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ORDINARY DIFFERENTIAL EQUATIONS

On the Multiplicative Property of Indicial Polynomials

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Abstract—The roots of the indicial polynomial constructed for a given linear ordinary differential operator provide information on the singularities of the solutions of the corresponding homogeneous differential equation. Operators and equations whose coefficients are formal Laurent series are discussed. Solutions of the same type are also considered. Under these assumptions, the structure of the indicial polynomial of the product of differential operators is described. This structural (multiplicative) property is preserved in the case of convergent series.

Keywords: linear ordinary homogeneous differential equations, indicial polynomial, formal Laurent series, induced recurrent operators

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1. INTRODUCTION

Solutions of a linear ordinary differential equation that have the form of formal Laurent series can be constructed using an induced recurrent operator that provides a relation for the coefficients of such solutions. Let us clarify that a *formal Laurent series* over a field K is an expression $s(x)$ (a formal sum—issues of convergence for such series are not considered) having the form

$$\sum_{n=-\infty}^{\infty} c(n)x^n, \tag{1}$$

where all $c(n)$, i.e., the coefficients of the series belong to K, and there exists $m \in \mathbb{Z}$ such that $c(n) = 0$ for $n < m$; for such m, the series (1) can be rewritten as $\sum_{n=m}^{\infty} c(n)x^n$, or as $c(m)x^m + c(m+1)x^{m+1} + \ldots$. It should be emphasized that, even with existence of such m, the series (1) is consider to be two-sided, as well as the sequence of its coefficients: in such a two-sided sequence, all elements that have, in particular, negative indices sufficiently large in absolute value are equal to zero.

In some publications on computer algebra, the term "a formal meromorphic series" is used; in this article, we use the term "a formal Laurent series", as in most publications.

For formal Laurent series, the basic arithmetic operations and differentiation are defined—details are given in Section 2.

In this context, consideration of recurrent operators arises in connection with the fact that, if there is a differential operator

$$L = a_r(x)D^r + \ldots + a_1(x)D + a_0(x) \tag{2}$$

(hereinafter, D is the operator of differentiation of a series with respect to x) and the role of the coefficients of the operator L is played by formal Laurent series over some field K, then this L can be associated with an induced recurrent operator

$$L^{①} = b_t\sigma^t + b_{t-1}\sigma^{t-1} + \ldots \in K[n]((\sigma^{-1})), \tag{3}$$

in which

- σ is the shift operator; i.e., $\sigma c(n) = c(n+1)$ for an arbitrary sequence $\{c(n)\}_{n\in\mathbb{Z}}$;
- the coefficients b_i are polynomials in n over K, having degree not higher than r—the maximum degree of D in L,

• the operator σ enters into $L^{①}$ with a finite number of integer nonnegative exponents and, generally speaking, with an infinite number of integer negative exponents. Details are given in Section 2.

Thus, $L^{①}$ is represented as a formal Laurent series in σ^{-1}. Then, if application of L to the series (1) yields a two-sided series $\sum_{n\in\mathbb{Z}}\tilde{c}(n)x^n$, then the two-sided sequence of coefficients $\{\tilde{c}(n)\}_{n\in\mathbb{Z}}$ is obtained by applying $L^{①}$ to the two-sided sequence $\{c(n)\}_{n\in\mathbb{Z}}$. In particular, any solution of the form (1) of the equation $L(y) = 0$ is associated with a solution $\{\tilde{c}(n)\}_{n\in\mathbb{Z}}$ of the equation $L^{①}(c) = 0$ and vice versa. It is essential that the solution of the equation $L(y) = 0$ is a formal series, while the solution of the equation $L^{①}(c) = 0$ is a sequence of coefficients.

The operator $L^{①}$ can be used in calculating the coefficients of the series solution. Of course, the induced operator $L^{①}$ has to be constructed. But, if it has already been constructed (taking into account that only a certain fragment of the operator $L^{①}$ rather that the entire operator may be required for the proposed calculations, only a finite number of terms of the operator $L^{①}$ with negative powers of σ are needed in the calculations), then, as will be established below, the indicial polynomial (see Section 2) and some other auxiliary objects have also been constructed, if the aforementioned fragment was sufficiently representative—these auxiliary objects are extracted from $L^{①}$ rather easily. Even before the direct computational use of the recurrent operator, its very form allows one to find some characteristics of the series solution, which additionally opens up an opportunity for proving some properties of the coefficients of the solutions, as well as various quantities and objects associated both with the differential equation itself and with its solutions. In particular, in Section 3, some multiplicative properties of the indicial polynomials are proved (Theorems 1 and 2).

It is worth emphasizing that the article does not propose new methods for solving differential equations. The aim of the article is different: to prove the existence of useful nontrivial properties for such an important tool for studying and solving differential equations as the indicial polynomial and, accordingly, the indicial equation.

The article uses the standard notation $K[x]$ for the ring of polynomials and $K((x))$ for the field or ring of formal Laurent series in x over a given field (or ring) K, i.e., polynomials and series with coefficients belonging to the field or ring K.

2. PRELIMINARY INFORMATION

2.1. Formal Laurent Series

For brevity, where it does not cause misunderstanding, we will write “Laurent series” or simply “series” instead of “formal Laurent series”. A series all of whose coefficients are zero is denoted by 0. There is also a series all of whose coefficients $c(n)$, except $c(0)$, are zero, while $c(0) = 1$; it is denoted by 1. A series $-s(x) = \sum_{n=-\infty}^{\infty}(-c(n))x^n$ is the opposite of the series $s(x) = \sum_{n=-\infty}^{\infty}c(n)x^n$. A series $\tilde{s}(x)$ is the inverse of a series $s(x)$ if $s(x)\tilde{s}(x) = 1$; for any nonzero series $s(x)$, there exists an inverse, $s^{-1}(x)$.

If $s(x) \neq 0$, then, as was already said in Section 1, there exists a minimum $m \in \mathbb{Z}$ such that the coefficient $c(m)$ of the series $s(x)$ is nonzero. We will denote this m by $\operatorname{val} s(x)$ and call it the valuation of the series $s(x)$ (in [2], Chapter 15, the term “lower power” is used). The coefficient $c(m)$ is called the lowest coefficient of the series $s(x)$ and is denoted by $\operatorname{tc} s(x)$. By definition, we assume that $\operatorname{val}(0) = \infty$ and $\operatorname{tc}(0) = 0$. It is easy to check that $\operatorname{val} s(x)t(x) = \operatorname{val} s(x) + \operatorname{val} t(x)$, $\operatorname{tc} s(x)t(x) = \operatorname{tc} s(x)\cdot\operatorname{tc} t(x)$, and $\operatorname{val}(s(x)+t(x)) \geqslant \min\{\operatorname{val} s(x), \operatorname{val} t(x)\}$ for any $s(x), t(x) \in K((x))$. It should be added that $\operatorname{val} s^{-1}(x) = -\operatorname{val} s(x)$ and $\operatorname{tc} s^{-1}(x) = (\operatorname{tc} s(x))^{-1}$ for any $s(x) \in K((x))\backslash\{0\}$.

Under the operations defined in this way, the formal Laurent series over the field K form the field $K((x))$. This and other properties of such series can be found in [2], Chapter 15; [3], Chapter 1, Section 1; and [4], Chapter 1. The derivative of a series $s(x) = \sum_{n=-\infty}^{\infty}c(n)x^n$ is defined as $Ds(x) = s'(x) = \sum_{n=-\infty}^{\infty}d(n)x^n$, where $d(n) = (n+1)c(n+1)$ for all n, which implies that the coefficient $d(-1)$ of the derivative is always zero.

Each of the series under consideration has a finite valuation, which allows one to apply to a series a differential operator. Such an application yields a zero series if and only if applying the induced operator to the sequence of coefficients of the series yields a zero sequence.

Below, we use the notation $O(x^m)$, where $m \in \mathbb{Z}$, for an unspecified formal series whose valuation is greater than or equal to m.

2.2. Valuation Block, Increment, and Indicial Polynomial

The indicial polynomial, associated with the equation $L(y) = 0$, where L has the form (2), contains among its roots all valuations of the Laurent solutions of the original equation. Such a polynomial may also have "extra" roots. There are known algorithms for checking the existence of a solution with a given valuation v and constructing such solutions if they exist. Extra roots can be rejected by these algorithms.

Definition 1. The *valuation block* of an operator L will be understood as any set of integers containing the valuations of all nonzero formal Laurent solutions of the operator L. (It is possible that the valuation block also contains some integers that are not valuations of the solutions under consideration.)

The indicial polynomial for a given L can be constructed in different ways. Usually, such a construction involves an integer, which below will be called an increment

Definition 2. We associate with the operator (2) the increment

$$\omega_L = \min_{0 \leqslant j \leqslant r}(\operatorname{val} a_j - j). \tag{4}$$

By equating to zero the lowest coefficient in the expression for the series $L(s(x))$, we obtain an algebraic indicial equation $I_L(n) = 0$ (see [1], Section 8), the left-hand side of which gives the indicial polynomial:

$$I_L(n) = \sum_{\substack{0 \leqslant j \leqslant r \\ \operatorname{val} a_j - j = \omega_L}} \operatorname{tc}(a_j) n^{\underline{j}}, \tag{5}$$

$n^{\underline{j}} = n(n-1)\ldots(n-j+1)$.

We have $\operatorname{val} L(s) \geqslant \omega_L + \operatorname{val} s(x)$. It is varified directly that, if $\operatorname{val} s(x) = n$, then

$$L(s(x)) = (\operatorname{tc} s(x)) I_L(n) x^{n+\omega_L} + O(x^{n+\omega_L+1}),$$

i.e., the series $L(s(x))$ has a coefficient $\operatorname{tc} s(x) I_L(n)$ at $x^{n+\omega_L}$. Thus, for a nonzero $s(x)$, the equality $I_L(n) = 0$ is a necessary condition for the equality $L(s(x)) = 0$.

This is illustrated by a simple example.

Example 1. Let $L = D$. In accordance with (4), we have

$$\omega_L = -1, \quad I_L(n) = n. \tag{6}$$

Thus, $L(s(x)) = 0$ implies $I_L(\operatorname{val} s(x)) = 0$; i.e., $\operatorname{val} s(x) = 0$. The conclusion that can be drawn here directly from the aforesaid is as follows: if the $D(y) = 0$ equation has a nonzero solution $s(x) \in K((x))$, then $s(x) = C + O(x)$, where C is a nonzero constant; the solution $s(x) \in K((x))$ cannot contain x in a negative power. The equalities (6) do not indicate anything more, in particular, that $O(x)$ for $s(x)$ is simply 0.

2.3. Induced Recurrent Operators

Let K be a field of characteristic 0, e.g., some number field, and let σ be the shift operator with respect to n: $\sigma(n) = n + 1$.

Proposition 1 ([5, 6]).

(i) *The correspondence*

$$x \to \sigma^{-1}, \quad x^{-1} \to \sigma, \quad D \to (n+1)\sigma \tag{7}$$

defines an isomorphism $\mathcal{F}_D^{\sigma}$ *of the ring* $K((x))[D]$ *onto the ring* $K[n]((\sigma^{-1}))$. *The inverse isomorphism is defined as follows*:

$$n \to xD, \quad \sigma \to x^{-1}, \quad \sigma^{-1} \to x. \tag{8}$$

(ii) *Let* $L \in K((x))[D]$ *and* $R = \mathcal{F}_D^{\sigma}(L)$. *Suppose that* $s(x) = \sum_n c(n)x^n$ *and* $\bar{s}(x) = \sum_n \bar{c}(n)x^n$, $s(x), \bar{s}(x) \in K((x))$, *are such that* $L(s(x)) = \bar{s}(x)$. *Then,* $R(c) = \bar{c}$, *where* c *and* $\bar{c}$ *are sequences of the coefficients of the series* $s(x)$ *and* $\bar{s}(x)$.

Example 2. Let $L = xD - (x-1)$; then

$$L^{\circledS} = \sigma^{-1}(n+1)\sigma - (\sigma^{-1} - 1) = n + 1 - \sigma^{-1}.$$

Let $y(x) = s(x)$, where $s(x)$ is a series (1). Then $L(y) = 0$ if and only if $L^{\circledS}(c) = 0$; i.e., the equality $(n+1)c(n) - c(n-1) = 0$, $n \in \mathbb{Z}$, is necessary and sufficient for the series (1) to satisfy the equation $L(y) = 0$. It is easy to see that $c(n) = 0$ for $n \leqslant -2$, and $c(-1)$ can be chosen arbitrarily and $c(n) = \dfrac{c(-1)}{(n+1)!}$ for $n \geqslant -1$.

3. RECURRENCE RELATION FOR THE SOLUTION COEFFICIENTS AND THE INDICIAL POLYNOMIAL

3.1. The Indicial Polynomial and the Increment in the Induced Operator

Proposition 2. *Let* $L^{\circledS}$ *have the form* (3). *Then*,

(i) $\omega_L = -t$,

(ii) $I_L(n) = b_t(n-t) = b_t(n+\omega_L)$.

Proof. Denote by j_0 the maximum integer such that $0 \leqslant j_0 \leqslant r$ and the corresponding value $j_0 - \operatorname{val} a_{j_0}$ is maximal. The maximum degree of σ in $L^{\circledS}$ is $t = j_0 - \operatorname{val} a_{j_0}$. The term with the highest power of σ in $L^{\circledS}$ is

$$b_t(n)\sigma^t = \sum_{\substack{0 \leqslant j \leqslant r \\ \operatorname{val} a_j - j = -t}} \operatorname{tc}(a_j)\sigma^{-\operatorname{val} a_j} \underbrace{(n+1)\sigma \cdots (n+1)\sigma}_{j} = \sum_{\substack{0 \leqslant j \leqslant r \\ \operatorname{val} a_j - j = -t}} \operatorname{tc}(a_j)\sigma^{-\operatorname{val} a_j}(n+1)\cdots(n+j)\sigma^j$$

$$= \sum_{\substack{0 \leqslant j \leqslant r \\ \operatorname{val} a_j - j = -t}} \operatorname{tc}(a_j)\sigma^{j - \operatorname{val} a_j}(n-j+1)\cdots n = \sigma^t \left(\sum_{\substack{0 \leqslant j \leqslant r \\ \operatorname{val} a_j - j = -t}} \operatorname{tc}(a_j) n^{\underline{j}} \right).$$

The coefficient at σ^t is the polynomial

$$\sum_{\substack{0 \leqslant j \leqslant r \\ \operatorname{val} a_j - j = \operatorname{val} a_{j_0} - j_0}} \operatorname{tc}(a_j)(n+t)^{\underline{j}} \tag{9}$$

in n. This polynomial has a degree j_0 and, consequently, is nonzero. Obviously, the maximality of $j - \operatorname{val} a_j$ for $0 \leqslant j \leqslant r$ implies the minimality of $\operatorname{val} a_j - j$, which corresponds to the definition (4) of the value of ω_L. We have (i).

Comparison of the coefficient (9) with (4) shows that (ii) is true.

Example 3. Consider again the case of $L = D$. We have (6). Here, $t = 1$ and $b_t = n + 1$. Equalities (i) and (ii) are obviously satisfied.

Proposition 2 (ii) implies that, if the indicial polynomial has integer roots, then the original equation $L(y) = 0$ has a solution in the form of a series $s(x)$ whose valuation is equal to the largest of the integer roots of the polynomial $I(n)$. Thus, we have proven the following proposition.

Proposition 3. *If the indicial polynomial* (5) *has integer roots, then the set of solutions of the original differential equation that have the form of nonzero formal series is not empty*.

3.2. Multiplicativity of the Indicial Polynomial

Let us formulate and prove the main result of the article.

Theorem 1. *Let L_1 and L_2 be differential operators of the form* (2). *Then*,

(i) $\omega_{L_1L_2} = \omega_{L_1} + \omega_{L_2}$;

(ii) $I_{L_1L_2} = I_{L_1}(n+\omega_{L_2})I_{L_2}(n)$.

Proof. By Proposition 1(i), we have

$$(L_1L_2)^{①} = L_1^{①}L_2^{①}.$$

We expand the product on the right-hand side of the last equality:

$$(\sigma^{-\omega_{L_1}}I_{L_1}(n) + b_{t_1-1}\sigma^{-\omega_{L_1}-1} + \ldots)(\sigma^{-\omega_{L_2}}I_{L_2}(n) + b_{t_2-1}\sigma^{\omega_{L_2}-1} + \ldots) = \sigma^{-\omega_{L_1}-\omega_{L_2}}I_{L_1}(n+\omega_{L_2})I_{L_2}(n) + \ldots .$$

Hence, we have (i) and (ii).

From the above proof of Theorem 1, it follows that the indicial polynomial has the multiplicative property specified in paragraph (ii) of this theorem.

Example 4. Let us return to Example 1 and consider the operator $L^k = \underbrace{D\cdots D}_{k}$. By Theorem 1, we have

$$\omega_{L^k} = -k, \quad I_{L^k}(n) = n^{\underline{k}}. \tag{10}$$

Theorem 1 (ii) admits the corollary:

Corollary. Let $L_1, L_2 \in k(x)[D]$ and $L = L_1L_2$. Let $\{\alpha_1,\ldots,\alpha_{N_1}\}$ and $\{\beta_1,\ldots,\beta_{N_2}\}$ be the sets of roots of the indicial polynomials I_{L_1} and I_{L_2} in some extension $\tilde{K}$ of the field K. Then, the set of roots of the polynomial I_L in $\tilde{K}$ is

$$\{\alpha_1-\omega_{L_2},\ldots,\alpha_{N_1}-\omega_{L_2},\beta_1,\ldots,\beta_{N_2}\}.$$

3.3. Valuation Blocks

Theorem 2. *Let L_1 and L_2 be operators and V_1 and V_2 valuation blocks* (*see Definition* 1) *of these operators. Suppose that $L = L_1L_2$ and $\tilde{V}_1$ is obtained from V_1 by subtracting ω_{L_2} from each of its elements. Then, $V = \tilde{V}_1 \cup V_2$ is a valuation block of the operator L*.

Proof. Valuations of the formal Laurent solutions of the operators L_1 and L_2 are roots of the polynomials I_{L_1} and I_{L_2}. The valuations of all solutions of the operator L are the roots of I_L. By Theorem 1 (ii), the valuations of all such solutions are contained in V.

3.4. Final Example and Remark

Example 5. For

$$L = (x^2 + O(x^3))D^3 - 1, \tag{11}$$

in accordance with (4) and (5),

$$\omega_L = -1, \quad I_L(n) = n^{\underline{3}} = n(n-1)(n-2). \tag{12}$$

Using Theorem 1, we can determine ω_{L^2}, $I_{L^2}(n)$ without using the explicit form of the operator L^2:

$$\omega_{L^2} = (-1) + (-1) = -2, \tag{13}$$

$$I_{L^2}(n) = \big((n-1)(n-2)(n-3)\big)n(n-1)(n-2) = n(n-1)^2(n-2)^2(n-3). \tag{14}$$

With the help of the Maple system ([7]) A.A. Ryabenko found an expression for L^2:

$$(x^4 + O(x^5))D^6 + (6x^3 + O(x^4))D^5 + (6x^2 + O(x^3))D^4 + O(x^2)D^3 + 1, \tag{15}$$

whence we can also obtain (13) and (14)—terms of the form $O(\ldots)$ in (15) do not affect the increment and the indicial polynomial associated with the operator L^2.

We can check the correctness of (12) by considering some specific variant of $O(x^3)$; i.e., the variant of the tail of the series in (11).

For example, replace $O(x^3)$ with a zero series. Thus, $L = x^2D^3 - 1$. For the operator

$$L^2 = x^4D^6 + 6x^3D^5 + 6x^2D^4 - 2x^2D^3 + 1$$

we find from (4) and (5) that

$$\omega_{L^2} = -2,$$

and

$$I_{L^2}(n) = n^{\underline{6}} + 6n^{\underline{5}} + 6n^{\underline{4}} = n(n-1)^2(n-2)^2(n-3).$$

The same is obtained by Theorem 1: see (13) and (14).

Remark 1. The proven multiplicative property of indicial polynomials (Theorem 1) is preserved when formal series are replaced with convergent ones, because the rules of addition, multiplication, and differentiation are the same for both types of series.

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