ISSN 0965-5425, Computational Mathematics and Mathematical Physics, 2019, Vol. 59, No. 10, pp. 1649–1659. © Pleiades Publishing, Ltd., 2019.
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Linear Ordinary Differential Equations and Truncated Series

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Received May 19, 2019; revised May 19, 2019; accepted June 10, 2019

Abstract—Linear ordinary differential equations with the coefficients in the form of truncated formal power series are considered. It is discussed what can be learned from the equation given in this from about its solutions belonging to the field of Laurent formal series. We are interested in the information about these solutions that is invariant to possible prolongations of those truncated series that represent the coefficients of the equation.

Keywords: differential equations, power series, Laurent series, truncated series, computer algebra systems

DOI: 10.1134/S0965542519100026

1. INTRODUCTION

Series play a fundamental role in the theory and application of differential equations. In particular, the coefficients of a linear ordinary differential equation are often represented by series and the task may be to find solutions of this equation in the form of series of some kind.

Below, the role of the coefficients of an equation will be played by formal power series and the role of the coefficients of these series will be played by elements of a given field K of characteristic 0. The solutions of our interest belong to the field of formal Laurent series over K. Such solutions will be called Laurent solutions. The questions of the convergence of these series will be beyond our interest.

The algorithmic aspect of this kind of problems involves the representation of infinite series, in particular, series playing the role of the coefficients of equations. In [1–3], an algorithmic representation was considered: the series $\sum a_n x^n$ was specified by an algorithm that determines a_n from a given n. It was found that some problems concerning the solutions of equations specified in this way are algorithmically unsolvable and, at the same time, other problems can be successfully solved. In particular, the problem of finding Laurent solutions is solvable. (In these works, not only individual scalar equations were discussed, but systems of equations as well.) In [3, 5], problems of constructing solutions were considered under the assumption that the series playing the role of the coefficients of a given equation or a system of equations are represented in approximate, namely, *truncated* form. For example, in [5], it was found out what truncation of the coefficients of a system will be sufficient for calculating a given number of the leading terms of the series entering into exponential and logarithmic solutions of the system. In [3], this problem was considered for constructing truncated Laurent solutions. In this paper, we are interested in information about Laurent solutions that are invariant to possible prolongations of truncated series representing the coefficients of the equation. Here, we propose an algorithm that calculates the maximum possible number of terms of Laurent solutions, the correctness of which is guaranteed in this sense.

Details of the problem statement are given in Section 2. The proposed algorithm is presented in Section 6. Its implementation in the Maple environment (see [6]) is described in Section 9.

2. PROBLEM STATEMENT

First, we introduce some concepts and notation. Let K be a field of characteristic 0. The following notation is standard:

$K[x]$ is the ring of polynomials with the coefficients from K;

$K[[x]]$ is the ring of formal power series with the coefficients from K;

$K((x))$ is the field of fractions of the ring $K[[x]]$.

In $K[x]$, $K[[x]]$, and $K((x))$ the differentiation $D = \frac{d}{dx}$ is defined.

Definition 1. The elements of the field $K((x))$ are formal *Laurent series*. For a nonzero element $a(x) = \sum a_i x^i \in K((x))$, its valuation $\operatorname{val} a(x)$ is defined as $\min\{i \mid a_i \neq 0\}$; and, $\operatorname{val} 0 = \infty$. Let $l \in \mathbb{Z} \cup \{-\infty\}$; the *l-truncation* $a^{\langle l \rangle}(x)$ is obtained by discarding all terms $a(x)$ of degree higher than l; if $l = -\infty$, then $a^{\langle l \rangle}(x) = 0$.

We will consider operators and differential equations written using the notation $\theta = xD$. In the original operator

$$L = \sum_{i=0}^{r} a_i(x)\theta^i \in K[x][\theta], \tag{1}$$

the coefficients have the form

$$a_i(x) = \sum_{j=0}^{t_i} a_{ij}x^j, \tag{2}$$

where t_i is a nonnegative integer greater than or equal to $\deg a_i(x)$, $i = 0, 1, \ldots, r$ (if $t_i > d_i = \deg a_i(x)$, then $a_{ij} = 0$ for $j = d_i + 1, d_i + 2, \ldots, t_i$). It is assumed that the constant term of at least one of the polynomials $a_0(x)$, ..., $a_r(x)$ is nonzero.

Definition 2. Let L have form (1) and the polynomial $a_r(x)$ (the *leading coefficient* of the differential operator L) be nonzero. A *prolongation* of the operator L will be understood as any operator $\tilde{L} = \sum_{i=0}^{r} b_i(x)\theta^i \in K[[x]][\theta]$ for which

$$b_i(x) - a_i(x) = O(x^{t_i+1}) \tag{3}$$

(i.e., $\operatorname{val}(b_i(x) - a_i(x)) > t_i$), $i = 0, 1, \ldots, r$.

If L is some differential operator, then *solutions of the operator* L will be understood as solutions of the equation $L(y) = 0$.

We will propose an algorithm the input of which is the operator $L \in K[x][\theta]$ and nonnegative integers $t_0, t_1, \ldots, t_r$ have the same meaning as in (2). As a result, the algorithm gives, in particular, finite sets of integers, $W = \{v_1, \ldots, v_k\}$ and $M = \{m_1, \ldots, m_k\}$,having the following properties (the solutions are assumed to belong to $K((x))$):

• for each $v_i \in W$, there is a solution $y(x)$ of the operator L, for which $\operatorname{val} y(x) = v_i$;

• if $y(x)$ is such a solution of the operator L that $\operatorname{val} y(x) = v_i$, $1 \le i \le k$, then, for any prolongation $\tilde{L}$ of the operator L, there is a solution $\tilde{y}(x)$ for which

$$\tilde{y}(x) = y^{\langle m_i \rangle}(x) + O(x^{m_i+1}); \tag{4}$$

• if $\tilde{y}(x)$ is a solution of some prolongation $\tilde{L}$ of the operator L and $\operatorname{val} \tilde{y}(x) = v_i$, $1 \le i \le k$, then there is a solution $y(x)$ of the operator L, for which

$$\tilde{y}(x) - y(x) = O(x^{m_i+1}) \tag{5}$$

and, consequently, (4) is satisfied;

• the values m_i, $i = 1, \ldots, k$, are the maximum possible values related in the aforementioned sense with L and W.

In this case, the sets W and M include all the values possessing these properties.

If we know the space of Laurent solutions of the operator L (this operator has polynomial coefficients; the algorithm for constructing its Laurent solutions is a simple modification of the algorithm from [4, Section 6]) and the sets W and M are also known, then we have a complete list of valuations of the solution and formulas of form (4), invariant to prolongations of the operator L.

Remark 1. When considering the Laurent solutions $\tilde{y}(x)$ of the prolongation $\tilde{L}$ of the operator L, for checking condition (5), it is essential that $\operatorname{val}\tilde{y}(x) = v_i$, i.e., that the operator L also have Laurent solutions with this valuation v_i. In this case, there may be Laurent solutions $\tilde{L}$ with valuations not belonging to W.

Remark 2. The last assertion is directly related to the question of representation of infinite series, raised in Introduction: the differential equation is given in the form

$$(a_r(x) + O(x^{t_r+1}))\theta^r y(x) + \ldots + (a_1(x) + O(x^{t_1+1}))\theta y(x) + (a_0(x) + O(x^{t_0+1}))y(x) = 0, \tag{6}$$

$t_i \geq \deg a_i(x)$, $i = 0,1,\ldots,r$. To it, we put into correspondence operator (1) and a set of numbers $t_0, t_1, \ldots, t_r$ and solve the problem of finding $W = \{v_1,\ldots,v_k\}$, $M = \{m_1,\ldots,m_k\}$, and formulas (4).

The prolongation of operator (1) in this case will also be called a *prolongation of equation* (6).

The form of representing the results of the algorithm will be established in Section 9.

3. SEQUENCES OF THE COEFFICIENTS OF LAURENT SOLUTIONS

Let σ denote the shift operator $\sigma c_n = c_{n+1}$ for any sequence (c_n). The transformation

$$x \to \sigma^{-1}, \quad \theta \to n \tag{7}$$

converts the initial differential equation

$$\sum_{i=0}^{r} a_i(x)\theta^i y(x) = 0, \tag{8}$$

where $a_i(x) \in K[[x]]$, into an *induced* recurrent equation (relation)

$$u_0(n)c_n + u_{-1}(n)\sigma^{-1}c_n + \cdots = 0.$$

Rewrite this equation in the form

$$u_0(n)c_n + u_{-1}(n)c_{n-1} + \cdots = 0. \tag{9}$$

Equation (8) has a Laurent solution $y(x) = c_v x^v + c_{v+1}x^{v+1} + \ldots$ if and only if the two-sided sequence

$$\ldots, 0, 0, c_v, c_{v+1}, \ldots \tag{10}$$

satisfies Eq. (9), i.e.,

$$\begin{gathered} u_0(v)c_v = 0, \\ u_0(v+1)c_{v+1} + u_{-1}(v+1)c_v = 0, \\ u_0(v+2)c_{v+2} + u_{-1}(v+2)c_{v+1} + u_{-2}(v+2)c_v = 0, \\ \ldots\ldots \end{gathered}$$

(see the proof in [4]).

Here, $(c_n)_{-\infty<n<\infty}$ is a sequence such that $c_n = 0$ for all negative integers n with sufficiently large $|n|$; each of the polynomials $u_0(n), u_{-1}(n), \ldots \in K[n]$ has a degree less than or equal to r; and $u_0(n)$ is the *leading* coefficient of relation (9).

Recall that, by our assumption, the constant term of at least one of the polynomials $a_0(x),\ldots,a_r(x)$ is not zero. Hence,

$$u_0(n) = \sum_{i=0}^{r} a_{i,0} n^i$$

is a nonzero polynomial. It does not depend on the prolongations of the original operator L.

We see that, if the coefficients of Eq. (8) are infinite series, then sum (9) contains an infinite number of terms. But, since there exists $v \in \mathbb{Z}$ such that $c_n = 0$ for all $n < v$, for each particular $n \geq v$, sum (9) is finite. The leading coefficient $u_0(n)$ can be considered as some variant of the *indicial polynomial* of the original equation. The finite set of integer roots of this polynomial contains all possible valuations v of the Laurent solutions of Eq. (8).

4. ADDITIONAL RELATIONS FOR CALCULATED COEFFICIENTS OF THE SERIES

If $u_0(n) \neq 0$ for some integer n, then (9) allows one to find c_n from $c_{n-1}, c_{n-2}, \ldots$. If $u_0(n) = 0$, then, we declare (possibly temporarily) c_n an *undetermined* coefficient entering into the solution under construction. In this case, it turns out that the previous values $c_{n-1}, c_{n-2}, \ldots$ must satisfy the relation

$$u_{-1}(n)c_{n-1} + u_{-2}(n)c_{n-2} + \ldots = 0. \quad (11)$$

Such relations have a finite number of terms on the left-hand side and, possibly, will help us to get rid of some previously introduced undetermined coefficients. Only when, after increasing step by step, n will exceed the maximum integer root of the equation $u_0(n) = 0$, there will be a guarantee that new undetermined coefficients and relations of form (11) will no more arise.

5. THE BASIS OF THE ALGORITHM

If the polynomial $u_0(n)$ has no integer roots, then none of the prolongations of the equation $L(y) = 0$ has solutions in $K((x))$. The algorithm reports this and stops working.

If there are integer roots $\alpha_1 < \ldots < \alpha_s$, then, when considering both the operator L and its prolongations, the existence of a Laurent solution with such a valuation is guaranteed only for α_s and $\alpha_1, \ldots, \alpha_{s-1}$ require special consideration. For each of these roots, there are three possibilities:

(a) Laurent solutions exist for all prolongations;

(b) Laurent solutions exist for some but not all prolongations;

(c) Laurent solutions do not exist for any prolongation.

First of all, we have to determine which of the three possibilities takes place. In such situations, we use symbolic *unspecified coefficients*: we add to each polynomial $a_i(x)$ of form (2), entering into the operator L, a certain number of higher degree terms $a_{i,t_i+1}x^{t_i+1} + a_{i,t_i+2}x^{t_i+2} + \ldots$, where $a_{i,t_i+1}, a_{i,t_i+2}, \ldots$ are symbols (letter designations). Having performed the calculations, we can see whether the values of the expressions from which we make conclusions (a), (b), or (c) depend on the values $a_{i,t_i+1}, a_{i,t_i+2}, \ldots$.

Thus, adding symbolic coefficients, we can establish for the considered α_i which of these three possibilities is realized. If (a), then we find m. If (b) or (c), then we exclude α_i from consideration.

As a result, the set of valuations $W = \{v_1, \ldots, v_k\}$ is determined. For each of these valuations v_i, a specific number m_i will be determined. In other words, we have a set $M = \{m_1, \ldots, m_k\}$ of integers for which the corresponding relation of form (4) are satisfied.

If we consider solutions having a valuation α_l, $l < s$, then, using the induced recurrent relation (9), we need to advance in constructing the solution at least to x^{α_s}, even if, before this moment, the coefficients of the series being constructed acquired symbolic unspecified coefficients of the prolongation. The use of the induced recurrent equation for n equal to one of the integer roots of its leading coefficient gives linear relation (11) for the already obtained coefficients of the series, symbolic or belonging to K (see Section 4). If we have reached the maximum root and have found out that these relations somehow limit the choice of the prolongation (prevents arbitrary choice), then we exclude from consideration the solution with valuation α_l. If the relations do not limit the choice of the prolongation (e.g., all the relation only express the existing undetermined coefficients via the added unspecified coefficients), then we select from the constructed segment its leading terms in accordance with formula (4) with $v_i = \alpha_l$.

After that, it is possible to construct a basis of the space of truncated Laurent solutions of a valuation greater than or equal to v_1 in the form of a finite set of finite series, which is represented by the union of k subsets; the ith subset, $1 \leq i \leq k$, consists of truncated solutions having a valuation v_i, and truncated series entering into some fixed subset are linearly independent over the field of constants.

6. STEPS OF THE ALGORITHM

Step 1. Input: differential operator (1) with polynomial coefficients and nonnegative integers $t_0, t_1, \ldots, t_r$.

Step 2. Set $W = M = \varnothing$.

Step 3. Construct the coefficient $u_0(n) = \sum_{i=0}^{r} a_{i,0}n^i$ of induced recurrent equation (9). Calculate the set $\alpha_1 < \ldots < \alpha_s$ of integer roots of $u_0(n)$. If this set is empty, then there are no Laurent solutions.

Step 4. Define d as the difference between the maximal and minimal integer roots of the polynomial $u_0(n)$. Calculate the coefficients $u_{-k}(n) = \sum_{i=0}^{r} a_{i,k}(n-k)^i$, $k = 1\ldots d$, of induced recurrent equation (9). Note that, for $k > t_i$, the found $u_{-k}(n)$ will contain symbolic unspecified coefficients $a_{i,k}$.

Step 5. For each α_i, calculate the initial coefficients c_n of the Laurent solution with a possible valuation α_i, setting $c_n = 0$ for all $n < \alpha_i$ and gradually increasing n, starting with $n = \alpha_i$.

5.1. Perform calculations up to n equal to α_s and determine which of possibilities (a), (b), or (c) from Section 5 is realized for the given α_i.

5.2. If (b) or (c), then pass to the next root or terminate the loop if the last root has been reached.

5.3. If (a), then add α_i to W and continue calculations for subsequent n, if necessary, calculating additional $u_{-k}(n)$, $k > d$. If the maximum integer root has been passed, then, for each subsequent n, the expression for c_n will be determined. Stop calculation when the expression for c_n acquires symbolic unspecified coefficients. Set $m_i = n - 1$.

5.4. Add m_i to M and assemble from the calculated coefficients $c_{\alpha_i} \ldots c_{m_i}$ the Laurent solution of the equation $L(y) = 0$ with valuation α_i having form (4), including arbitrary constants. After that, pass to the next root or terminate calculation if the last root has already been reached.

Step 6. Result: the found Laurent solutions together with the sets W and M.

Example 1. Let us follow the steps of the proposed algorithm on the example of the operator

$$L = -\theta^2 - 2\theta, \quad t_0 = t_1 = t_2 = 0. \tag{12}$$

At step 3, we obtain the indicial polynomial $u_0(n) = -n^2 - 2n$ and the set of its integer roots: $\alpha_1 = -2$ and $\alpha_2 = 0$. At step 4, we construct $u_{-1}(n)$ and $u_{-2}(n)$, which contain symbolic unspecified coefficients in the prolongation of the operator L.

At step 5, for the possible valuation $\alpha_1 = -2$, we successively calculate c_{-2}, c_{-1}, and c_0:

- $n = -2$: $u_0(-2)c_{-2} = 0 \cdot c_{-2} = 0$: the coefficient c_{-2} remains undetermined;
- $n = -1$: $u_0(-1)c_{-1} + u_{-1}(-1)c_{-2} = c_{-1} + u_{-1}(-2)c_{-2} = 0$. We have $c_{-1} = -u_{-1}(-1)c_{-2}$;
- $n = 0$: $u_0(0)c_0 + u_{-1}(0)c_{-1} + u_{-2}(0)c_{-2} = 0 \cdot c_0 + u_{-1}(0)c_{-1} + u_{-2}(0)c_{-2} = -u_{-1}(0)u_{-1}(-1)c_{-2} + u_{-2}(0)c_{-2} = (-u_{-1}(0)u_{-1}(-1) + u_{-2}(0))c_{-2} = 0$. The coefficient c_0 remains undetermined, while it turns out that $c_{-2} = 0$ if $-u_{-1}(0)u_{-1}(-1) + u_{-2}(0) \neq 0$. Since $u_{-1}(n)$ and $u_{-2}(n)$ depend on unspecified coefficients, here we have case (b); i.e., Laurent solutions with valuation $\alpha_1 = -2$ exist only if $-u_{-1}(0)u_{-1}(-1) + u_{-2}(0) = 0$. This value is discarded.

Step 5 for $\alpha_2 = 0$:

- $n = 0$: $u_0(0)c_0 = 0 \cdot c_0 = 0$: the coefficient c_0 remains undetermined;
- $n = 1$: $u_0(1)c_1 + u_{-1}(1)c_0 = -3c_1 + u_{-1}(1)c_0 = 0$. We have $c_1 = \frac{u_{-1}(1)c_0}{3}$. In this case, $u_{-1}(1)$ and, therefore, c_1 also depend on unspecified coefficients. Since the root $n = 0$, which corresponds to the maximum possible valuation, has already been passed, further calculations are not required.

Therefore, for the equation $L(y) = 0$, we have $W = \{0\}$ and $m_1 = 0$. For any prolongation of the coefficients, there is a Laurent solution:

$$y(x) = C + O(x), \tag{13}$$

where C is an arbitrary constant. For $C = 0$, this solution is equal to zero, since the valuation of the Laurent solution of operator (12) cannot be positive. Solution (13) can be written in the form $C(1 + O(x))$.

Example 2. Add to the coefficients of operator (12) several terms:

$$\tilde{L} = (-1 + x + x^2)\theta^2 - 2\theta + (x + 6x^2), \quad t_0 = 3, \quad t_1 = t_2 = 2. \tag{14}$$

Step 3 for $\tilde{L}$ gives the same results as for (12). At step 4, we obtain

$$u_{-1}(n) = (n-1)^2 + 1, \quad u_{-2}(n) = (n-2)^2 + 6.$$

Step 5 for possible valuation $\alpha_1 = -2$ is performed similarly to (12). But, for $n = 0$, we have $-u_{-1}(0)u_{-1}(-1) + u_{-2}(0) = -((0-1)^2+1)((-1-1)^2+1) + ((0-2)^2+6) = 0$, which means that relation (11) is an identity 0 = 0 and, therefore, the coefficient c_{-2} remains undetermined. Accordingly, for the valuation $v_1 = -2$, Laurent solutions exist for any prolongation $\tilde{L}$. Here we have case (a). We obtain $c_{-1} = -u_{-1}(-1)c_{-2} = -((-1-1)^2+1)c_{-2} = -5c_{-2}$. Further calculations are not required, since the root $n = 0$, which corresponds to the maximum possible valuation, has been passed and the expressions for all subsequent coefficients of the solution will contain unspecified coefficients. Accordingly, for the equation $\tilde{L}(y) = 0$ with any prolongation of the coefficients, there is a Laurent solution with valuation $v_1 = -2$:

$$\frac{C_1}{x^2} - \frac{5C_1}{x} + C_2 + O(x),$$

where C_1 and C_2 are arbitrary constants.

Step 5 for $\alpha_2 = 0$:

- $n = 0$: $u_0(0)c_0 = 0 \cdot c_0 = 0$, and the coefficient c_0 remains undetermined;
- $n = 1$: $u_0(1)c_1 + u_{-1}(1)c_0 = -3c_1 + 1c_0 = 0$. We have $c_1 = \frac{1}{3}c_0$;
- $n = 2$: $u_0(2)c_2 + u_{-1}(2)c_1 + u_{-2}(2)c_0 = -8c_2 + 2c_1 + 6c_0 = 0$. We have $c_2 = \frac{5}{6}c_0$;
- $n = 3$: Calculate $u_{-3}(n) = a(n-3)^2 + b(n-3)$, where a and b are symbolic unspecified coefficients.

Then, $u_0(3)c_3 + u_{-1}(3)c_2 + u_{-2}(3)c_1 + u_{-3}(3)c_0 = -15c_3 + 5c_2 + 7c_1 + 0 \cdot c_0 = 0$. We have $c_3 = \frac{13}{30}c_0$. Note that, in this case, c_3 has been calculated despite the fact that $u_{-3}(n)$ contains unspecified coefficients, since $u_{-3}(3) = 0$ for any values of these unspecified coefficients.

Further calculations are not required, since the root $n = 0$, which corresponds to the maximum possible valuation, has been passed and the expressions for all subsequent coefficients of the solution will contain unspecified coefficients. Accordingly, for the equation $\tilde{L}(y) = 0$ with any prolongation of the coefficients, there is a Laurent solution with valuation $v_2 = 0$:

$$C + \frac{1}{3}Cx + \frac{5}{6}Cx^2 + \frac{13}{30}Cx^3 + O(x^4). \tag{15}$$

Thus, for the equation $\tilde{L}(y) = 0$, we have found $W = \{-2, 0\}$, $m_1 = 0$, and $m_2 = 3$.

Example 3. If we add other new terms to the coefficients of the initial operator (12),

$$\tilde{\tilde{L}} = (-1 + x + x^2)\theta^2 + (-2 + x^2)\theta, \quad t_0 = 3, \quad t_1 = t_2 = 2, \tag{16}$$

then similar calculations show that $W = \{0\}$ and $m_1 = 3$ for the equation $\tilde{\tilde{L}}(y) = 0$. For any prolongation of the coefficients, there is a Laurent solution $C + O(x^4)$. Here, we have case (c), which led to discarding the possible valuation –2.

Let us summarize the consideration of examples 1, 2, and 3. We have the equations $L(y) = 0$, $\tilde{L}(y) = 0$, and $\tilde{\tilde{L}}(y) = 0$, the last two of which are prolongations of the first equation in the sense of Definition 2. As shown in example 1, except for 0, there is no integer value for which any prolongation of the equation $L(y) = 0$ would have a Laurent solution with a valuation equal to this value. Moreover, although the equation $\tilde{L}(y) = 0$ has a Laurent solution with valuation –2, the equation $\tilde{\tilde{L}}(y) = 0$ has no solutions with such valuation. This confirms the correctness of the answer obtained in example 1.

Proposition 1. *Suppose that the values $v_1, \ldots, v_k$, $m_1, \ldots, m_k$ have been found by the proposed algorithm for the equation $L(y) = 0$, where L has form* (1) *and $t_i \geq \deg a_i(x)$, $i = 0, 1, \ldots, r$. Let m be a positive integer such that $m > m_i$ for some $1 \leq i \leq k$. Then, for the equation $L(y) = 0$, there exists a prolongation $\tilde{L}(y) = 0$ such* *that, for some its solution* $\tilde{y}(x) \in K((x))$, $\operatorname{val} \tilde{y}(x) = v_i$, *the equality* $\tilde{y}(x) = y^{\langle m \rangle}(x) + O(x^{m+1})$ *is not satisfied for any solution* $y(x) \in K((x))$, $\operatorname{val} y(x) = v_i$, *of the equation* $L(y) = 0$.

Proof. The proposed algorithm finds each of the values m_i, using, in particular, the fact that the terms with symbolic coefficients added to $a_j(x)$ appear in expressions for the coefficients of solutions. For different prolongations of the operator L, we can obtain different coefficients of the solution $\tilde{y}$ for x^j when $j > m_i$.

Remark 3. In Proposition 1, it is essential that both $\operatorname{val} \tilde{y}(x) = v_i$ and $\operatorname{val} y(x) = v_i$, i.e., that L also has Laurent solutions with such valuation v_i. In this case, $\tilde{L}$ may have Laurent solutions with valuations that are not included in $v_1, \dots, v_k$. This is illustrated by examples 1 and 2.

Remark 4. We assume that, in the operator L of form (1), the constant term of at least one polynomial coefficient is nonzero. This assumption guarantees that the polynomial $u_0(n)$ in (9) is nonzero. In addition, it should be noted that, if the minimum valuation of nonzero coefficients of the equation is positive and equal to β and also if $t_i \geq \beta$ for each coefficient $a_i(x)$, including zero coefficients, then we can divide both sides of the equation by x^β and replace each t_i with $t_i - \beta$ and then apply our algorithm.

7. RELATION WITH THE TRUNCATION PROBLEM

It is easy to prove that an equation of the form $L(y) = 0$, $L \in K((x))[\theta]$, has a solution in $K((x))$ if and only if the indicial polynomial of the operator L has integer roots (for example, if there are such roots, then there is a Laurent solution with a valuation equal to the maximum of them). This fact is discussed, in particular, in [3], where, in addition, the so-called truncation problem is considered: how many leading terms of the coefficients of the operator L influence a given number of leading terms of the Laurent solution of the equation $L(y) = 0$?

Proposition 2 (see [3, Proposition 1 (ii)]). *Let l be a nonnegative integer,* $L \in K[[x]][\theta]$, $f(x) \in K((x))$, $v = \operatorname{val} f(x)$. *Let the indicial polynomial of the operator L have integer roots, and let* e^*, e_* *be the largest and smallest of them. Let* $s_l = \max\{e^* - e_*, l - 1\}$ *and* $s \geq s_l$. *Then the equation* $L(y) = 0$ *has a solution of the form* $f(x) + O(x^{v+l})$ *if and only if the truncated equation* $L^{\langle s \rangle}(y) = 0$ *has a solution of the same form.*

Remark 5. In [3] and in this article, the concept of truncation of a series is defined in different ways. In the statement of Proposition 2, this concept is interpreted in accordance with Definition 1.

Proposition 2 enables one to formulate another approach to solving the problem considered in this paper. Let L be an operator of form (1) with coefficients (2). Let us set $t = \min_{i=0}^{r} t_i$ and find the maximum integer l for which $s_l \leq t$. Then, the set of valuations of the Laurent solutions of the equation $L^{\langle s_l \rangle}(y) = 0$ and the first l terms of these solutions give the required solution (with this approach, $m_i = v_i + l - 1$).

This approach does not require the addition of symbolic coefficients. However, if, e.g., $s_1 > t$ (the values of t_i are too small), we cannot choose l—the approach turns out to be inapplicable. However, the algorithm presented in Section 6 copes with this situation. Let us return to example 1, where $t_0 = t_1 = t_2 = 0$ and, therefore, $t = 0$. In this case, $s_l = \max\{2, l - 1\} \geq 2$. The approach based on Proposition 2 does not give a solution to the problem, but the algorithm presented in Section 6 allowed us to obtain a solution in Example 1.

It is also easy to show that the algorithm from Section 6 can find more terms of the Laurent solutions than the algorithm based on Proposition 2. For (14), the algorithm based on Proposition 2 instead of (15) will obtain one fewer terms: $C + \frac{1}{3}Cx + \frac{5}{6}Cx^2 + O(x^3)$.

8. DIFFERENTIATION OPERATIONS θ AND D

If the original differential equation is written using D instead of θ, then, by the substitution $D = x^{-1}\theta$, this equation can be transformed into an equivalent equation written using θ: here, we have a useful equality for the composition of the operators θ and x^{-1}: $\theta x^{-1} = x^{-1}(\theta - 1)$. It is clear that, if the initial equation is given in the form

$$(w_r(x) + O(x^{\tau_r+1}))D^r y(x) + \ldots + (w_1(x) + O(x^{\tau_1+1}))Dy(x) + (w_0(x) + O(x^{\tau_0+1}))y(x) = 0, \quad (17)$$

then, as a result of this transformation, we will come to Eq. (6), but $t_0, t_1, \ldots, t_r$ may differ from $\tau_0, \tau_1, \ldots, \tau_r$. The coefficients

$$a_i(x) + O(x^{t_i+1}), \quad i = 0, 1, \ldots, r \quad (18)$$

for the new equation can be found using symbolic unspecified coefficients, which must not affect (18).

The transition from D to θ can be performed without using symbolic unspecified coefficients: we consider operator (17) as an operator with polynomial coefficients $w_0(x), \ldots, w_r(x)$, but the degree of each $w_i(x)$ is assumed to be equal to t_i, setting, if necessary, the coefficients multiplying some of the highest degrees to zero; recall that $w_0(x), \ldots, w_r(x)$ are truncated series and each t_i specifies the corresponding truncation. It cannot be ruled out that, when passing from D to θ, the coefficient multiplying some θ^i will arise as a sum of polynomials of various degrees (in this case, the degrees are understood in the aforementioned sense). In this case, the degrees of these truncated polynomials should be aligned with their lowest degree. This is fully consistent with the approach based on the use of symbolic unspecified coefficients.

Example 4. When passing from D to θ in the operator

$$L = (-x + x^2 + x^3)D^2 + (-3 + x)D, \quad \tau_0 = 2, \quad \tau_1 = 1, \quad \tau_2 = 3, \quad (19)$$

the resulting coefficient multiplying θ is represented by the sum of $1 - x - x^2$ with $t_1 = 2$ and $-3 + x$ with $t_1 = 1$. When summing, it is necessary to pass to the lower of the truncations. In the given case, the sum is equal to -2 with $t_1 = 1$.

Application of the algorithm to the resulting operator

$$(-1 + x + x^3)\theta^2 + (-2)\theta, \quad t_0 = 3, \quad t_1 = 1, \quad t_2 = 2,$$

shows that, for any prolongation of the coefficients of the operator L, there is a Laurent solution:

$$C + O(x^4), \quad W = \{0\}, \quad m_1 = 3.$$

When passing from an equation given in form (17) to an equivalent equation given in form (6), it may occur that, for some i $(0 \le i \le r)$, $t_i < 0$ even if all $\tau_0, \tau_1, \ldots, \tau_r$ are nonnegative integers. This will mean that the given equation with truncated coefficients does not contain enough information to obtain a indicial polynomial as a polynomial of n with coefficients from K. In particular, the equation

$$(x^2 + O(x^3))D^2 y(x) + O(x)Dy(x) + (1 + O(x))\, y(x) = 0$$

is equivalent to

$$(1 + O(x))\theta^2 y(x) + O(1)\theta y(x) + (1 + O(x))y(x) = 0 \quad (20)$$

and the indicial polynomial is

$$u_0(n) = n^2 + a_{1,0}n + 1, \quad (21)$$

where the coefficient $\theta y(x)$ of Eq. (20) is assumed to be equal to $a_{1,0} + O(x)$; in this case, $a_{1,0}$, i.e., the constant term of this coefficient, acts as another variable of the polynomial u_0. Here, we cannot apply the algorithm from Section 6.

9. PROGRAM IMPLEMENTATION AND EXAMPLES OF USE

The algorithm was implemented in the Maple environment (see [6]) in the form of the `LaurentSolution` procedure. The first argument of the procedure is a differential equation of form (6). An application of θ^k to an unknown function $y(x)$ is written as `theta(y(x),x,k)`. It is also possible to use ordinary differentiation (operator D: see Section 8); in this case, the application of the operator D^k to an unknown function $y(x)$ is defined in the form `diff(y(x),x$k)`, standard for Maple. The truncated coefficients of the equation are given in the form $a_i(x) + O(x^{t_i+1})$, where $a_i(x)$ is a polynomial of degree no higher than t_i over the field of rational numbers. As the second argument of the procedure, an unknown function is specified.

The result of the procedure is a list of truncated Laurent solutions corresponding to valuations $v_i \in W$. Each element of the list is represented as

$$c_{v_i}x^{v_i} + c_{v_i+1}x^{v_i+1} + \ldots + c_{m_i}x^{m_i} + O(x^{m_i+1}), \tag{22}$$

where $v_i \in W$ is a valuation that guarantees the existence of a Laurent solution for any prolongation of the given equation, m_i has the previous meaning, and c_i are the calculated coefficients of the Laurent solution, which can be linear combinations of arbitrary constants of the form $\_c_j$.

Below we present eight examples, which we combine into one, containing paragraphs 1–8.

Example 5.

1. Each of the equations

$$\sin(x)\theta y(x) - x\cos(x)y(x) = 0, \tag{23}$$

$$(e^x - 1)\theta y(x) - xe^x y(x) = 0 \tag{24}$$

can be represented in the form

$$(x + O(x^2))\theta y(x) + (-x + O(x^2))y(x) = 0. \tag{25}$$

Apply the implemented procedure to (25):

```
> eq1:=(x+O(x^2))*(theta(y(x),x,1))+(-x+O(x^2))*y(x);
```

$$eq1 := (x + O(x^2))\theta(y(x), x, 1) + (-x + O(x^2))y(x)$$

```
> LaurentSolution(eq1,y(x));
```

$$[x\_c_1 + O(x^2)]$$

The answer obtained here means that $W = \{1\}$ and $m_1 = 1$.

2. Add to the coefficients of Eq. (25) some terms corresponding to coefficients (23). We obtain a truncation of the solution to the degree x^2, which corresponds to the expansion in a power series of the function $\sin(x)$, which is a solution of (23):

```
> eq2:=(x+O(x^3))*(theta(y(x),x,1))+(-x+x^3/2+O(x^4))*y(x);
```

$$eq2 := (x + O(x^3))\theta(y(x), x, 1) + \left(-x + \frac{x^3}{2} + O(x^4)\right)y(x)$$

```
> LaurentSolution(eq2,y(x));
```

$$[x\_c_1 + O(x^3)]$$

The answer obtained also means that, here, $W = \{1\}$ and $m_1 = 2$.

3. Now add to the coefficients of Eq. (25) some terms corresponding to coefficients (24). We obtain a truncation of the solution up to the degree x^2, which corresponds to an expansion in power series of the function $\exp(x) - 1$, which is a solution of (24):

```
> eq3:=(x+x^2/2+O(x^3))*theta(y(x),x,1)+(-x-x^2-x^3/2+O(x^4))*y(x);
```

$$eq3 := \left(x + \frac{x^2}{2} + O(x^3)\right)\theta(y(x), x, 1) + \left(-x - x^2 - \frac{x^3}{2} + O(x^4)\right)y(x)$$

```
> LaurentSolution(eq3, y(x));
```

$$\left[x\_c_1 + \frac{x^2\_c_1}{2} + O(x^3)\right]$$

The answer obtained also means that $W = \{1\}$ and $m_1 = 2$.

4. For each of the equations in paragraphs 1–3, there is only one valuation for which Laurent solutions exist for any prolongation of the equation. Consider application of the procedure to an equation specified by operator (14):

```
> eq4:=(-1+x+x^2+O(x^3))*theta(y(x),x,2)+(-2+O(x^3))*theta(y(x),x,1)+
      (x+6*x^2+O(x^4))*y(x);
```

$$eq4 := (-1 + x + x^2 + O(x^3))\theta(y(x), x, 2) + (-2 + O(x^3))\theta(y(x), x, 1) + (x + 6x^2 + O(x^4))y(x)$$

```
> LaurentSolution(eq4,y(x));
```

$$\left[\frac{\_c_1}{x^2} - \frac{5\_c_1}{x} + \_c_2 + O(x),\ \_c_1 + \frac{x\_c_1}{3} + \frac{5x^2\_c_1}{6} + \frac{13x^3\_c_1}{30} + O(x^4)\right]$$

The answer obtained means that $W = \{-2, 0\}$, $m_1 = 0$, and $m_2 = 3$.

5. Does it make sense to consider, e.g., the case of pairwise different $t_0, t_1, \ldots, t_r$ entering into (6), or is it enough to restrict the consideration to the case of equality of these numbers? In other words, can the replacement in (6) of each t_i by $t = \min_{i=0}^{r} t_i$ lead to a reduction in the accuracy of the algorithm? The following example shows that such a reduction is possible. Thus, the time expenditures caused by the rejection of the a priori assumption of equality of all t_i may be justified.

For the following equation, we obtain five terms of the solution:

```
> eq5:=(1+O(x))*(theta(y(x),x,1))+(x^4+O(x^5))*y(x);
```

$$eq5 := (1 + O(x))\,\theta(y(x), x, 1) + (x^4 + O(x^5))y(x)$$

```
> LaurentSolution(eq5,y(x));
```

$$\left[\_c_1 - \frac{\_c_1 x^4}{4} + O(x^5)\right]$$

If we set $t = 0$, then we will obtain one term of the solution:

```
> eq6:=(1+O(x))*(theta(y(x),x,1))+O(x)*y(x);
```

$$eq6 := (1 + O(x))\,\theta(y(x), x, 1) + O(x)y(x)$$

```
> LaurentSolution(eq6,y(x));
```

$$[\_c_1 + O(x)]$$

6. An example of an equation that has no Laurent solutions under any prolongations:

```
> eq7:=(2+O(x))*(theta(y(x),x,1))+(1+O(x))*y(x);
```

$$eq7 := (2 + O(x))\,\theta(y(x), x, 1) + (1 + O(x))\,y(x)$$

```
> LaurentSolution(eq7,y(x));
```

$$[\,]$$

The answer is an empty list, i.e., the absence of solutions.

7. If the indicial polynomial depends on the unspecified coefficients of the equation (see Section 8), then the answer will be *FAIL*:

```
> eq8:=(x^2+O(x^3))*diff(y(x),x,x)+O(x)*diff(y(x),x)+(1+O(x))*y(x);
```

$$eq8 := (x^2 + O(x^3))\left(\frac{d^2}{dx^2}y(x)\right) + O(x)\left(\frac{d}{dx}y(x)\right) + (1 + O(x))y(x)$$

```
> LaurentSolution(eq8,y(x));
```

$$FAIL$$

In this case, as a result of the transition to the operation θ in the equation, the constant term of the coefficient multiplying the first degree of θ is unspecified.

8. Apply the procedure to operator (19) with $D = \dfrac{d}{dx}$:

```
> eq9:=(-x+x^2+x^3+O(x^4))*(diff(y(x),x,x))+(-3+x+O(x^2))*
      (diff(y(x),x))+O(x^3)*y(x);
```

$$eq9 := (-x + x^2 + x^3 + O(x^4))\left(\frac{d^2}{dx^2}y(x)\right) + (-3 + x + O(x^2))\left(\frac{d}{dx}y(x)\right) + O(x^3)y(x)$$

```
> LaurentSolution(eq9,y(x));
```

$$[\_c_1 + O(x^4)]$$

The answer obtained also means that $W = \{0\}$ and $m_1 = 3$.

ACKNOWLEDGMENTS

We are grateful to the Maplesoft (Waterloo, Canada) for consultations and discussions.

FUNDING

This work was supported by the Russian Foundation for Basic Research, project no. 19-01-00032.

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Translated by E. Chernokozhin