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ORDINARY DIFFERENTIAL EQUATIONS

On Ranks of Matrices over Noncommutative Domains

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Abstract—We consider matrices with entries in some domain R, i.e., in a ring, not necessarily commutative, not containing non-trivial zero divisors. The concepts of the row rank and the column rank are discussed. (Coefficients of linear dependencies belong to the domain R; left coefficients are used for rows, right coefficients for columns.) Assuming that the domain satisfies the Ore conditions, i.e., the existence of non-zero left and right common multiples for arbitrary non-zero elements, it is proven that these row and column ranks are equal, which allows us to speak about the rank of a matrix without specifying which rank (row or column) is meant. In fact, the existence of non-zero left and right common multiples for arbitrary non-zero elements of R is a necessary and sufficient condition for the equality of the row and column ranks of an arbitrary matrix over R. An algorithm for calculating the rank of a given matrix is proposed. Our Maple implementation of this algorithm covers the domains of differential and (q-)difference operators, both ordinary and with partial derivatives and differences.

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1. INTRODUCTION

Operations with matrices are widely used in basic and applied research. In carrying them out, it is important to bear in mind that their properties depend on the properties of the algebraic structure to which the matrix entries belong.

A *domain* in this paper is a ring, not necessarily commutative, which contains no nontrivial zero divisors (in literature the terms *entire ring, integral domain* etc.are also used). In the sequel, R always denotes a domain.

Definition 1. Let A be a matrix over R. The rows $u_1, u_2, \ldots, u_r$ of A are *linearly dependent over R* if there are $f_1, f_2, \ldots, f_r \in R$, not all zero, such that $f_1u_1 + f_2u_2 + \ldots + f_ru_r = 0$; otherwise, these rows are *linearly independent* over R. The columns $v_1, v_2, \ldots, v_s$ of A are *linearly dependent over R* if there are $g_1, g_2, \ldots, g_s \in R$, not all zero, such that $v_1g_1 + v_2g_2 + \ldots + v_sg_s = 0$; otherwise, these columns are *linearly independent* over R. The maximum number of linearly independent rows resp. columns of A is called the *row rank* or the *left rank*, resp. the *column rank* or the *right rank*, of A.

The definition of the rank of a matrix as the maximum number of its linearly independent rows, and the proof that this number is equal to the maximum number of its linearly independent columns, is the approach in classical linear algebra. But there are examples of non-commutative domains for which these ranks need not coincide (see, e.g., Example 1 in Section 2), so it is natural to ask for a characterization of domains R such that the row and column ranks of matrices over R do coincide. We provide such a characterization in Theorem 1 in Section 3 where we show that domains with this property are exactly those satisfying the *Ore conditions*, i.e., where non-zero left and right common multiples exist for arbitrary non-zero $a, b \in R$.

Let us review here some of the definitions and theorems from the literature on these and related topics, and compare them with our approach.

† Deceased.

In order to determine the row resp. column rank of A, we need to test linear dependence of sets of rows resp. columns of A. Note however that in a general domain R, the familiar linear-algebraic methods for testing linear dependence of a set $S \subseteq R^k$ may not work. In the literature, one can find different approaches to defining the rank of a matrix over a domain. Often the definition is selected based on the convenience of proving the desired theorems, which may lead to apparent inconsistencies with results from sources using non-equivalent definitions. Even for equivalent definitions, the proofs of their equivalence can be rather complicated and cumbersome; an example is presented later in this section.

The questions of embeddability of a domain into a skew field (a non-commutative field) and, in particular, of existence of a skew field of fractions, are discussed in many publications: see, e.g., [4, 7, 8, 15]. In [4], the rank concept is given in Definition 1.1: *Let $A \in K^{m\times n}$, where K is a skew field. The subspace of $K^{1\times n}$ spanned by the row vectors of A is called the row space of A, and the subspace of $K^{m\times 1}$ spanned by the column vectors of A is called the column space of A. The dimension of the row space of a matrix A is called the row rank of A, and the dimension of the column space of a matrix A is called the column rank of A. The following assertion is then proved as Theorem* 1.2: *Let A be an arbitrary $m \times n$-matrix over a skew field K. Then the row rank of A and the column rank of A are equal*. Notice that since K is a skew field, the dimension of the row resp. column space equals the maximum number of linearly independent rows resp. columns of A. An example of such a skew field K is the left skew field Q of quotients of R, i.e., the skew field of formal fractions of the form $b^{-1}a$ with $a, b \in R$, $b \neq 0$. The existence of this skew field is guaranteed if R satisfies the Ore conditions (see, e.g., [12], Section 2, [7], or [14], part II, §9). We prove Theorem 1 in Section 3 by showing that the row and column ranks over R coincide with the corresponding ranks over Q, and hence with each other. Note that instead of with the left skew fields of quotients, it is equally possible to work with the right skew fields of formal fractions of the form ab^{-1} with $a, b \in R$, $b \neq 0$. It is also known that for domains R satisfying the Ore conditions, their skew fields of left and right quotients are isomorphic, but this is not essential here.

To illustrate the claim that proofs of equivalence of various definitions of the concept of rank of matrices over non-commutative domains can be complicated, we turn to [2], where the rank of a matrix over an Ore polynomial ring (see [13] or [3]) is defined as the maximum number of its linearly independent rows. The authors of [2] note that their definition differs from the one in [8], Section 0.6, where the rank of a matrix A over R is defined as the rank of the left module M generated by the rows of A over R, i.e., as the cardinality of a maximal R-linearly independent subset of M. Theorem A.2 in [2] states that for matrices over an Ore polynomial ring, the two quantities ultimately are the same. However, the part of the proof of this theorem which shows that the rank of M does not exceed the maximum number of R-linearly independent rows of A certainly appears to be non-trivial.

In Lopatinsky's book [10], the importance of the concept of rank for the study of integral manifolds of systems of linear partial differential equations is emphasized, and the proof of equality of the row and column ranks is given there only for differential operators. In [2], the proof of equality of the row and column ranks of matrices over a ring of (non-commutative) Ore polynomials equipped with an automorphism σ and a self-map δ which is a differentiation with respect to σ is given (see, e.g., [13] or [3]). Such one-variable Ore polynomials do not cover, say, partial differential operators. As for the ring of one-variable Ore polynomials over a commutative field of coefficients like the one considered in [2], this ring is Euclidean.

The articles [16, 17] also give examples of proofs of the equality of different ranks defined for matrices over domains having some specific properties. The definitions considered differ from our Definition 1. Let us add that Definition 1 is natural and agrees with that accepted in classical linear algebra.

In our proofs in Section 3 we proceed from the more general assumption that for the domain R, the Ore conditions are satisfied. For a multivariate Ore polynomial ring (in particular, for the ring of partial differential operators), this condition is satisfied, as shown in [5]. As a special case, our Theorem 1 covers matrices with ordinary differential operators as elements. In [15], equality of the row and column ranks of such matrices is proved using the fact that the ring of ordinary differential operators is Euclidean. In [8], Subsection 8.1, Theorem 1.1, a more general case is considered with R a principal ideal ring. Note that every Euclidean ring is a principal ideal ring, but the Ore conditions can be satisfied also for a domain that is not a principal ideal ring.

The theory of left and right determinants proposed by Ore in [12] does not seem to provide a quick way to prove equality of the row and column ranks, although it allows one to establish that the rows of a matrix A over R are linearly dependent if and only if so are the columns of A. It seems that the available literature lacks a complete proof of equality of the row and column ranks (in the sense of Defnition 1) for matrices over a domain R that satisfies the Ore conditions.

The contents of the rest of the paper are as follows. Section 2 lists the main properties of the ranks of matrices over R, and presents two examples: one of a matrix with differing row and column ranks, and one which demonstrates why it is opportune to define the column ranks by using linear combinations with right scalar factors, and the row ranks by using linear combinations with left scalar factors – as we do in Definition 1. Section 3 contains Lemmas 1, 2 and Theorem 1 which prove the main theoretical result of the paper – namely, that the row and column ranks of matrices over domains satisfying the Ore conditions are equal. It also shows that if one has an algorithm for calculating non-zero left common multiples of domain elements, one can compute matrix ranks by Gaussian elimination. In Section 4, a Maple implementation of this approach is proposed, oriented towards matrices whose entries are either partial differential or partial difference or partial q-difference operators, and showcased by several computational examples.

A preliminary version of this paper was published as [1], where instead of the Ore conditions a more complex necessary and sufficient condition for equality of the row and column ranks of a matrix A over R was proposed, namely that for any positive integer n, the rows of any matrix from $R^{(n+1)\times n}$ as well as the columns of any matrix from $R^{n\times(n+1)}$ are linearly dependent over R. In fact, in the current paper this condition is strengthened: the only value of n that should be considered for $R^{n\times(n+1)}$, $R^{(n+1)\times n}$ is $n = 1$.

2. PRELIMINARIES AND MOTIVATING EXAMPLES

Considering a matrix $A \in R^{m\times n}$ of the form

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}, \tag{1}$$

we will denote by $A_{i,*}$ the i-th row, $1 \le i \le m$, and by $A_{*,j}$ the jth column, $1 \le j \le n$, of A.

If the given domain R is embedded in a skew field F, then the left and right ranks over F of a matrix $A \in R^{m\times n}$ are equal, but the question of equality of ranks of this matrix over R remains open. If we consider the skew field F_l of left fractions (i.e. fractions of the form $b^{-1}a$), or the skew field F_r of right fractions of the form ab^{-1} as an extension of R, then given that over each of these skew fields the left and right ranks of A are equal, it is not clear a priori whether these ranks over F_l and F_r are equal. Lemma 2 proved below in Section 3 gives an affirmative answer about the equality of all left and right ranks over R, F_l, F_r.

Recall that the skew field axioms differ from the field axioms in the absence of the commutativity axiom of multiplication; skew fields are sometimes called *non-commutative fields*. The equality of the left and right ranks over an arbitrary skew field is proved in, e.g., [4], Theorem 1.2.

The main properties of the ranks of matrices of the form (1) are the following:

(P1) From [4, 6] it follows that if R is embedded in some skew field F and if the linear dependencies of the rows (columns) are considered with coefficients belonging to F (one can speak of a *dependence over* F), then the left and right ranks over F coincide.

The notation $\mathrm{lRank}_G\, A$ and $\mathrm{rRank}_G\, A$ will be used for the left and right ranks of the matrix A over the domain or skew field G.

(P2) It is evident that if F is a skew field and $R \subseteq F$, then

$$\mathrm{lRank}_R\, A \ge \mathrm{lRank}_F\, A, \qquad \mathrm{rRank}_R\, A \ge \mathrm{r\,Rank}_F\, A.$$

(P3) Conditions for the existence of a skew field were suggested by O. Ore in [12]:

In order for a domain R to have the left skew field F_l (with elements of the form $b^{-1}a$), it is necessary and sufficient that for any $a, b \in R\backslash\{0\}$ there exists a left common multiple $c \in R\backslash\{0\}$, i.e., that $va = wb = c$ for some $v, w \in R$.

Ore himself uses the term *common multiplum* for such a c. We will write $c = \mathrm{lCM}(a, b)$. It is established inductively that if $a_1, \ldots, a_n \in R\backslash\{0\}$, then $\mathrm{lCM}(a_1, \ldots, a_n)$ exists: for each i with $1 \le i \le n$ we have $\mathrm{lCM}(a_1, \ldots, a_i, \ldots, a_n) = v_i a_i$ for some $v_i \in R$.

Similarly, we can consider the right common multiple $\mathrm{rCM}(a, b)$ and, respectively, the right skew field of fractions F_r i.e., the skew field of fractions of the form ab^{-1}.

We present here two examples which show that the row and column ranks of matrices over a domain R behave differently when R is a non-commutative domain than in the familiar case when R is a field. The first example shows that the row and column ranks of matrices over a non-commutative domain need not coincide.

Example 1. Let R be the ring of polynomials in non-commuting variables x and y (hence $xy \neq yx$) over some field, e.g., the field of rational numbers $\mathbb{Q}$. Consider the matrix

$$A = \begin{bmatrix} x \\ y \end{bmatrix} \in R^{2\times 1}.$$

Its rows $[x]$ and $[y]$ are linearly independent over R. So, the row rank of A is 2, while its column rank is 1.

Example 2 features a domain R such that, according to Lemma 2 in Section 3, the row and column ranks of every matrix over R coincide, but this would be false if linear dependence of a set of columns were defined alternatively using the multipliers f_i in Definition 1 as *left* rather than right factors.

Example 2. Let $K = \mathbb{Q}(x)$, and let R be the ring $K\left[\frac{d}{dx}\right]$ of linear ordinary differential operators with coefficients in K. The columns $A_{*,1}$, $A_{*,2}$ of the matrix

$$A = \begin{bmatrix} x\frac{d}{dx} + 1 & x \\ \frac{d}{dx} & 1 \end{bmatrix}$$

satisfy $A_{*,1} = \frac{d}{dx} A_{*,2}$, so under the above alternative definition of linear dependence they would be linearly dependent, and the column rank of A would be 1. On the other hand, assume that $L_1 A_{1,*} + L_2 A_{2,*} = 0$ for some $L_1, L_2 \in R$. Then

$$L_1 x \frac{d}{dx} + L_1 + L_2 \frac{d}{dx} = 0, \tag{2}$$

$$L_1 x + L_2 = 0, \tag{3}$$

hence $L_2 = -L_1 x$ by (3), so $L_1 = 0$ by (2), and $L_2 = 0$ by (3), implying that the rows of A are linearly independent, and the row rank of A is 2.

However, if we consider the rank by columns as the right-hand rank, then for A this rank will be equal to 2. Indeed, if $A_{*,1} M_1 + A_{*,2} M_2 = 0$ then

$$\left(x\frac{d}{dx} + 1\right) M_1 + x M_2 = 0, \tag{4}$$

$$\frac{d}{dx} M_1 + M_2 = 0. \tag{5}$$

Multiplying (5) on the left by x and subtracting from (4) we obtain $M_1 = 0$. From (5) we now get $M_2 = 0$.

3. THE RANKS AND THEIR COMPUTATION

Lemma 1. *Let the domain R satisfy the Ore conditions for existence of the left and right skew fields of fractions, and let F_l, F_r be those left and right skew fields of fractions for R. Then*

(i) $\mathrm{lRank}_{F_l} A = \mathrm{lRank}_R A$.

(ii) $\mathrm{rRank}_{F_r} A = \mathrm{rRank}_R A$.

Proof. (i) It is sufficient to show that rows of A are linearly dependent over F_l if and only if these rows are linearly dependent over R.

It is evident that dependency over R implies dependency over F_l. To prove the converse, let

$$c_1 A_{1,*} + \ldots + c_m A_{m,*} = 0, \tag{6}$$

$c_i = b_i^{-1} a_i \in F_l$, $a_i \in R$, $b_i \in R \backslash \{0\}$, $i = 1, \ldots, m$. Let $b = \mathrm{lCM}(b_1, \ldots, b_m)$, and let $v_1, \ldots, v_m \in R \backslash \{0\}$ be such that $b = v_i b_i$, $i = 1, \ldots, m$. Then the left multiplication of (6) by b gives $v_1 a_1 A_{1,*} + \ldots + v_m a_m A_{m,*} = 0$, and since $a_i \neq 0$ for some i, the absence of zero divisors in R guarantees that $v_i a_i \neq 0$.

(ii) Can be proved similarly: each element of the skew field F_r is represented as ab^{-1}, $a \in R$, $b \in R \backslash \{0\}$.

Lemma 2. *Let the domain R satisfy the Ore conditions for existence of the left and right skew fields of fractions, and let F_l, F_r be those left and right skew fields of fractions for R. Let A be a matrix over R. Then* $\mathrm{lRank}_R\, A$, $\mathrm{rRank}_R\, A$, $\mathrm{lRank}_{F_l}\, A$, $\mathrm{rRank}_{F_l}\, A$, $\mathrm{lRank}_{F_r}\, A$, $\mathrm{rRank}_{F_r}\, A$ *are all equal.*

Proof. It follows from Lemma 1(i) and **(P2)** that

$$\mathrm{lRank}_{F_l}\, A = \mathrm{lRank}_R\, A \geq \mathrm{lRank}_{F_r}\, A. \tag{7}$$

It can be proved similarly that

$$\mathrm{rRank}_{F_r}\, A = \mathrm{rRank}_R\, A \geq \mathrm{rRank}_{F_l}\, A. \tag{8}$$

Taking into account the equalities $\mathrm{lRank}_{F_l}\, A = \mathrm{rRank}_{F_l}\, A$, $\mathrm{lRank}_{F_r}\, A = \mathrm{rRank}_{F_r}\, A$ (see **(P1)**), we get from (7), (8) that $\mathrm{lRank}_{F_l}\, A \geq \mathrm{rRank}_{F_r}\, A$, and, at the same time, $\mathrm{rRank}_{F_r}\, A \geq \mathrm{lRank}_{F_l}\, A$. Hence the sign $\geq$ in (7), (8) can be replaced by $=$. Thus

$$\mathrm{lRank}_R\, A, \quad \mathrm{rRank}_R\, A, \quad \mathrm{lRank}_{F_l}\, A, \quad \mathrm{rRank}_{F_l}\, A, \quad \mathrm{lRank}_{F_r}\, A, \quad \mathrm{rRank}_{F_r}\, A \tag{9}$$

are all equal among themselves.

Remark 1. For the special case of Ore algebra (an algebra of skew polynomials in several indeterminates), F. Chyzak and B. Salvy proposed in [5] an algorithm for computing non-zero right and left common multiples of non-zero elements, for example, it works for linear differential and (q-)difference operators, both ordinary and those with partial derivatives and differences. F. Chyzak implemented this algorithm as the command `annihilators` of Maple package `Ore_algebra` [11].

Lemma 2 shows that by computing the value of one of the ranks in (9), we find out the value of each of them. Therefore, to calculate the value of any one of these ranks we can focus on the rank whose calculation looks the simplest. Since F_l is a (skew) field, the calculation of $\mathrm{lRank}_{F_l} A$ looks more convenient than, say, the calculation of $\mathrm{lRank}_R\, A$. We concentrate on the calculation of $\mathrm{lRank}_{F_l}\, A$ by Gaussian elimination, using the following elementary rank-preserving transformations:

(T1) Swapping two rows in a matrix.

(T2) Left multiplication of a row by a nonzero element of F_l.

(T3) Replacing a row with the sum of the replaced row itself and some other row, multiplied from the left by a nonzero element F_l.

These transformations are formulated in [8], §8.1 for matrix columns; right multiplication is used instead of left.

Gaussian elimination will make it possible to bring the matrix A into a stepped form, where the nonzero rows have different numbers of initial zero elements. The number of such rows is the desired rank.

We emphasize that it is possible to compute lRank_{F_l} of a matrix $A \in R^{m \times n}$ using only operations from R, i.e., Gaussian elimination can be performed in a fraction-free way. Let some row have the first nonzero element a, and let another row have the first nonzero element b in the same column. Then according to the Ore conditions there exist $v, w \in R \backslash \{0\}$ such that

$$va = wb. \tag{10}$$

Multiply the first row from the left by v, and the second one by $-w$. The sum of the obtained rows gives a row having zero as its first element. The rank computed by this kind of eliminations is equal to $\mathrm{lRank}_{F_l}\, A$. By **T2**, the multiplication of a row of A by v from the left is allowed for invertible v. Since $v \neq 0$, v is invertible in F_l even if v is not invertible in R. All the calculations involved in computing $\mathrm{lRank}_{F_l} A$ are carried out without leaving R. Equality of the resulting value to $\mathrm{lRank}_R A$ and $\mathrm{rRank}_R A$ is assured by Lemma 2.

Theorem 1. (i) *Let R be a domain satisfying the Ore conditions for the existence of left and right skew fields of fractions, i.e., for any $a, b \in R\backslash\{0\}$ there exist $v_1, w_1, v_2, w_2 \in R\backslash\{0\}$ such that $v_1 a = w_1 b$, $av_2 = bw_2$. Then for any matrix $A \in R^{m\times n}$, the values $\mathrm{lRank}_R A$ and $\mathrm{rRank}_R A$ are equal; as a consequence, we can simply speak about the rank $\mathrm{Rank}_R A$ of the matrix A over R.*

(ii) *If there exists an algorithm for finding a non-zero left common multiple of arbitrary $a, b \in R\backslash\{0\}$, or a similar algorithm for finding a non-zero right common multiple of arbitrary $a, b \in R\backslash\{0\}$, then there is an algorithm for computing $\mathrm{Rank}_R A$ of any matrix $A \in R^{m\times n}$, $m, n \in \mathbb{Z}_{>0}$.*

Proof. (i) By Lemma 2.

(ii) Assume that we know an algorithm for calculating $v, w \neq 0$ for which (10) holds. Let the elements a_{ij}, a_{kj} of the matrix A be nonzero, and let $v, w \neq 0$ be such that $va = wb$. Then replacing the row $A_{k,*}$ with

$$vA_{i,*} - wA_{k,*} \tag{11}$$

results in elimination of the jth element in the kth row using the ith row. If the first $j-1$ elements in both $A_{k,*}$, $A_{i,*}$ are equal to zero, then the first j elements of (11) are equal to zero.

Similarly, ranks can be computed by Gaussian elimination in the columns of the matrix using right common multiples of the matrix elements.

Corollary 1. Let R be such a domain that for any $a, b \in R\backslash\{0\}$ there exist $v_1, w_1, v_2, w_2 \in R\backslash\{0\}$ such that $v_1 a = w_1 b$, $av_2 = bw_2$. Then multiplication of any row (column) of a matrix on the left (right) by a nonzero element $p \in R$ does not change $\mathrm{Rank}_R A$.

Proof. Multiplying a row (column) by p does not change $\mathrm{lRank}_{F_l} A$ (resp., $\mathrm{rRank}_{F_r} A$). Hence $\mathrm{lRank}_{F_l} A = \mathrm{rRank}_{F_r} A = \mathrm{Rank}_R A$ do not change either.

Let us add to assertion (i) of Theorem 1 that the Ore conditions are not only sufficient, but also necessary for equality of ranks over R of any matrix with entries in R. A proof of necessity of these conditions is obtained by considering matrices

$$[a, b] \in R^{1\times 2}, \quad \begin{bmatrix} a \\ b \end{bmatrix} \in R^{2\times 1}$$

with arbitrary nonzero $a, b \in R$.

Remark 2. As a special case, Theorem 1 covers matrices with ordinary differential operators as elements. This case has been repeatedly discussed in the literature. For example, in [15] the equality of the row and column ranks of such matrices was proved using the fact that the ring of ordinary differential operators is Euclidean. In [8], Section 8.1, Theorem 1.1, a more general case is considered with R a principal ideal ring (note that every Euclidean ring is a principal ideal ring).

4. IMPLEMENTATION AND USE OF THE ALGORITHM

Using commands of the Maple system mentioned in Remark 1, we have implemented Gaussian elimination on matrices with entries in an Ore algebra. The algorithm is run by invoking the command

`OreAlgebraGaussianElimination`

and is available at

`http://www.ccas.ru/ca/orealgebragaussianelimination.`

Let us consider some examples of its applications.

Declare an Ore algebra by the differential type, which is predefined in the package `Ore_algebra`:

> *DiffOA* := *Ore_algebra* :– *skew_algebra*(*diff* = $[D_1, x_1]$, *diff* = $[D_2, x_2]$) :

Here we introduce differential operators $D_1 = \dfrac{\partial}{\partial x_1}$ and $D_2 = \dfrac{\partial}{\partial x_2}$. This Ore algebra consists of linear differential operators that are linear combinations of monomials of the form $D_1^{v_1} D_2^{v_2}$ where $v_1, v_2 \in \mathbb{Z}_{\geq 0}$, with coefficients which are polynomials in the variables x_1, x_2. For the matrix

$$> A_1 := \begin{bmatrix} D_1 + D_2 & x_1 + x_2 & 0 \\ D_1 & x_2 & 1 \\ D_2 & x_1 & -1 \end{bmatrix}:$$

Gaussian elimination yields

$> OreAlgebraGaussianElimination(A_1, DiffOA)$

$$\begin{bmatrix} D_1 + D_2 & x_1 + x_2 & 0 \\ 0 & D_1 x_1 - x_2 D_2 & -D_1 - D_2 \\ 0 & 0 & 0 \end{bmatrix}$$

This result has rank two, hence the source matrix A_1 has rank two as well.

The next example presents an Ore algebra of linear difference operators:

$> ShiftOA := Ore\_algebra :- skew\_algebra(shift = [E_1, n_1],\ shift = [E_2, n_2]):$

Here we introduce operators E_1, where $E_1 y(n_1, n_2) = y(n_1 + 1, n_2)$, and E_2, where $E_2 y(n_1, n_2) = y(n_1, n_2 + 1)$. For the matrix

$$> A_2 := \begin{bmatrix} E_1 + E_2 & n_1 + n_2 & 0 \\ E_1 & n_2 & 1 \\ E_2 & n_1 & -1 \end{bmatrix}:$$

we find out that its rank is two:

$> OreAlgebraGaussianElimination(A_2,\ ShiftOA)$

$$\begin{bmatrix} E_1 + E_2 & n_1 + n_2 & 0 \\ 0 & E_1 n_1 - E_2 n_2 + E_1 - E_2 & -E_1 - E_2 \\ 0 & 0 & 0 \end{bmatrix}$$

The next matrix has entries which are ordinary difference operators:

$$> A_3 := \begin{bmatrix} E_1 & n_1 & 0 \\ E_1 & 0 & 1 \\ 0 & n_1^2 + 1 & E_1 - 1 \end{bmatrix}:$$

$> OreAlgebraGaussianElimination(A_3, ShiftOA)$

$$\begin{bmatrix} E_1 & n_1 & 0 \\ 0 & -n_1 & 1 \\ 0 & 0 & -E_1 n_1 - n_1^2 + n_1 - 1 \end{bmatrix}$$

We see that the rank of A_3 is three, i.e., that A_3 has full rank.

The last example demonstrates an Ore algebra of linear q-difference operators:

$> QShiftOA := Ore\_algebra :- skew\_algebra(qdilat = [Q_1, x_1, q],\ qdilat = [Q_2, x_2, q]):$

Here we introduce operators Q_1, where $Q_1 y(x_1, x_2) = y(q x_1, x_2)$, and Q_2, where $Q_2 y(x_1, x_2) = y(x_1, q x_2)$. For the matrix

$$> A_4 := \begin{bmatrix} Q_1 + 2Q_2 & x_1 + x_2 & 0 \\ Q_1 & x_2 & 1 \\ 0 & 1 & 0 \end{bmatrix}:$$

we obtain the full-rank result again:

> $OreAlgebraGaussianElimination(A_4, QShiftOA)$

$$\begin{bmatrix} Q_1 + 2Q_2 & x_1 + x_2 & 0 \\ 0 & qx_1Q_1 - 2qx_2Q_2 & -Q_1 - 2Q_2 \\ 0 & 0 & -Q_1 - 2Q_2 \end{bmatrix}$$

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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