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ORDINARY DIFFERENTIAL EQUATIONS

Counterexamples to the Assumption on the Possibility of Prolongation of Truncated Solutions of a Truncated LODE

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Abstract—Previously, the authors proposed algorithms making it possible to find exponential-logarithmic solutions of linear ordinary differential equations with coefficients in the form of power series in which only the initial terms are known. The solution includes a finite number of power series, and the maximum possible number of their terms is calculated. Now, these algorithms are supplemented with the option to confirm the impossibility of obtaining a larger number of terms in the series without using additional information about the given equation a counterexample is constructed to the assumption that it is possible to obtain uniquely defined additional terms. In previous papers, the authors proposed such confirmations for the cases of Laurent and regular solutions.

Keywords: differential equations, truncated power series, computer algebra systems

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1. INTRODUCTION

This article deals with linear ordinary differential equations (LODEs) with coefficients that have the form of power series in which only their initial terms are known and their "tails" are unknown. Thus, there is only incomplete information about these equations. In [1–6], algorithms were proposed for finding solutions of such equations in the form of Laurent series, as well as for finding regular and exponential-logarithmic solutions. It was proved that these algorithms make it possible to find the maximum possible number of terms of those series involved in the solutions. The algorithms were implemented by the authors in a package of procedures (see [7–10]). For the user of these procedures, it may be desirable to obtain some demonstrative arguments in favor of the assumption that the number of terms found for the series is maximal. Evident tools of this kind are offered below: an algorithm is described that, for an arbitrary equation with truncated coefficients, presents two variants for the prolongation of the original equation the solutions of which differ from each other in subsequent terms, not included in the number of terms found for the series involved in the solutions.

Let us clarify the essence of the problem with a simple example. Using an algorithm from [4], it is found that the equation

$$\left(x^3 + \frac{x^5}{3} + O(x^6)\right)y'(x) + (1 + 3x + O(x^3))y(x) = 0 \tag{1}$$

(here, $O(x^k)$ denotes some unknown terms of the power series with powers of x not lower than k) has a solution

$$e^{\frac{1}{2x^2} + \frac{3}{x}} x^{\frac{1}{3}} (C + O(x)), \tag{2}$$

where C is an arbitrary constant. Can we, based on (1), find more terms of the series that is represented in (2) as $(C + O(x))$? A negative answer is justified by presenting two variants for the prolongation of Eq. (1):

$$\left(x^3 + \frac{x^5}{3} + 4x^6 + O(x^7)\right)y'(x) + (1 + 3x + x^3 + O(x^4))y(x) = 0 \tag{3}$$

and

$$\left(x^3+\frac{x^5}{3}-4x^6+O(x^7)\right)y'(x)+(1+3x+O(x^4))y(x)=0. \tag{4}$$

The algorithm from [4] finds the solutions of these two equations:

$$e^{\frac{1}{2x^2}+\frac{3}{x}}x^{\frac{1}{3}}(C+4Cx+O(x^2)),$$
$$e^{\frac{1}{2x^2}+\frac{3}{x}}x^{\frac{1}{3}}(C-3Cx+O(x^2)).$$

This shows that, for the series $(C+O(x))$ involved in (2), we cannot find its coefficient at x without using any additional information on Eq. (1). The pair of equations (3) and (4) form, in the terminology of this article, a *counterexample* to the assumption that the subsequent terms of the series involved in the exponential-logarithmic solutions of Eq. (1) can be found only from this truncated equation.

In Section 6, we demonstrate the construction of this counterexample, i.e., Eqs. (3) and (4), using our algorithm implemented in the Maple environment.

A preliminary version of this work was reported in [11].

2. TRUNCATED EQUATION

Let K be an algebraically closed field of characteristic 0. For the ring of polynomials in x over K, we will use the notation $K[x]$. The ring of formal power series in x over K is denoted by $K[[x]]$, and the field of formal Laurent series, by $K((x))$. Obviously, $K[x]\subset K[[x]]\subset K((x))$. For a nonzero element $a(x)=\sum a_i x^i$ belonging to $K((x))$, its *valuation* $\operatorname{val} a(x)$ is defined by the equality $\operatorname{val} a(x)=\min\{i\,|\,a_i\neq 0\}$, while $\operatorname{val} 0=\infty$.

We consider equations of the form

$$a_r(x)y^{(r)}(x)+a_{r-1}(x)y^{(r-1)}(x)+\cdots+a_0(x)y(x)=0, \tag{5}$$

where $y(x)$ is an unknown function of x. Regarding $a_0(x),a_1(x),\ldots,a_r(x)$ (*coefficients of the equation*), it is assumed that, for each $i=0,1,\ldots,r$, the coefficient $a_i(x)$ is a *truncated series*

$$a_i(x)=\sum_{j=0}^{t_i}a_{ij}x^j+O(x^{t_i+1}), \tag{6}$$

where $a_{ij}\in K$ and t_i is an integer such that $t_i\geqslant -1$ (if $t_i=-1$, then the sum in (6) is assumed to be equal to 0). Hereinafter, the symbol $O(x^t)$ occurring in formal expressions denotes some series whose valuation is not smaller than t. We call t_i the *truncation degree* of a series $a_i(x)$ represented in the form (6). Note that any coefficient in (5) can have the form $O(x^m)$, $m\geqslant 0$.

A prolongation of Eq. (5) is any equation

$$\tilde{a}_r(x)y^{(r)}(x)+\tilde{a}_{r-1}(x)y^{(r-1)}(x)+\cdots+\tilde{a}_0(x)y(x)=0$$

for which $\tilde{a}_i(x)-a_i(x)=O(x^{t_i+1})$; i.e., $\operatorname{val}(\tilde{a}_i(x)-a_i(x))>t_i$, $i=0,1,\ldots,r$.

3. TRUNCATED SOLUTIONS

Formal *exponential-logarithmic* solutions of the equation

$$\left(\sum_{j=0}^{\infty}\tilde{a}_{rj}x^j\right)y^{(r)}(x)+\left(\sum_{j=0}^{\infty}\tilde{a}_{r-1,j}x^j\right)y^{(r-1)}(x)+\cdots+\left(\sum_{j=0}^{\infty}\tilde{a}_{0j}x^j\right)y(x)=0, \tag{7}$$

having completely defined series-represented coefficient are understood as solutions of the form

$$e^{Q(x^{-1/q})}x^{\lambda}w(x^{1/q}), \tag{8}$$

where Q is a polynomial with coefficients from K, $q \in \mathbb{Z}_{>0}$, $\lambda \in K$,

$$w(x) = \sum_{s=0}^{m} w_s(x) \ln^s x,$$

$m \in \mathbb{Z}_{\geq 0}$, $w_s(x) \in K((x))$, $s = 0,\dots,m$, and $w_m(x) \neq 0$. In this case, $x^{\lambda} w(x^{1/q})$ is called the *regular part*; $Q(x^{-1/q})$, the *exponent of the irregular part*; and q, the *ramification index* of solution (8).

If $q = 1$ and $Q \in K$, then solution (8) is called *formally regular*; otherwise, it is called *irregular*. For $q = 1$, $Q \in K$, $\lambda \in \mathbb{Z}$, and $w(x) \in K((x))$, a formal regular solution (8) is called *Laurent one*. When discussing solutions of equations, we omit the word "formal", but mean it.

Let the leading coefficient $\tilde{a}_r(x)$ in Eq. (7) be nonzero. It is known (see, e.g., [12, Ch. V; 13–16]) that there are r solutions of the form (8) for Eq. (7) that are linearly independent over K. In [13–17], algorithms were proposed for finding the ramification index q and the exponent of the irregular part $Q(x^{-1/q})$ of r linearly independent solutions of the form (8). Let the valuation of at least one of the coefficients in (7) be equal to 0. Then, to construct the ramification index q and the exponent of the irregular part $Q(x^{-1/q})$ for all solutions, it suffices to know $r \operatorname{val} \tilde{a}_r(x)$ values of the initial coefficients of all $\tilde{a}_i(x)$, $i = 0,1,\dots,r$ (see, e.g., [18]). To construct the regular part of the solution with any given truncation degree of the series involved in $w(x)$, the algorithms proposed in [12, Ch. IV; 19; 20, Ch. II, VIII] can be used. For this construction, it is also sufficient to know some finite number of initial coefficients of all $\tilde{a}_i(x)$ (see [21, Prop. 1]).

Let $Q(x^{-1/q}) \in K[x^{-1/q}]$, $q \in \mathbb{Z}_{>0}$, $\lambda \in K$, and

$$w_s^{\langle k_s \rangle}(x) = \sum_{j=j_s}^{k_s} w_{s,j} x^j + O(x^{k_s+1}),$$

$j_s, k_s \in \mathbb{Z}$, $k_s \geq j_s$, $s = 0,\dots,m$, and $w_{m,j_m} \neq 0$. For Eq. (5) with truncated coefficients, we understand the *solution with a truncated regular part* as an expression

$$e^{Q(x^{-1/q})} x^{\lambda} \sum_{s=0}^{m} w_s^{\langle k_s \rangle}(x^{1/q}) \ln^s x, \tag{9}$$

if any equation that is a prolongation of (5) has a solution $e^{Q(x^{-1/q})} x^{\lambda} \tilde{w}(x^{1/q})$ that is a prolongation of solution (9); i.e., $\tilde{w}(x)$ has the form

$$\tilde{w}(x) = \sum_{s=0}^{m} \tilde{w}_s(x) \ln^s x$$

and $\tilde{w}_s(x) - w_s^{\langle k_s \rangle}(x) = O(x^{k_s+1})$ is satisfied; i.e., $\operatorname{val}(\tilde{w}_s(x) - w_s^{\langle k_s \rangle}(x)) > k_s$, $s = 0,1,\dots,m$. We say that the truncated solution is *invariant* to any prolongation of Eq. (5).

4. SOLUTIONS WITH THE MAXIMUM TRUNCATION DEGREE

It was shown in [1–4, 7] that, for an equation of the form (5), it is possible to construct all invariant truncated solutions with the *maximum truncation degree* of the series involved in the solution. The maximum truncation degree in $s_{\max}$ means that there is no invariant solution s that is a prolongation of $s_{\max}$, such that the truncation degree of at least one series in s is greater than the truncation degree of the corresponding series in $s_{\max}$. In this case, we are talking about the exhaustive use of information about a given equation in constructing truncated solutions. In the above-cited articles, algorithms for solving this problem and their implementation in Maple are presented.

In [22, 23], we considered the issue of automatic confirmation of such an exhaustive use of information about a given equation in the construction of Laurent and regular truncated solutions. It is confirmed by a counterexample consisting of two different prolongations of the given equation, which lead to the appearance of different additional terms in the solutions.

Algorithms for constructing both the truncated solutions and counterexamples mentioned above are based on finding solutions with *literals*, i.e., symbols used to represent the unspecified coefficients of the series in the equation (see [7]). Literals denote the coefficients in the terms of the series, the degrees of which are greater than the truncation degree of the series. Finding solutions using literals means representing subsequent (non-invariant to all possible prolongations) terms of the series by expressions containing literals, i.e., unspecified coefficients. This makes it possible to clarify the effect of unspecified coefficients on the subsequent terms of the series in the solution.

Below, we extend the results of [22, 23] to the case of exponential-logarithmic solutions with a truncated regular part. We solve the problem of finding two different prolongations of the original equation that give a counterexample to the assumption about the possibility of adding invariant terms to the truncated solutions of the given truncated equation.

5. CONSTRUCTING A COUNTEREXAMPLE

The prolongation of Eq. (5) containing literals has the form

$$\left(\sum_{j=0}^{t_r} a_{rj}x^j + \sum_{j=t_r+1}^{\infty} U_{rj}x^j\right)y^{(r)}(x) + \left(\sum_{j=0}^{t_{r-1}} a_{r-1,j}x^j + \sum_{j=t_{r-1}+1}^{\infty} U_{r-1,j}x^j\right)y^{(r-1)}(x) + \dots + \left(\sum_{j=0}^{t_0} a_{0j}x^j + \sum_{j=t_0+1}^{\infty} U_{0j}x^j\right)y(x) = 0, \tag{10}$$

where U_{ij} denotes a literal. The algorithm from [15], presented in a more general form in [17], makes it possible to construct the irregular parts $e^{Q(x^{-1/q})}$ of all solutions of the form (8) for Eq. (10). We will be interested only in those for which the ramification index q and the coefficients of the polynomial Q do not depend on literals. For each such pair q, Q, we perform in (10) the substitution

$$x = t^q, \quad y(x) = e^{Q(1/t)}z(t),$$

where t is a new independent variable and $z(t)$ is a new unknown function. As a result of substituting and multiplying the equation by $e^{-Q(1/t)}$, we obtain a new equation, whose coefficients are Laurent series in t. The coefficients of these series are polynomials in literals over K. For the new equation, we construct regular solutions $t^{\lambda}w(t)$, using a variant of the algorithm from [3, Section 4.2]. This variant, for each series involved in the regular solution, calculates the maximum number of terms that are invariant to all prolongations of the equation and one additional term the coefficient of which depends on the literals. This coefficient is a polynomial over K in a finite number of literals.

Thus, for an exponential-logarithmic solution with a truncated regular part (9), we obtain a finite set of polynomials in literals, which can be used to construct a counterexample.

In [23], when considering truncated Laurent and regular solutions, we proved the following lemma.

Lemma 1 (see [23, Lemma 1]). *For any integer* $m > 0$ and $p_i(x_1, \dots, x_l) \in K[x_1, \dots, x_k]\backslash K$, $i = 1, \dots, m$, *there exist* $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l \in K$ *such that*

$$p_i(\alpha_1, \dots, \alpha_l) \neq p_i(\beta_1, \dots, \beta_l), \quad i = 1, \dots, m. \tag{11}$$

From the proof given in [23], it follows that

$$\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l \tag{12}$$

can be integers (the ring of integers is naturally embedded in any field K of characteristic 0). It is possible to iterate over all sets of integers (12) up to the first one satisfying (11). This will make it possible to find the desired set. It is also possible to involve heuristics and random choice.

Based on this, we can describe an algorithm for constructing a counterexample to the assumption that it is possible to obtain uniquely determined additional terms of the series involved in the solutions.

Theorem 1. *Let* $\mathscr{E}$ *be an equation of the form* (5) *and* s *be its truncated solution found using the algorithm from* [4]. *Then, for* $\mathscr{E}$, *there are two different prolongations* $\mathscr{E}_1$ *and* $\mathscr{E}_2$ *that have truncated solutions* s_1 *and* s_2, *respectively, which are such prolongations of* s *that any truncated series involved in* s *has a prolongation both in* s_1 *and in* s_2 *and even the very first additional terms in* s_1 *and* s_2 *are different.*

Proof. Each series involved in the truncated solution of the form (9) is constructed by the algorithm from [4] up to the first term containing literals, which is no longer included in the final truncated solution. Before discarding the terms with literals, the series in the truncated solution can be written as

$$c_{i0} + c_{i1}x + \ldots + c_{ik_i}x^{k_i} + p_i(u_1,\ldots,u_l)x^{k_i+1} + O(x^{k_i+2}),$$

where

$u_1,\ldots,u_l$ are some of the literals found in (10);

$c_{i0}, c_{i1},\ldots,c_{ik_i}$ are literal-independent constants;

$p_i(u_1,\ldots,u_l)$ is a nonconstant polynomial over K in the literals $u_1,\ldots,u_l$, $i = 1,\ldots,m$.

To the polynomials $p_i(u_1,\ldots,u_l)$, $i = 1,\ldots,m$, Lemma 1 can be applied. Thus, there exist and can be found two different sets of (integer) values $\alpha_1,\ldots,\alpha_l$, $\beta_1,\ldots,\beta_l \in K$, of the literals $u_1,\ldots,u_l$ that are used to construct prolongations $\mathcal{E}_1$ and $\mathcal{E}_2$ that have truncated solutions s_1 and s_2 with different additional terms $p_i(\alpha_1,\ldots,\alpha_l)x^{k_i+1}$ and $p_i(\beta_1,\ldots,\beta_l)x^{k_i+1}$, respectively, not containing literals. This proves the theorem.

6. EXTENDING THE CAPABILITIES OF THE PROCEDURE FOR CONSTRUCTING SOLUTIONS

The construction of a counterexample is implemented in the Maple 2021 computer algebra system as an extension of the capabilities of the procedure *FormalSolution* from the *TruncatedSeries* package. This package contains our Maple implementations of the algorithms presented in [1–9, 22, 23]. Maple library files containing the package and Maple session files with examples of using the procedures of the package can be found on the web-page [10].

The first parameter of the *FormalSolution* procedure is differential equation (5). The derivative $y(x)$ of order i is written in the standard Maple form: diff$(y(x), x\$i)$. Truncated coefficients of the form (6) are written as $a_i(x) + \mathrm{O}(x^{t_i+1})$, where $a_i(x)$ is a polynomial of degree not higher than t_i over the field of algebraic numbers.

The name of the unknown function is specified by the second parameter of the procedure.

To work with the package procedures, one needs to download the TruncatedSeries2021.zip archive located on the web-page [10]. This archive contains two files: maple.ind and maple.lib. It is necessary to place these files in some directory, e.g., "/usr/userlib", and, in the Maple session, perform the assignment

$>$*libname* := "/usr/userlib", *libname*:

The following command in the session makes it possible to call *TruncatedSeries* package procedures in short form:

$>$*with*(*TruncatedSeries*):

The Maple 2021 system interface allows one to enter equations in mathematical form. We assign to the variable *eq* an expression denoting Eq. (1):

$$> eq := \left(1 + 3x + \mathrm{O}(x^3)\right) y(x) + \left(x^3 + \frac{1}{3}x^5 + \mathrm{O}(x^6)\right)\left(\frac{d}{dx}y(x)\right) = 0 :$$

As a result of the following call of the *FormalSolution* procedure, a truncated solution with the maximum truncation degree will be obtained:

$> FormalSolution(eq, y(x), 'counterexample' = 'Eqs')$

$$\left[\mathrm{e}^{\frac{1}{2x^2} + \frac{3}{x}} x^{\frac{1}{3}} \left(\_c_1 + \mathrm{O}(x)\right) \right]$$

and the variable *Eqs* will be assigned a pair of equations comprising a counterexample:

$> Eqs[1]$

$$\left(1 + 3x + x^3 + \mathrm{O}(x^4)\right) y(x) + \left(x^3 + \frac{x^5}{3} + 4x^6 + \mathrm{O}(x^7)\right)\left(\frac{d}{dx}y(x)\right) = 0$$

$> Eqs[2]$

$$\left(1+3x+O(x^4)\right)y(x)+\left(x^3+\frac{x^5}{3}-4x^6+O(x^7)\right)\left(\frac{d}{dx}y(x)\right)=0$$

For the equations of the counterexample, we construct truncated solutions:

$>FormalSolution(Eqs[1], y(x))$

$$\left[e^{\frac{1}{2x^2}+\frac{3}{x}}x^{\frac{1}{3}}\left(\_c_1+4\_c_1x+O(x^2)\right)\right]$$

$>FormalSolution(Eqs[2], y(x))$

$$\left[e^{\frac{1}{2x^2}+\frac{3}{x}}x^{\frac{1}{3}}\left(\_c_1-3\_c_1x+O(x^2)\right)\right]$$

It can be seen that the coefficients at x in the series involved in these solutions coincide only if both these solutions are zero.

Consider another equation, of the second order:

$$>eq := O(x^{10})y(x)+\left(1+3x+O(x^3)\right)\left(\frac{d}{dx}y(x)\right)+\left(x^3+\frac{x^5}{3}+O(x^6)\right)\left(\frac{d^2}{dx^2}y(x)\right)=0:$$

Using the *FormalSolution* procedure, we obtain exponential-logarithmic solutions in which the regular parts are calculated to the maximum possible degree:

$>FormalSolution(eq, y(x))$

$$\left[\_c_1+O(x^{11})+e^{\frac{1}{2x^2}+\frac{3}{x}}x^{\frac{10}{3}}\left(\_c_2+O(x)\right)\right] \tag{13}$$

The first two terms in (13), i.e., $\_c_1+O(x^{11})$, mean that all prolongations of the equation eq have Laurent solutions whose valuation is 0; here, the initial series truncated up to a power of 10 is equal to $\_c_1$, where $\_c_1$ is an arbitrary constant.

The third term means that all prolongations of the equation eq have irregular solutions with the exponent of the irregular part $1/(2x^2)+3/x$, the exponent $\lambda = 10/3$, and the initial truncated series $\_c_2$, where $\_c_2$ is an arbitrary constant.

If, when calling the *FormalSolution* procedure, an optional parameter $'output' = 'literal'$ is specified, then the regular parts of the solution are calculated to the maximum degree and terms with coefficients depending on literals are added:

$>FormalSolution(eq, y(x), 'output' = 'literal')$

$$\_c_1-\frac{U_{[0,10]}\_c_1x^{11}}{11}+O(x^{12})+e^{\frac{1}{2x^2}+\frac{3}{x}}x^{\frac{10}{3}}\left(\_c_2+\left(-U_{[1,3]}\_c_2+U_{[2,6]}\_c_2-2\_c_2\right)x+O(x^2)\right)$$

In our implementation of literals, the coefficient at $x^k\theta^i$ is denoted $U_{[i,k]}$. There are two sets of integer values of literals, such that the expressions

$$-\frac{U_{[0,10]}\_c_1}{11} \quad \text{and} \quad -U_{[1,3]}\_c_2+U_{[2,6]}\_c_2-2\_c_2$$

take different values. These two sets correspond to two prolongations of the equation eq, which comprise a counterexample. Indeed, the solutions of these two equations will be different prolongations of solution (13) and all the regular parts of the solution will be prolonged. Obviously, there are an infinite number of counterexamples. As a result of the *FormalSolution* procedure called with the optional parameter $'counterexample' = 'Eqs'$, the variable *Eqs* will be assigned a pair of equations, which is the desired counterexample:

$>FormalSolution(eq, y(x), 'counterexample' = 'Eqs'):$

For the first equation of this counterexample,

$> Eqs[1]$

$$\left(x^{10}+\mathrm{O}(x^{11})\right)y(x)+\left(1+3x+4x^3+\mathrm{O}(x^4)\right)\left(\frac{d}{dx}y(x)\right)+\left(x^3+\frac{x^5}{3}+\mathrm{O}(x^7)\right)\left(\frac{d^2}{dx^2}y(x)\right)=0$$

using *FormalSolution*, we obtain a truncated solution

$> FormalSolution(Eqs[1], y(x))$

$$\left[\_c_1-\frac{\_c_1x^{11}}{11}+\mathrm{O}(x^{12})+\mathrm{e}^{\frac{1}{2x^2}+\frac{3}{x}}x^{\frac{10}{3}}\left(\_c_2-6\_c_2x+\mathrm{O}(x^2)\right)\right] \tag{14}$$

For the second equation,

$> Eqs[2]$

$$\left(-4x^{10}+\mathrm{O}(x^{11})\right)y(x)+\left(1+3x+5x^3+\mathrm{O}(x^4)\right)\left(\frac{d}{dx}y(x)\right)$$
$$+\left(x^3+\frac{x^5}{3}-2x^6+\mathrm{O}(x^7)\right)\left(\frac{d^2}{dx^2}y(x)\right)=0$$

we obtain

$> FormalSolution(Eqs[2], y(x))$

$$\left[\_c_1+\frac{4\_c_1x^{11}}{11}+\mathrm{O}(x^{12})+\mathrm{e}^{\frac{1}{2x^2}+\frac{3}{x}}x^{\frac{10}{3}}\left(\_c_2-9\_c_2x+\mathrm{O}(x^2)\right)\right] \tag{15}$$

It can be seen that (14) and (15) are prolongations of (13) and the truncated series involved in them are different.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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