually m-hypergeometric m-sparse power series for some m.

Note that the mentioned algorithm from [14] allows one to find only primitive m-hypergeometric solutions of a recurrence. But it is easy to prove that an m-hypergeometric m-sparse solution having $c_n \neq 0$ with arbitrary large n is primitive m-hypergeometric. Thus the algorithm from [14] together with an algorithm to search for all m-points is sufficient for the search of all m-hypergeometric m-sparse local solutions of the given differential equation.

Looking through the list of solved problems of the search for local solutions one can detect two gaps in it. In [14, 6, 2, 3] the following two concrete cases have not been considered.

G1. $(c_0, c_1, \ldots)$ is an eventually nonpolynomial rational sequence, i.e., we have $c_n = R(n)$ for all large enough n, where $R(n)$ is a nonpolynomial rational function of n.

G2. $(c_0, c_1, \ldots)$ is eventually m-sparse (in particular m-hypergeometric m-sparse), i.e., there exists an integer N such that (4) holds for all large enough n.

Concerning **G1**, note that any rational sequence is hypergeometric. But there is no method in [6, 7] which lets one select such ordinary points at which a local solution with a rational coefficient sequence exists.

It is possible to give examples showing that series with the coefficient sequences mentioned in **G1**, **G2** exist at points which algorithms from [6, 2, 3] do not find.

Example 1 The equation

$$(1-x)y'' - y' = 0 \tag{7}$$

has the local solution

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \tag{8}$$

with nonpolynomial rational function coefficients for $n \geq 1$, while the point $a = 0$ is not a singularity of (7).

Example 2 The equation

$$(x^5 - 2x^3 - x^2 + x + 1)y' - (x^4 - 2x^2 + 2x + 1)y = 0 \tag{9}$$

has the local solution

$$x + \frac{1}{1-x^2} = 1 + x + x^2 + x^4 + x^6 + \cdots \tag{10}$$

which is 2-sparse (and 2-hypergeometric as well) for $n \geq 2$. But applying the algorithm from [2, 3] to (9) with $m = 2$ results only in the information that (9) has no 2-points, and does not yield the point $a = 0$.

Below we will fill in the two indicated gaps (**G1** and **G2**). The result is that either only a finite set of candidates for suitable points exist, or all points are suitable. In the first case each candidate can be checked by solving a simple linear algebra problem. We will not discuss this check because it is very similar to the one described in [14].

A preliminary version of this paper has appeared as [4].

2 Generalities

We can write (2) in the operator form $L(y) = 0$, where L is equal to (5). The recurrence (3) for coefficients of a local solution at 0 can be written as $R(c) = 0$ where R is a difference (recurrence) operator

$$q_l(n)E^l + q_{l-1}(n)E^{l-1} + \cdots + q_t(n)E^t \tag{11}$$

with $l \geq t$; $q_l(n), \ldots, q_t(n) \in \mathbb{C}[n]$; $q_l(n), q_t(n) \neq 0$. The operator R is the $\mathcal{R}$-image of L where $\mathcal{R}$ is the isomorphism of $\mathbb{C}[x, x^{-1}, D]$ onto $\mathbb{C}[n, E, E^{-1}]$:

$$\mathcal{R}D = (n+1)E, \ \mathcal{R}x = E^{-1}, \ \mathcal{R}x^{-1} = E;$$

resp.

$$\mathcal{R}^{-1}E = x^{-1}, \ \mathcal{R}^{-1}E^{-1} = x, \ \mathcal{R}^{-1}n = xD$$

(see [7]).

It can be useful to consider sequences of the form

$$c = (c_k, c_{k+1}, \ldots) \tag{12}$$

where k is an integer, possibly negative. If c has the form (12) then we write $\nu(c) = k$. A sequence of the form (12) can be multiplied by any $\alpha \in \mathbb{C}$ and, therewith, $\nu(\alpha c) = \nu(c)$. The sum of two sequences c and c' is such that $\nu(c + c') = \max\{\nu(c), \nu(c')\}$. The actions of the shift operator E and its inverse E^{-1} are defined in the natural way, $\nu(Ec) = \nu(c) - 1$, $\nu(E^{-1}c) = \nu(c) + 1$. Finally, if c is of the form (12) and a function $f(n)$ is defined for all $n \geq k$ then $f(n)c = (f(k)c_k, f(k+1)c_{k+1}, \ldots)$ and $\nu(f(n)c) = \nu(c)$. We say that c of the form (12) satisfies the equation $R(z) = 0$ if applying R to c gives the sequence $(d_{k-l}, d_{k-l+1}, \ldots)$ with zero elements.

If the coefficient of x^i in the polynomial $p_j(x)$ is not equal to zero in (5) then we write $x^iD^j \in L$. It is easy to check that if L is of the form (5) and $R = \mathcal{R}L$ then

$$l = \max_{x^iD^j \in L} \{j - i\}, \ t = \min_{x^iD^j \in L} \{j - i\}. \tag{13}$$

We set $\omega^*(R) = l, \omega_*(R) = t$. In the case $R = \mathcal{R}L$ we write

$$\omega^*(L) = \omega^*(R), \ \omega_*(L) = \omega_*(R).$$

Let $c = (c_0, c_1, \ldots)$. Denote by (c, x) the formal series $c_0 + c_1x + \cdots$ and by $(c)_{\geq k}$ the sequence $(c_k, c_{k+1}, \ldots)$ with $c_k = c_{k+1} = \ldots = c_{-1} = 0$ if $k < 0$. It can be shown that if $R = \mathcal{R}L$ and R is of the form (11), $t = \omega_*(L)$, then

$$L((c, x)) = 0 \Longleftrightarrow R((c)_{\geq t}) = 0 \tag{14}$$

(see [5, 7]). Let R be of the form (11) and ρ_0 be the maximal nonnegative integer root of $q_l(n)$ if such roots exist, and -1 otherwise. Set

$$\iota^*(R) = l + \rho_0 = \omega^*(R) + \rho_0.$$

Let $L \in \mathbb{C}[x, D]$ and $R = \mathcal{R}L$, then we set $\iota^*(L) = \iota^*(R)$. For any $c = (c_0, c_1, \ldots)$ such that $L((c, x)) = 0$ the values $c_0, \ldots, c_{\iota^*(L)}$ allow one to compute (by means of $\mathcal{R}L$) the values $c_{\iota^*(L)+1}, c_{\iota^*(L)+2}, \ldots$ (these latter values are uniquely determined because the leading coefficient of the recurrence $\mathcal{R}L$ does not vanish when we compute c_n with $n > \iota^*(L)$). Let $a \in \mathbb{C}$. Let L be of the form (5). Observe that the formal power series y_a of the form (1) is such that $L(y_a) = 0$ iff $L^a(y) = 0$, where y is equal to

$$\sum_{n=0}^{\infty} c_n x^n \tag{15}$$

and

$$L^a = p_r(x+a)D^r + \cdots + p_1(x+a)D + p_0(x+a). \tag{16}$$

So, the general case of a fixed a can be reduced to the case $a = 0$.

Lemma 1 *[2, 3] Let L be an operator of the form (5). Let a either be a parameter or belong to $\mathbb{C}$. Let $R^a = \mathcal{R}L^a$ and R^a be equal to*

$$g_{l'}(n, a)E^{l'} + \cdots + g_{t'}(n, a)E^{t'}.$$

Then $t' = \omega_*(L)$ and $g_{t'}$ does not depend on a. If $a \in \mathbb{C}$ then $l' \le r$; otherwise $l' = r$. □

Let R be of the form (11). Let ρ_1 be the maximal nonnegative integer root of $q_t(n)$ if such roots exist, and -1 otherwise. Set

$$\iota_*(R) = \max\{t + \rho_1, -1\} = \max\{\omega_*(R) + \rho_1, -1\}.$$

Let $L \in \mathbb{C}[x, D]$ and $R = \mathcal{R}L$, then we set $\iota_*(L) = \iota_*(R)$.

We formulate three properties of the value ι_* which will be useful later.

1. For any $(c_0, c_1, \ldots)$ such that $L((c, x)) = 0$ the values $c_k, c_{k+1}, \ldots$ with $k > \iota_*(L) + 1$ let one compute (by means of $\mathcal{R}L$) the values $c_{\iota_*(L)+1}, c_{\iota_*(L)+2}, \ldots, c_{k-1}$ (these latter values are uniquely determined because the trailing coefficient of the recurrence $\mathcal{R}L$ does not vanish when we compute c_n with $n > \iota_*(L)$).

2. $\iota_*(L^a) = \iota_*(L)$ (by Lemma 1).

3. Let L have the form (5), $R = \mathcal{R}L$ and R be of the form (11). Let $R(d) = 0$ where $d = (d_s, d_{s+1}, \ldots)$, $s = \iota_*(L) + 1$. Let (15) satisfy equation $L(y) = 0$ and $(c_0, c_1, \ldots)$ be the coefficient sequence of (15). Let

$$c_n = d_n \tag{17}$$

for all large enough n. Then (17) holds for all $n = s, s+1, \ldots$ (by property 1).

3 Eventually rational points of operators

If an equation $L(y) = 0$ of the form (5) has a local solution (1) at a such that $(c_0, c_1, \ldots)$ is a rational sequence for all n (resp. for all large enough n), then we call a a *rational* point (resp. an *eventually rational* point) of L and of $L(y) = 0$. It is evident that any rational point is eventually rational.

Lemma 2 *Let $L \in \mathbb{C}[x, D]$. Then there exists $L^{[1]} \in \mathbb{C}[x, D]$ such that for any point a the operator of differentiation D maps the space $\mathcal{O}_a(L)$ onto the space $\mathcal{O}_a(L^{[1]})$.*

Proof: Due to Ore's theory [11, 12] the operator $L^{[1]}$ is defined by the equality

$$\mathrm{LCM}(L, D) = L^{[1]} \circ D$$

(LCM is the least common left multiple). In practice it is convenient to construct $L^{[1]}$ directly, without using the Euclidean algorithm: let L have the form (5). If $p_0(x)$ is the zero polynomial then $L^{[1]} = p_r(x)D^{r-1} + \cdots + p_1(x)$, otherwise one can construct

$$p_0(x)D \circ L - p_0'(x)L \tag{18}$$

which has the form $\tilde{p}_r D^{r+1} + \cdots + \tilde{p}_0 D$ and set $L^{[1]} = \tilde{p}_r D^r + \cdots + \tilde{p}_0$. □

One can construct operators $L^{[2]} = (L^{[1]})^{[1]}, L^{[3]} = ((L^{[1]})^{[1]})^{[1]}, \ldots$ as well.

Lemma 3 *Let $L \in \mathbb{C}[x, D]$ and a either belong to $\mathbb{C}$ or be a parameter. Then $(L^a)^{[1]} = (L^{[1]})^a$.*

Proof: This is evident if $p_0(x)$ is the zero polynomial. Otherwise observe that for the operator M which is equal to (18) we have

$$M^a = p_0(x + a)D \circ L^a - p_0'(x + a)L^a,$$

and at the same time $p_0(x + a)$ is the coefficient of D^0 in the operator L^a. □

Let $U(n)$ be a rational function such that for the series y_a defined by (1) the equality $c_n = U(n)$ holds for $n \geq k$ where k is a nonnegative integer. Then the series

$$y_a' = \sum_{n=0}^{\infty} f_n (x - a)^n$$

is such that $f_n = (n + 1)U(n + 1)$ for $n \geq \max\{0, k - 1\}$. It is clear that $V(n) = (n + 1)U(n+1)$ is a rational function of n (it is possible that $V(n)$ is a polynomial while $U(n)$ is a nonpolynomial rational function). It is easy to show that if y_a satisfies $L(y) = 0$ of the form (2) then $c_n = U(n)$ for all $N > \iota_*(L)$: in [1] a description of an algorithm to find rational solutions of a linear recurrence with polynomial coefficients was given; it was shown there that if $S(c) = 0$ is such a recurrence then any pole of a rational function which satisfies the recurrence is $\leq \iota_*(S)$. Therefore in the case $S = R^a = \mathcal{R}L^a,\ a \in \mathbb{C}$, the poles are $\leq \iota_*(L^a)$. But by property 2 of the value ι_* (see Section 2) we have $\iota_*(L^a) = \iota_*(L)$. We can use further property 3 of ι_*. We get the following theorem.

Theorem 1 *Let a be an eventually rational point of $L \in \mathbb{C}[x, D]$. Then a is a rational point of $L^{[\iota_*(L)+1]}$.* □

Therefore, to find all eventually rational points of L it is sufficient to construct the operator $M = L^{[\iota_*(L)+1]}$ and to investigate all points a such that either a itself or $a+1$ is a singularity of the operator M.

Going back to Example 1 we see that the recurrent operator $(n+1)(n+2)E^2 - (n+1)^2E$ corresponds to equation (7). Therefore $\iota_*(L) = 0$ where $L = (1-x)D^2 - D$. We have $L^{[1]} = (1-x)D - 1$. The set of singularities and of points a such that $a+1$ is a singularity of $L^{[1]}$ is $\{0, 1\}$. Further investigation shows that 0 is an eventually rational point of L.

4 Eventually m-points of operators

As noted in Section 1, a point a is an m-point of an operator L if the equation $L(y) = 0$ has a local solution at a with m-sparse sequence of coefficients. If the sequence is m-sparse for all large enough n then we will call a an *eventually m-point* of L (hence, any m-point of L is at the same time an eventually m-point).

We will consider along with operators L and $R = \mathcal{R}L$ the set of operators $L_0, \dots, L_{m-1}$ and $R_0, \dots, R_{m-1}$ which are called an *m-splitting* of the operators L and R ([2, 3]). If L and R are of the form (5) and, resp., (11) then

$$L_\tau = \sum_{\substack{x^iD^j \in L \\ j-i-t \equiv \tau \pmod m}} p_{ji}x^iD^j, \tag{19}$$

$$R_\tau = \sum_{\substack{t \le j \le l \\ j-t \equiv \tau \pmod m}} q_j(n)E^j, \tag{20}$$

$\mathcal{R}L_\tau = R_\tau,\ \tau = 0, \dots, m-1,\ l = \omega^*(R) = \omega^*(L),\ t = \omega_*(R) = \omega_*(L)$. We call a difference operator of the form (11) m-sparse if for some N

$$(q_j(n) \ne 0) \Rightarrow (j \equiv N \pmod m)$$

and we call a differential operator M m-sparse if for some N

$$(x^iD^j \in M) \Rightarrow (j - i \equiv N \pmod m).$$

It is easy to see that any differential and any difference operator defined by (19) and by (20) are m-sparse. It is also easy to show that the $\mathcal{R}$-image of a differential operator is an m-sparse difference operator iff the original differential operator is m-sparse.

In [8] some properties of the sequences that satisfy equalities $T_1(c) = T_2(c) = \cdots = T_k(c) = 0$, where $T_1, \dots, T_k \in \mathbb{C}[n, E]$, are proven. Those results can trivially be extended to the case $T_1, \dots, T_k \in \mathbb{C}[n, E, E^{-1}]$. We will use a theorem from [8] that after extending to operators from $\mathbb{C}[n, E, E^{-1}]$ can be presented in the following form:

Theorem 2 *Let $T_1, \ldots, T_k \in \mathbb{C}[n, E, E^{-1}]$, $s = \min\{\iota_*(T_1), \ldots, \iota_*(T_k)\} + 1$. Let a sequence $d = \{d_w, d_{w+1}, \ldots\}$, $w > s$, satisfy equalities $T_1(d) = T_2(d) = \cdots = T_k(d) = 0$. Then the sequence d uniquely can be extended to the sequence*

$$d' = \{d_s, d_{s+1}, \ldots, d_{w-1}, d_w, d_{w+1}, \ldots\}$$

such that

$$T_1(d') = T_2(d') = \cdots = T_k(d') = 0. \tag{21}$$

□

Observe that the uniqueness of such an extention is a trivial fact: suppose $s = \iota_*(T_u) + 1$, $1 \leq u \leq k$, then d' can uniquely be constructed by means of T_u. The nontrivial part of the theorem is (21).

This theorem allows us to establish an important property of eventually m-sparse sequences.

Theorem 3 *Let $R \in \mathbb{C}[n, E, E^{-1}]$, $t = \iota_*(R) + 1$. Let an eventually m-sparse sequence $\{c_0, c_1, \ldots\}$ satisfy the equality $R(c) = 0$. Then the sequence $c_{\geq t} = \{c_t, c_{t+1}, \ldots\}$ is m-sparse.*

Proof: For any large enough non-negative integer w the sequence $\{c_w, c_{w+1}, \ldots\}$ is m-sparse. Suppose w is such an integer. If $w \leq t$ then there is nothing to prove. Suppose $w > t$. If $R_0, \ldots, R_{m-1}$ is the m-splitting of R, then by (20), $\iota_*(R) = \iota_*(R_0)$.

Set $d_i = c_i, i = w, w+1, \ldots$ The sequence

$$d = \{d_w, d_{w+1}, \ldots\}$$

satisfies the equalities

$$R(d) = R_0(d) = \cdots = R_{m-1}(d) = 0.$$

By Theorem 2 there exists the uniquely-defined sequence d', $\nu(d') = s = \min\{\iota_*(R), \iota_*(R_0), \ldots, \iota_*(R_{m-1})\} + 1$, such that $d'_{\geq w} = d$ and

$$R(d') = R_0(d') = \cdots = R_{m-1}(d') = 0.$$

So we have $R_0(d') = 0$ and $\iota_*(R_0) + 1 = \iota_*(R) + 1 = t$. Since the sequence d and the operator R_0 are m-sparse this implies that the sequence $\{d_t, d_{t+1}, \ldots\}$ is m-sparse. By $R(d') = 0$ and $t \geq s$ we have $d_i = c_i$ for all $i = t, t+1, \ldots$ □

As consequence we get the following

Theorem 4 *Let a be an eventually m-point of $L \in \mathbb{C}[x, D]$. Then a is an m-point of $L^{[\iota_*(L)+1]}$.* □

In [2, 3] was shown that for any fixed m either there exist only finitely many m-points a and they can be found explicitly, or all points $a \in \mathbb{C}$ are m-points of the given equation and the operator L can be factored as (6) where C is an m-sparse differential operator with constant coefficients, $\operatorname{ord} C > 0$. If $a \in \mathbb{C}$ is an m-point then $L_0^a, \ldots, L_{m-1}^a$, i.e., the elements of the m-splitting of the operator L^a are such that $\operatorname{ord}\operatorname{GCD}(L_0^a, \ldots, L_{m-1}^a) > 0$, where GCD is the greatest common right divisor.

Note that in the situation where any point is an m-point of L it is possible that at some points there exist more linearly independent (eventually) m-sparse local solutions than at others. To select such points one can find an m-sparse differential operator C with constant coefficients such that (6) takes place with some $\tilde{L} \in \mathbb{C}[x, D]$ (using the algorithm from [3] one can find such an operator C of the greatest possible order). It is easy to see that applying C to an eventually m-sparse series gives an eventually m-sparse series. It means that it would pay to consider especially the eventually m-points of $\tilde{L}$. If the set of such points is empty then the only eventually m-sparse solutions of $L(y) = 0$ are solutions of $C(y) = 0$ and all points are interchangeable.

According to [2, 3] we can assume m to satisfy

$$2 \leq m \leq \operatorname{ord} L - \omega_*(L).$$

Going back to Example 2, we see that $2 \leq m \leq 5$. For $m = 2$ we have $\operatorname{GCD}(L_0^{a_0}, L_1^{a_0}) = 1$ for all $a_0 \in \mathbb{C}$ and by [2, 3] the equation $L(y) = 0$ has no 2-sparse solution. We find $\iota_*(L) = 1$,

$$M = L^{[2]} = (12x^3 + 12x)D + (3x^4 - 2x^2 - 1).$$

We have

$M_0^a = (3x^4 + (18a - 2)x^2 + (3a^4 - 2a^2 - 1))D + (12x^3 + (36a^2 + 12)x)$,

$M_1^a = (12ax^3 + (12a^3 - 4a)x)D + (36ax^2 + 12a^3 + 12a)$.

The algorithm [10] allows to determine that $\operatorname{GCD}(M_0^a, M_1^a)$ is

$$(3x^4 - 2x^2 - 1)D + (12x^3 + 12x)$$

if $a = 0$ and 1 otherwise. Therefore the point 0 is the only candidate for eventually 2-points. There is no such candidate if $m \in \{3, 4, 5\}$. Further investigation shows that 0 is an eventually 2-point of L.

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