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Regular Solutions of Linear Ordinary Differential Equations and Truncated Series

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Abstract—Linear ordinary differential equations with coefficients in the form of truncated formal power series are considered. Earlier, it was discussed what can be found from an equation specified in this way about its solutions belonging to the field of formal Laurent series. Now a similar question is discussed for regular solutions. We are still interested in information about these solutions that is invariant under possible prolongations of truncated series representing the coefficients of the equation. The possibility of including in the solutions symbolic unspecified coefficients of possible prolongations of the equation is also considered.

Keywords: computer algebra, differential equations, power series, truncated series, regular solutions

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1. INTRODUCTION

In this article, as in [1], the linear differential equation is defined in an "approximate" form: its coefficients are truncated series; i.e., each coefficient is an expression

$$p(x) + O(x^{t+1}), \tag{1}$$

where $p(x)$ is a polynomial and $t \geq \deg p(x)$. We are interested in information about regular (representable by power-logarithmic expansions; the definition is given in Section 3) solutions that are invariant to all possible prolongations of the truncated series representing the coefficients of the equation. The algorithm proposed allows one to build the longest possible segments of the series arising in the solutions: the terms entering into these segments do not depend on possible prolongations ("tails") of the truncated coefficients of the equation, i.e., on the unspecified terms that are hidden in expressions (1) under the symbol O. In a certain sense, the algorithm can also clarify the influence of these terms on the subsequent (not invariant to all possible prolongations) terms of the series entering into regular solutions. We mean formulas expressing these terms via unspecified coefficients. The unspecified coefficients are denoted by symbols. These unspecified coefficients will be called *literals*.

As to previous studies, it should be said that, in Bruno's works (see, e.g., [2]), a method for constructing regular solutions is proposed, which, for all series entering into this type of solution, enables one to find, in particular, any specified number of terms. The equations, generally speaking, are nonlinear. They may have a very general form and are specified using explicit analytical functions of several variables. In this case, as is well known, linear (completely specified) equations are solved using classical approaches: the Frobenius [3; 4, Ch. 4, § 8] and Heffter [5] methods, as well as their modern versions [6–8]; for most of them, there is a computer implementation.

It only remains to repeat that, in contrast to previous works, in this article, linear differential equations are specified not completely but in "approximate", i.e., truncated, form. In this case, for the Laurent series entering into the representation of regular solutions, the proposed algorithm finds the maximum possible number of terms invariant with respect to the unknown terms of the series representing the coefficient of the equation.

The implementation of the proposed algorithm in the Maple environment [9] is described in Section 6.

It should be emphasized that the term “regular solutions” for the solutions that are considered in this article and whose exact definition is given in Section 3 has entrenched in computer algebra [9–11]. Originally, it appeared because, in the analytical theory of linear differential equations, it is customary to subdivide (see [12, Ch. 3, § 10]) the singular points of the equations into regular (or weakly singular) and irregular (or strongly singular). In the neighborhood of a regular singular point, there exists a basis for the space of solutions of the equation, consisting of regular (in the above sense) solutions, and this was the reason for introducing the concept of a regular solution. It should be noted that, in this article, we consider solutions as formal expressions. The series that enter into these expressions, in turn, are formal Laurent series in x. In this sense, we consider solutions at the point 0 and look for the maximum possible number of linearly independent regular solutions. This number does not exceed the order of the equation and can be smaller. We do not deal with the convergence of the series. The role of constants is played by abstract quantities: elements of an algebraically closed field of characteristic 0, which will be discussed in the next section.

2. PRELIMINARY INFORMATION

2.1. Basic Concepts

First, let us recall some concepts and standard notation. Let K be some field. The following notation is standard:

$K[x]$ is the ring of polynomials with coefficients from K;

$K[[x]]$ is the ring of formal power series with coefficients from K;

$K((x))$ is the quotient field of the ring $K[[x]]$.

In $K[x]$, $K[[x]]$, and $K((x))$, the differentiation $D = \frac{d}{dx}$ is defined. We will consider operators and differential equations written with the designation $\theta = x\frac{d}{dx}$.

Definition 1. The elements of the field $K((x))$ are formal *Laurent series*. For a nonzero element $a(x) = \sum a_i x^i \in K((x))$, its *valuation* $\operatorname{val} a(x)$ is defined as $\min\{i \mid a_i \neq 0\}$; in this case, $\operatorname{val} 0 = \infty$. Let $t \in \mathbb{Z} \cup \{-\infty\}$; the t-*truncation* $a^{\langle t \rangle}(x)$ is obtained by discarding all terms in $a(x)$ of degree higher than t; if $t = -\infty$, then $a^{\langle t \rangle}(x) = 0$. The number t is called the *degree of truncation*.

Henceforward, the field K, by default, will be assumed to be algebraically closed and having a characteristic 0.

In the original operator

$$L = \sum_{i=0}^{r} a_i(x)\theta^i \in K[x][\theta], \tag{2}$$

the polynomial coefficient $a_i(x)$ will be assumed to have the form

$$a_i(x) = \sum_{j=0}^{t_i} a_{ij}x^j, \tag{3}$$

where t_i is a nonnegative integer greater than or equal to $\deg a_i(x)$, $i = 0, 1, \ldots, r$ (if $t_i > d_i = \deg a_i(x)$, then $a_{ij} = 0$ for $j = d_i + 1, d_i + 2, \ldots, t_i$). It is assumed that the constant term of at least one of the polynomials $a_0(x), \ldots, a_r(x)$ is nonzero.

Definition 2. A polynomial $a_r(x)$ (the *leading coefficient* of the differential operator L from (2)) is assumed to be nonzero. A *prolongation* of the operator L is any operator

$$\tilde{L} = \sum_{i=0}^{r} \tilde{a}_i(x)\theta^i \in K[[x]][\theta],$$

for which $\tilde{a}_i(x) - a_i(x) = O(x^{t_i+1})$, i.e., $\operatorname{val}(\tilde{a}_i(x) - a_i(x)) > t_i$, $i = 0, 1, \ldots, r$.

In what follows, to a *truncated differential equation*

$$(a_r(x) + O(x^{t_r+1}))\theta^r y(x) + \ldots + (a_1(x) + O(x^{t_1+1}))\theta y(x) + (a_0(x) + O(x^{t_0+1}))y(x) = 0, \tag{4}$$

$t_i \geq \deg a_i(x)$, $i = 0, 1, \dots, r$, we assign operator (2) and the set of numbers $t_0, t_1, \dots, t_r$. In this case, the prolongation of operator (2) will also be called a *prolongation of equation* (4).

Below, to denote an operator with the coefficient in the form of series

$$\sum_{i=0}^{r} a_i(x)\theta^i \in K[[x]][\theta], \tag{5}$$

where

$$a_i(x) = \sum_{j=0}^{\infty} a_{i,j}x^j \in K[[x]],$$

we use the letter $\mathcal{L}$. For $\mathcal{L}$, we also assume that there is a number i such that $a_{i,0} \neq 0$. To denote an operator with polynomial coefficients (e.g., for an operator with truncated coefficients) we use the letter L.

If L (or $\mathcal{L}$) is some differential operator, then *solutions of the operator* L (or $\mathcal{L}$) will be understood as the solutions of the equation $L(y) = 0$ (correspondingly, $\mathcal{L}(y) = 0$).

If L is a truncated variant of the operator $\mathcal{L}$, we will call L and $L(y) = 0$ *truncations* of the operator $\mathcal{L}$ and, correspondingly, the equation $\mathcal{L}(y) = 0$.

2.2. Laurent Solutions

A solution of the equation in the form of a Laurent series will be called a *Laurent solution*.

First of all, for a truncated equation $L(y) = 0$, the algorithm from [1] finds a finite set of candidates for all possible valuations of Laurent solutions. This set contains all valuations of Laurent solutions for all prolongations of this equation. For each element of this set, it is then checked whether, for any prolongation of the equation, there is a Laurent solution having such a valuation. If the answer is 'no', this valuation is no more considered. If the answer is 'yes', it is possible to calculate an integer m such that the terms of all Laurent solutions having this valuation coincide (up to a common nonzero constant factor, since the equations are homogeneous) up to terms of the order x^m. In this case, the maximum possible m is chosen. As a result, in addition to the set of valuations $\{v_1, v_2, \dots\}$, we obtain the set $\{m_1, m_2, \dots\}$ of the corresponding values of m.

The aforementioned actions: discarding excess valuations, finding the values of m, etc., are performed using the *induced recurrence equation* that is assigned to the differential equation.

2.3. Induced Recurrence Equation

Let σ denote the shift operator $\sigma c_n = c_{n+1}$ for any sequence (c_n). The transformation

$$x \to \sigma^{-1}, \quad \theta \to n \tag{6}$$

puts into correspondence to the differential equation

$$\sum_{i=0}^{r} a_i(x)\theta^i y(x) = 0, \tag{7}$$

where $a_i(x) \in K[[x]]$, to the induced recurrence equation (relation)

$$u_0(n)c_n + u_{-1}(n)c_{n-1} + \dots = 0. \tag{8}$$

Let $\mathcal{L} \in K[[x]][\theta]$, $g(x) \in K((x))$ then, $\mathcal{L}(g(x)) = b(x) \in K((x))$. In this case, applying the induced recurrence operator $u_0(n) + u_{-1}(n)\sigma^{-1} + \dots$ to the sequence (g_n) of coefficients of the series $g(x)$ gives the sequence (b_n) of the coefficients $b(x)$ of the series: formulas (6) clearly indicate how the sequence of coefficients is transformed when the series is multiplied by x and when the operation θ is applied to it. All this makes induced equations a useful tool when considering inhomogeneous equations of the form $\mathcal{L}(y) = b(x)$ with Laurent right-hand sides.

Let the equation

$$\sum_{i=0}^{r} a_i(x)\theta^i y(x) = b(x) \tag{9}$$

have the right-hand side in the form of a Laurent series:

$$b(x) = \sum_{n=\nu}^{\infty} b_n x^n.$$

Then, the right-hand side of the induced recurrence equation will be equal to b_n ($b_n = 0$ for $n < \nu$):

$$u_0(n)c_n + u_{-1}(n)c_{n-1} + \cdots = b_n. \tag{10}$$

Homogeneous equation (7) (and inhomogeneous equation (9)) has a Laurent solution $y(x) = c_v x^v + c_{v+1}x^{v+1} + \ldots$ if and only if the two-sided sequence ..., 0, 0, c_v, c_{v+1}, ... satisfies Eq. (8) (correspondingly, Eq. (10)) (see the proof in [13]).

It should be recalled that, by our assumption, the constant term of at least one of the polynomials $a_0(x),\ldots,a_r(x)$ is not zero. Hence,

$$u_0(n) = \sum_{i=0}^{r} a_{i,0} n^i \tag{11}$$

is a nonzero polynomial. It can be considered a variant of the *indicial polynomial* of the original equation. The finite set $n_1,\ldots,n_\iota$ of integer roots of this polynomial contains all possible valuations v of the Laurent solutions of Eq. (7). The valuations of Laurent solutions (9) are determined by both $n_1,\ldots,n_\iota$ and the valuation ν of the right-hand side $b(x)$.

The calculation of c_v, $c_{v+1},\ldots$ is performed by successively increasing n, starting with $n = v$, the minimum integer root of the polynomial $u_0(n)$ (starting from $v = \min\{\nu, n_1,\ldots,n_\iota\}$ for (9)). If, for some integer n, we have $u_0(n) \neq 0$, then (8) and (10) allow us to find c_n from $c_{n-1}, c_{n-2},\ldots$ (since $c_{v-1}, c_{v-2},\ldots$ are equal to zero, for each integer n, the induced recurrence equation has a finite number of nonzero terms on the left-hand side). If $u_0(n) = 0$, then we declare c_n an *unknown constant*. Then, the previously calculated values $c_{n-1}, c_{n-2},\ldots,c_v$, in the homogeneous case, must satisfy the relationship

$$u_{-1}(n)c_{n-1} + u_{-2}(n)c_{n-2} + \ldots + u_{-n+v}(n)c_v = 0 \tag{12}$$

and, in the inhomogeneous case,

$$u_{-1}(n)c_{n-1} + u_{-2}(n)c_{n-2} + \cdots + u_{-n+v}(n)c_v = b_n. \tag{13}$$

Perhaps, such relationships will make it possible to calculate the value of some previously unknown constants. If relationship (13) for some integer n turns into a false identity, then (9) does not have Laurent solutions. After n exceeds the largest integer root $u_0(n)$, new unknown constants and relationships of the form (12) and (13) will not arise. The unknown constants that have not obtained values during the calculations are declared *arbitrary constants* entering into the Laurent solution of the differential equation.

If a truncated equation $L(y) = 0$ is specified under the condition that the constant term of at least one of the polynomials $a_0(x),\ldots,a_r(x)$ is nonzero, then, obviously, (11) does not depend on the prolongation.

3. REGULAR SOLUTIONS

3.1. Power Factors

Definition 3. A solution of the equation $\mathscr{L}(y) = 0$, having the form

$$x^\lambda \sum_{s=0}^{k} g_{k-s}(x)\frac{\ln^s x}{s!}, \tag{14}$$

where $\lambda \in K$, $k \in \mathbb{Z}_{\geq 0}$, and $g_s(x) \in K((x))$, $s = 0,1,\ldots,k$, will be called a *regular solution*. We say that x^λ is a *power factor* of solution (14). The set

$$x^{\lambda_1}, x^{\lambda_2},\ldots,x^{\lambda_\rho} \tag{15}$$

is called a *complete* set of power factors of regular solutions of the equation $\mathcal{L}(y) = 0$ if

—among $\lambda_1, \dots, \lambda_\rho$, none differ by an integer;

—each element of set (15) is a power factor for some nonzero regular solution of the equation $\mathcal{L}(y) = 0$;

—for each nonzero regular solution of the equation $\mathcal{L}(y) = 0$, among (15), there is a power factor for this solution.

Let $\mathcal{L}$ have form (5). It is known (see [3, 4]) that, if $\lambda_1, \dots, \lambda_\rho$ is the set of all roots of indicial polynomial (11) such that $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $i \neq j$, then (15) is a complete set of power factors of regular solutions of $\mathcal{L}(y) = 0$. Moreover, for each power factor x^λ, the value of k in (14) such that $g_0(x) \neq 0$ is smaller (with allowance for multiplicity) than the number of roots of the indicial polynomial that differ from λ by an integer number.

Remark 1. For the equation $\mathcal{L}(y) = 0$, there exist as many linearly independent solutions of the form (14) as the number of roots λ (with allowance for the multiplicity) of the indicial polynomial. These solutions form a basis of the linear space of regular solutions; i.e., any linear combination of solutions of the form (14) is called a regular solution, but, up to Section 6, regular solutions will be understood as expressions of the form (14).

Example 1. Regular solutions of the equation

$$\left(-1 + x + \sum_{j=3}^{\infty} x^j\right)\theta^2 y(x) + \left(-1 - x - \frac{3}{2}x^2 + \sum_{j=3}^{\infty} \frac{(-1)^j}{2} x^j\right)\theta y(x)$$
$$+ \left(\frac{3}{4} + \frac{1}{4}x + \frac{3}{4}x^2 + \sum_{j=3}^{\infty} j x^j\right) y(x) = 0$$

have the form $y(x) = \sqrt{x}\left(g_0(x)\ln x + g_1(x)\right)$; in this case,

$$g_0(x) = C_1 + \frac{C_1}{5}x^3 + \sum_{n=4}^{\infty} g_{0,n}x^n,$$

$$g_1(x) = -\frac{2C_1}{x^2} + \frac{8C_1}{x} + C_2 + \sum_{n=2}^{\infty} g_{1,n}x^n,$$

where C_1, C_2 are arbitrary constants.

Remark 2. In [15–17], an algorithmic representation of infinite series was considered: a series $\sum a_n x^n$ was specified by an algorithm determining a_n from n. It was found that, in the case of a differential equation with coefficient in the form of series defined algorithmically, the problem of finding regular solutions, i.e., the problem of constructing algorithms for representing $g_0(x)$, $g_1(x), \dots, g_k(x)$, is solvable (see also [14], where not only individual scalar equations but also systems were discussed). Since the coefficients of the equation from Example 1 are specified algorithmically, then $g_{0,n}$ and $g_{1,n}$ can be calculated for any n.

Based on the truncated equation $L(y) = 0$, we cannot expect finding its regular solutions in the complete form (14). The algorithm proposed in this article makes it possible to obtain expressions for solutions in the form of truncated Laurent series $g_0(x), g_1(x), \dots, g_k(x)$. These truncated series are constructed based on the same principle of selection of valuations and degrees of truncation as in the construction of Laurent solutions (Section 2).

3.2. General Scheme for Finding Regular Solutions (*Heffter's Approach*)

Let $\mathcal{L}$ have the form (5). For $k = 0, 1, \dots, r$, we can construct the operators

$$\mathcal{L}_k = \sum_{i=k}^{r} a_i(x)\binom{i}{k}\theta^{i-k}, \tag{16}$$

where $\binom{i}{k}$ is a binomial coefficient and $\mathcal{L}_0 = \mathcal{L}$. Based on Heffter's approach [5], the general scheme for finding regular solutions with $\lambda = 0$ consists in considering for $k = 0,1,\ldots$ systems of the form

$$\begin{gathered} \mathcal{L}_0(g_0) = 0, \\ \mathcal{L}_0(g_1) = -\mathcal{L}_1(g_0), \\ \mathcal{L}_0(g_2) = -(\mathcal{L}_1(g_1) + \mathcal{L}_2(g_0)), \\ \ldots \\ \mathcal{L}_0(g_k) = -(\mathcal{L}_1(g_{k-1}) + \ldots + \mathcal{L}_k(g_0)) \end{gathered} \tag{17}$$

(for $k = 0$, the system consists of one equation $\mathcal{L}(g_0) = 0$). A Laurent solution of system (17) will be understood as any solution $(g_0(x), g_1(x), \ldots, g_k(x))$ with the components belonging to $K((x))$.

Proposition 1 (see [5]). *The set of nonnegative integers k for which system* (17) *has a Laurent solution* $(g_0(x), g_1(x), \ldots, g_k(x))$, $g_0(x) \neq 0$, *is finite* (*such a solution can exist for some* $k > 0$ *only if the solution exists also for* $k - 1$). *If this set is empty, then* $\mathcal{L}(y) = 0$ *does not have nonzero solutions in* $K((x))[\ln x]$. *If this set is not empty and* $\tilde{k}$ *is its maximum element, then any solution of the equation* $\mathcal{L}(y) = 0$, *belonging to* $K((x))[\ln x]$, *has the form*

$$\sum_{s=0}^{\tilde{k}} g_{\tilde{k}-s}(x) \frac{\ln^s x}{s!}, \tag{18}$$

where

$$\left(g_0(x), g_1(x), \ldots, g_{\tilde{k}}(x)\right), \quad g_0(x) \neq 0, \tag{19}$$

is a Laurent solution of system (17) *for* $k = \tilde{k}$. *At the same time, any Laurent solution of the form* (19) *to system* (17), *for* $k = \tilde{k}$, *generates a solution* (18) *of the equation* $\mathcal{L}(y) = 0$.

If λ is known, then the substitution

$$y(x) = x^{\lambda} w(x), \tag{20}$$

reduces the search for regular solutions to the search for solutions belonging to $K((x))[\ln x]$. For λ, we take the roots of the indicial polynomial.

We obtain the following scheme (considered in detail in [14]):

1. For the equation $\mathcal{L}(y) = 0$ with operator (5), find indicial polynomial (11). Considering two roots $\lambda, \lambda' \in K$ of this polynomial equivalent if $\lambda - \lambda' \in \mathbb{Z}$, construct a set Λ containing one representative from each equivalence class.

2. For each $\lambda \in \Lambda$, find regular solutions having a power factor x^{λ}:

(a) Construct the equation $\mathcal{L}_{\lambda}(y) = 0$ by substituting (20) into $\mathcal{L}(y) = 0$ and then multiplying by $x^{-\lambda}$.

(b) Construct Laurent solutions of systems (17) for $k = 0,1,\ldots$, where $\mathcal{L}_k = \mathcal{L}_{\lambda,k}$ are obtained by formula (16), up to $k = \tilde{k} + 1$, when (17) has no longer Laurent solutions such that $g_0(x) \neq 0$. This gives regular $\tilde{k} + 1$ solutions $y(x)$ in the form (18) to the equation $\mathcal{L}_{\lambda}(y) = 0$.

(c) Multiply the resulting regular solutions by x^{λ}.

Remark 3. At step 2a, instead of the substitution into the differential equation, an equivalent operation on the induced recurrence equation can be performed. If (8) is the induced recurrence equation for the original equation $\mathcal{L}(y) = 0$, then the induced recurrence equation for $\mathcal{L}_{\lambda}(y) = 0$ has the form

$$u_0(n + \lambda)c_n + u_{-1}(n + \lambda)c_{n-1} + \cdots = 0.$$

For more detail, see, e.g., [14].

3.3. Working with Inhomogeneous Equations

The equation

$$L_0(g_k) = -\sum_{j=1}^{k} L_j(g_{k-j}), \quad k = 0, 1, \ldots, r,$$

will be denoted by S_k (the equation $L_0(g_0) = 0$ is denoted by S_0).

Suppose that we have constructed a Laurent solution $g_0(x)$ of the equation S_0, i.e., of the equation $L_0(g_0) = 0$ (see Subsection 2.3). This solution contains several unknown constants. We use $g_0(x)$ to obtain the right-hand side of the equation S_1, i.e., the equation $L_0(g_1) = -L_1(g_0)$, and the aforementioned unknown constants enter into this right-hand side linearly. As soon as, when constructing $g_1(x)$, relationship (13) arises (its right-hand side b_n linearly depends on the constants entering into $g_0(x)$), this relationship is used, if possible, to calculate the value of some unknown constant. If it turns out that $g_0(x) = 0$, this means that $\tilde{k} = 0$ and the construction of regular solutions is completed.

We continue constructing $g_k(x)$ for the next equation S_k by calculating the unknown constants entering into $g_0(x)$, $g_1(x)$, ..., $g_{k-1}(x)$ until $g_0(x) \neq 0$. According to Proposition 1, this process terminates. The unknown constants among $g_0(x)$, $g_1(x)$, ..., $g_{\tilde{k}}(x)$ that have not obtained values, are declared arbitrary.

Remark 4. For a Laurent solution, its m-truncation is constructed using the induced recurrence equation for n not exceeding m. When constructing an m-truncation of a Laurent solution $g_k(x)$ to the equation S_k, it is necessary to know the elements of the sequence (b_n), the right-hand side of induced recurrence equation (10) up to $n = m$. It is easy to show that, for this, it suffices to construct m-truncations of $g_0(x)$, $g_1(x), \ldots, g_{k-1}(x)$.

4. HEFFTER'S SCHEME FOR TRUNCATED COEFFICIENTS

4.1. Constructing Laurent Solutions of the Truncated Equation

The algorithm from [1] constructs for Eq. (4) a finite set of m_i-truncations of Laurent solutions. Any element $c_{v_i}x^{v_i} + c_{v_i+1}x^{v_i+1} + \ldots + c_{m_i}x^{m_i} + O(x^{m_i+1})$ of this set does not contain literals, i.e., unspecified coefficients of Eq. (4). Each coefficient c_n was calculated, as described in Subsection 2.3, successively in n starting from n_j, where n_j is an integer root of the indicial polynomial, to a value $n = m_i$ such that c_{m_i} does not depend on literals but c_{m_i+1} does depend. When searching for regular solutions, it is preferable to construct the Laurent solutions of the homogeneous equation S_0 (and the inhomogeneous equation S_k, $k > 0$) in the form of one expression:

$$c_v x^v + c_{v+1}x^{v+1} + \ldots + c_m x^m + O(x^{m+1}), \tag{21}$$

where $v = \min v_i$ and $m = \max m_i$. Here, the coefficients c_n may contain literals.

The representation in the form of one truncation with the use of literals makes it possible, if necessary, to proceed to the representation of the solution in the form of a set of truncations such as in [1]. In order to obtain this representation, it is necessary for each valuation v_i to calculate the values of arbitrary constants for which c_v, c_{v+1}, ..., c_{v_i-1} are equal to zero and, after substituting these values into (21), to discard all terms whose coefficients contain literals. This will give a m_i-truncation for the valuation v_i, invariant with respect to the prolongation of the original equation.

Expression (21) is constructed, as described in Subsection 2.3, successively in n starting from $v = \min n_j$ to $w = \max n_j$, where n_j are all integer roots of the indicial polynomial (for $k > 0$, it is easy to show that $\operatorname{val} b(x) \geq v$, where $b(x)$ is the right-hand side of the inhomogeneous equation S_k). In the course of calculations, at $n = n_j$ such that $u_0(n_j) = 0$, it is necessary to consider relationship (12) (correspondingly, (13) for $k > 0$). If this relationship with $n = n_j$ is not an identity, then, in contrast to the case of a completely specified equation, if (12) and (13) depends on literals, we do not use it to calculate the values of the unknown constants. In the end, we obtain c_v, c_{v+1}, ..., c_w, the set of unknown constants, and a set of relationships for the unknown constants containing literals. Based on this set of relationships, we find the values of the unknown constants that are invariant with respect to all prolongations of the specified truncated equation (e.g., as described in Remark 5).

Next, we calculate the values of c_n as long as there exists a nontrivial set of values of the remaining unknown constants (i.e., not all constants are equal to zero; we can restrict the consideration to sets in which one element is equal to unity and the rest to zero) for which c_n does not depend on literals.

Remark 5. Let us consider the set of relationships (12) and (13) that arise when constructing Laurent solutions of homogeneous and inhomogeneous equations in Heffter's scheme for a truncated equation. The left- and right-hand sides of this relationship are polynomials in literals whose coefficients are linear combinations over K of the unknown constants introduced when solving the equations $S_0, S_1, \ldots, S_k$. We equate the coefficients multiplying the same monomials on the right- and left-hand sides. We obtain a linear homogeneous system with respect to the unknown constants. When solving this system, a part of the unknown constants will obtain values and the rest will remain unknown.

Example 2. Let us trace the work of the proposed algorithm on the example of the operator

$$L = (-1 + x + x^2)\theta^2 - 2\theta + (x + 6x^2), \quad t_0 = 3, \quad t_1 = t_2 = 2. \tag{22}$$

This operator was considered in [1, Example 2]. Here, the defining polynomial is $u_0(n) = -n^2 - 2n$, the set $\{-2, 0\}$ of its integer roots contain all possible valuation of the Laurent solutions to the equation $L(y) = 0$. Calculations using the induced recurrence equation begin with the minimal integer root of the indicial polynomial, i.e., with $n = -2$. In the end of the calculations, we obtain a 4-truncation of the Laurent solution, written using literals:

$$\begin{gathered}
\frac{C_1}{x^2} - \frac{5C_1}{x} + C_2 + x\left(\frac{4}{3}C_1U_{2,3} + \frac{1}{3}C_2 - \frac{35}{3}C_1 - \frac{2}{3}C_1U_{1,3}\right) + x^2\left(\frac{11}{24}C_1U_{1,3} - \frac{7}{24}C_1U_{2,3} + \frac{5}{6}C_2\right. \\
\left. - \frac{35}{12}C_1 + \frac{1}{8}C_1U_{0,4} - \frac{1}{4}C_1U_{1,4} + \frac{1}{2}C_1U_{2,4}\right) + x^3\left(-\frac{77}{12}C_1 - \frac{19}{120}C_1U_{1,3} + \frac{21}{40}C_1U_{2,3} + \frac{13}{30}C_2\right. \\
\left. + \frac{1}{15}C_1U_{0,5} - \frac{7}{24}C_1U_{0,4} + \frac{1}{4}C_1U_{1,4}\frac{1}{6}C_1U_{2,4} - \frac{2}{15}C_1U_{1,5} + \frac{4}{15}C_1U_{2,5}\right) \\
+ x^4\left(\frac{1}{18}C_1U_{2,3}^2 - \frac{35}{9}C_1 - \frac{13}{36}C_1U_{1,3} - \frac{7}{18}C_1U_{2,3} + \frac{19}{36}C_2 - \frac{13}{72}C_1U_{0,5} - \frac{5}{72}C_1U_{0,4}\right. \\
+ \frac{5}{36}C_1U_{2,4} + \frac{11}{72}C_1U_{1,5} - \frac{7}{72}C_1U_{2,5} + \frac{1}{36}U_{1,3}C_1U_{2,3} + \frac{1}{72}C_2U_{1,3} - \frac{1}{36}C_1U_{1,3}^2 - \frac{1}{12}C_1U_{1,6} \\
\left. + \frac{1}{72}C_2U_{2,3} + \frac{1}{24}C_2U_{0,4} + \frac{1}{24}C_1U_{0,6} + \frac{1}{6}C_1U_{2,6}\right) + O(x^5).
\end{gathered} \tag{23}$$

Hereinafter, the literal denoted by $U_{i,j}$ corresponds to an unspecified coefficient of x^j in the coefficient multiplying θ^i in the original equation; C_1 and C_2 are arbitrary constants. The calculations are carried out to a power of 4. The set of solutions of our interest can be described more briefly by discarding the term of degree 4 in (23) and replacing $O(x^5)$ by $O(x^4)$. We have written out this term in order that the reason for terminating the calculations be visible.

We pass from (23) to the representation in the form of a set of invariant truncations for each of the valuations. For the valuation $v_1 = -2$ (i.e., for $C_1 \neq 0$ and $C_2 \neq 0$), discarding the terms starting with the first whose coefficient contains literals gives $m_1 = 0$:

$$\frac{C_1}{x^2} - \frac{5C_1}{x} + C_2 + O(x).$$

For the valuation $v_2 = 0$ (i.e., for $C_1 = 0$ and $C_2 \neq 0$), such a discard gives $m_2 = 3$:

$$C_2 + \frac{1}{3}C_2x + \frac{5}{6}C_2x^2 + \frac{13}{30}C_2x^3 + O(x^4).$$

These invariant truncations were also obtained in [1, Example 2]. Representation (23) also makes it possible to obtain a prolongation of the truncation for each individual valuation. For the valuation $v_1 = -2$, it coincides with (23). For $v_2 = 0$, it is equal to

$$C_2 + \frac{1}{3}C_2x + \frac{5}{6}C_2x^2 + \frac{13}{30}C_2x^3 + x^4\left(\frac{19}{36}C_2 + \frac{1}{72}C_2U_{1,3} + \frac{1}{72}C_2U_{2,3} + \frac{1}{24}C_2U_{0,4}\right) + O(x^5).$$

4.2. The Algorithm

For the equation $L(y) = 0$ with coefficients (3), the indicial polynomial is constructed as the coefficient $u_0(n)$ of the induced recurrence equation. The set of its roots is found, and the set Λ is formed. This calculation corresponds to step 1 of Heffter's scheme (Subsection 3.2).

For each noninteger $\lambda \in \Lambda$, we perform step 2(a), i.e., we obtain the equation L_λ. At step 2(b), the solution of system (17) with a current k consists in finding the truncation of the Laurent solution of the equation S_k (in this case, possibly, the values of some unknown constants entering into $g_0(x)$, $g_1(x), \dots, g_{k-1}(x)$ are also calculated). This truncation contains literals; the degree of truncation is determined as described in Subsection 4.1. The truncation of the Laurent solution to S_k is constructed successively using induced recurrence equation (8) (or (10) for $k > 0$). For $k > 0$, in order to obtain the right-hand side of the induced recurrence equation, a successive calculation of the coefficients of the Laurent series on the right-hand side of the equation S_k is needed. The search for Laurent solutions terminates either if k is equal to the number (with allowance for the multiplicity) of integer roots of the indicial polynomial for S_0 or when it is found that the next system does not have a Laurent solution with $g_0(x) \neq 0$. Based on the obtained truncations of Laurent solutions with literals, a final set of invariant truncations of regular solutions to the original equation is formed.

Proposition 2. *Each of the segments of the series found by the proposed algorithm as a truncation of one or another series $g_i(x)$ in solution* (14) *of the original truncated equation $L(y) = 0$ has the maximum possible length: adding terms of a higher degree to any of these segments entails a loss of invariance with respect to possible prolongations of the equation $L(y) = 0$.*

Proof. Each of these segments is arranged so that the termination occurs at the moment when the next term w_sx^s of the series has a coefficient w_s that depends on some literals. Such a coefficient will be a polynomial over K in a finite number of literals. Since the characteristic of the field K is zero, the field K is infinite (it contains a subfield isomorphic to the field of rational numbers) and, for the above polynomial, one can find two different sets of values of the literals entering into it, such that the values of the polynomial w_s on these sets do not coincide. This means that the equation $L(y) = 0$ has two prolongations that lead to different w_s. Indeed, as the first continuation, we take that in which the literals entering into w_s are replaced by the values from the first set and the remaining terms of the prolongation are set to zero. Similarly, but using the values from the second set, we construct the second prolongation. Therefore, the segment of the series entering into the solution and containing the term w_sx^s is not invariant to all possible prolongations of the equation $L(y) = 0$.

5. EXAMPLES

Example 3. Let us trace the steps of the proposed algorithm on the example of the operator

$$L = (-1 + x + x^2)\theta^2 - 2\theta, \quad t_0 = 3, \quad t_1 = 1, \quad t_2 = 2. \tag{24}$$

The indicial polynomial is $u_0(n) = -n^2 - 2n$. Based on the set of its roots, $\{-2, 0\}$, we obtain $\Lambda = \{0\}$. The search for Laurent solutions to S_0, as described in Subsection 4.1, gives

$$g_0(x) = C_{0,1} + \frac{1}{24}x^4C_{0,1}U_{0,4} + O(x^5), \tag{25}$$

where $C_{0,1}$ is an unknown constant. Further on, for S_1, we obtain a 4-truncation of the right-hand side of the equation:

$$2C_{0,1} + \frac{5}{12}x^4C_{0,1}U_{0,4} + O(x^5).$$

Here, the induced recurrence equation (10) for $n = 0$ has the form

$$-2C_{1,1}U_{1,2} = 2C_{0,1},$$

where $C_{1,1}$ is the unknown constant arising at $n = -2$ and corresponding to the coefficient c_{-2} of the solution. This relationship has the form (13), which cannot be used to calculate the values of unknown constants $C_{1,1}$ and $C_{0,1}$, since this relationship includes the literal $U_{1,2}$. For the prolongations of operator (24) for which $U_{1,2} = 0$, we find that $C_{0,1} = 0$ and $C_{1,1}$ remains an unknown constant. In this case, $g_0(x) = 0$; i.e., $\tilde{k} = 0$. For prolongations with $U_{1,2} \neq 0$, calculations will be continued and $\tilde{k} = 1$ will be obtained. We set $C_{1,1} = 0$ and $C_{0,1} = 0$. Thus, we obtain $\tilde{k} = 0$ and truncated regular solution (25). We pass from it to the set of invariant truncations:

$$C + O(x^4),$$

where C is an arbitrary constant.

Example 4. Add to the coefficients of operator (24) one term (x^2 in the coefficient multiplying θ^1):

$$\tilde{L} = (-1 + x + x^2)\theta^2 + (-2 + x^2)\theta, \quad t_0 = 3, \quad t_1 = t_2 = 2. \tag{26}$$

This operator was considered in [1, Example 3], and it was determined that, for any prolongation of the coefficients, this operator, like operator (24), has a Laurent solution $C + O(x^4)$, where C is an arbitrary constant.

The indicial polynomial $u_0(n) = -n^2 - 2n$, as well as for operator (24), has the roots $\{-2, 0\}$, and, therefore, $\Lambda = \{0\}$.

For S_0, we obtain the same truncation of solution (25). Finding a Laurent solution for S_1 gives

$$\begin{aligned} g_1(x) = {} & -\frac{C_{0,1}}{x^2} + 4\frac{C_{0,1}}{x} + C_{1,1} + x\left(\frac{2}{3}C_{0,1}U_{1,3} - \frac{4}{3}C_{0,1}U_{2,3}\right) + x^2\left(-\frac{5}{12}C_{0,1}U_{1,3} + \frac{1}{3}C_{0,1}U_{2,3} - \frac{1}{8}C_{0,1}U_{0,4}\right. \\ & \left. + \frac{1}{4}C_{0,1}U_{1,4} - \frac{1}{2}C_{0,1}U_{2,4} + \frac{1}{8}C_{0,1}\right) + x^3\left(\frac{2}{45}C_{0,1}U_{1,3} - \frac{4}{45}C_{0,1}U_{2,3} + \frac{7}{30}C_{0,1}U_{0,4} - \frac{1}{5}C_{0,1}U_{1,4} + \frac{2}{15}C_{0,1}U_{2,4}\right. \\ & \left. + \frac{1}{30}C_{0,1} - \frac{1}{15}C_{0,1}U_{0,5} + \frac{2}{15}C_{0,1}U_{1,5} - \frac{4}{15}C_{0,1}U_{2,5}\right) + x^4\left(-\frac{7}{80}C_{0,1}U_{1,3} + \frac{1}{20}C_{0,1}U_{2,3} + \frac{7}{180}C_{0,1}U_{0,4}\right. \\ & + \frac{7}{240}C_{0,1}U_{1,4} - \frac{3}{40}C_{0,1}U_{2,4} + \frac{7}{160}C_{0,1} + \frac{17}{120}C_{0,1}U_{0,5} - \frac{7}{60}C_{0,1}U_{1,5} + \frac{1}{15}C_{0,1}U_{2,5} + \frac{1}{36}C_{0,1}U_{1,3}^2 \\ & \left. - \frac{1}{36}C_{0,1}U_{1,3}U_{2,3} - \frac{1}{18}C_{0,1}U_{2,3}^2 + \frac{1}{24}C_{1,1}U_{0,4} - \frac{1}{24}C_{0,1}U_{0,6} + \frac{1}{12}C_{0,1}U_{1,6} - \frac{1}{6}C_{0,1}U_{2,6}\right) + O(x^5). \end{aligned} \tag{27}$$

The truncation $g_1(x)$, like the truncation $g_0(x)$, was calculated up to the power x^4, since, both for $C_{0,1} = 0$ and $C_{1,1} \neq 0$, only the coefficient multiplying x^4 contains literals. For the calculation of $g_1(x)$, the 4-truncation of the right-hand side of the equation S_1 was constructed:

$$2C_{0,1} - x^2C_{0,1} - x^3C_{0,1}U_{1,3} + x^4\left(\frac{5}{12}C_{0,1}U_{0,4} - C_{0,1}U_{1,4}\right) + O(x^5).$$

The further search for the Laurent solution of the equation for S_2 is not performed. We obtain $\tilde{k} = 1$ and a regular solution $g_1(x) + g_0(x)\ln x$, where $g_0(x)$ and $g_1(x)$ are defined by (25) and (27).

Let us pass to the solution in the form of a set of invariant truncated regular solutions. For the valuation $v_1 = -2$ in $g_1(x)$, we obtain an expression containing two truncated series:

$$-\frac{C_1}{x^2} + 4\frac{C_1}{x} + C_2 + O(x) + (C_1 + O(x^4))\ln x,$$

where C_1 and C_2 are arbitrary constants. Setting $C_{0,1} = 0$ and $C_{1,1} \neq 0$ in (25) and (27), we obtain the truncation

$$C_2 + O(x^4).$$

This truncation corresponds to the truncation of the Laurent solution of the equation S_1 with the valuation $v_2 = 0$.

Remark 6. The solutions $g_k(x), \dots, g_0(x)$ of the equations $S_k, \dots, S_0$ may contain the same arbitrary constants; therefore, the transition from the representation using literals to invariant truncations is performed not for each Laurent solution $g_k(x), \dots, g_0(x)$ separately, but for the entire regular solution (18) with the factor x^λ including these Laurent solutions. In this example, the Laurent solution $g_0(x) = C_{0,1} + O(x^4)$ is discarded when constructing the invariant truncation $g_1(x)$ with $C_{0,1} = 0$.

6. COMPUTER IMPLEMENTATION AND EXAMPLES OF USE

The algorithm is implemented in the Maple environment [9] in the form of the `RegularSolution` procedure. The main arguments of the procedure are the same as the argument of the `LaurentSolution` procedure presented in [1], which implements the algorithm of searching for Laurent solutions. The first argument to the procedure is a differential equation of the form (4). The application of θ^k to an unknown function $y(x)$ is written as `theta(y(x),x,k)`. As in the case of `LaurentSolution`, it is also possible to use the ordinary differentiation (the operator $D = \frac{d}{dx}$); in this case, the application of the operator D^k to an unknown function $y(x)$ is specified in the standard Maple form `diff(y(x),x$k)`. The truncated coefficients of the equation have the form $a_i(x) + O(x^{t_i+1})$, where $a_i(x)$ is a polynomial of degree not higher than t_i over the field of algebraic numbers. Irrational algebraic numbers in Maple are represented in the form `RootOf(p(_Z),index=k)`, where `p(_Z)` is an irreducible polynomial, the kth root of which is a given algebraic number. For example, $\mathrm{RootOf}(\_Z^2 - 2, index = 2) = -\sqrt{2}$.

The second argument of the procedure is the unknown function.

The result of the procedure is a list of truncated regular solutions that are invariant to prolongations of the coefficients of a given equation. In truncations, arbitrary constants of the form $\_c_j$ may occur.

Additionally, the optional parameter `` `output`=`literal` `` can be specified to get the answer not in the form of a list of invariant truncations but as a single truncation with literals. It is also possible to specify the optional parameter `` `degree`=n ``, where `n` is an integer for obtaining a truncation of a given degree (in this case, coefficients expressed via literals will be added to the truncations; the degree of the truncation can be greater than the specified n; at least as many coefficients as required to determine all the possible valuations of Laurent solutions that arise in the calculations will be calculated). It should be noted that the `LaurentSolution` procedure is also supplemented with similar options.

Below we present six examples, which we combine into one, containing in paragraphs 1–6.

Example 5.

1. The equation specified by operator (24):

```
> eq1:=(-1+x+x^2+O(x^3))*theta(y(x),x,2)+(-2+O(x^2))*theta(y(x),x,1)+
        (O(x^4))*y(x);
```

$$eq1 := (-1 + x + x^2 + O(x^3))\theta(y(x), x, 2) + (-2 + O(x^2))\theta(y(x), x, 1) + O(x^4)y(x)$$

```
> RegularSolution(eq1, y(x));
```

$$[\_c_1 + O(x^4)]$$

The answer is the same as in Example 3.

2. An equation specified by operator (26):

```
> eq2:=(-1+x+x^2+O(x^3))*theta(y(x),x,2)+(-2+x^2+O(x^3))*theta(y(x),x,1)+
        O(x^4)*y(x);
```

$$eq2 := (-1 + x + x^2 + O(x^3))\theta(y(x), x, 2) + (-2 + x^2 + O(x^3))\theta(y(x), x, 1) + O(x^4)y(x)$$

```
> RegularSolution(eq2, y(x));
```

$$\left[-\frac{\_c_1}{x^2} + \frac{4\_c_1}{x} + \_c_2 + O(x) + \ln(x)(\_c_1 + O(x^4)), \_c_2 + O(x^4)\right]$$

The answer is the same as in Example 4.

We apply the procedure again to the same equation with the option of representing the result in literals; the result is a regular solution containing truncations of the form constructed as described in Subsection 4.1, i.e., to such a degree that the reason for terminating the calculations is visible:

```
> RegularSolution(eq2,y(x),'output'='literal');
```

$$\Big[\ln(x)\Big(\_c_1 + \frac{1}{24}x^4\_c_1U_{[0,4]} + O(x^5)\Big) - \frac{\_c_1}{x^2} + \frac{4\_c_1}{x} + \_c_2 + x\Big(\frac{2}{3}\_c_1U_{[1,3]} - \frac{4}{3}\_c_1U_{[2,3]}\Big) + x^2\Big(\frac{1}{4}\_c_1U_{[1,4]} - \frac{5}{12}\_c_1U_{[1,3]} + \frac{1}{3}\_c_1U_{[2,3]} - \frac{1}{2}\_c_1U_{[2,4]} - \frac{1}{8}\_c_1U_{[0,4]} + \frac{1}{8}\_c_1\Big)$$

$$+ x^3\Big(\frac{2}{15}\_c_1U_{[2,4]} - \frac{1}{5}\_c_1U_{[1,4]} + \frac{2}{45}\_c_1U_{[1,3]} - \frac{4}{45}\_c_1U_{[2,3]} + \frac{2}{15}\_c_1U_{[1,5]} + \frac{7}{30}\_c_1U_{[0,4]} + \frac{1}{30}\_c_1 - \frac{4}{15}\_c_1U_{[2,5]} - \frac{1}{15}\_c_1U_{[0,5]}\Big) + x^4\Big(\frac{1}{24}U_{[0,4]}\_c_2 - \frac{3}{40}\_c_1U_{[2,4]} + \frac{7}{240}\_c_1U_{[1,4]}$$

$$- \frac{7}{80}\_c_1U_{[1,3]} + \frac{1}{20}\_c_1U_{[2,3]} - \frac{7}{60}\_c_1U_{[1,5]} + \frac{7}{180}\_c_1U_{[0,4]} + \frac{7}{160}\_c_1 + \frac{1}{15}\_c_1U_{[2,5]} + \frac{17}{120}\_c_1U_{[0,5]} + \frac{1}{36}\_c_1U_{[1,3]}^2 - \frac{1}{36}U_{[1,3]}\_c_1U_{[2,3]} - \frac{1}{18}\_c_1U_{[2,3]}^2 - \frac{1}{24}\_c_1U_{[0,6]} + \frac{1}{12}\_c_1U_{[1,6]} - \frac{1}{6}\_c_1U_{[2,6]}\Big) + O(x^5)\Big]$$

The answer is the same as the answer in literals in Example 4.

We apply the procedure again to the same equation with the option of specifying the degree of truncation:

```
> RegularSolution(eq2,y(x),'degree'=2);
```

$$\Big[-\frac{\_c_1}{x^2} + \frac{4\_c_1}{x} + \_c_2 + x\Big(\frac{2}{3}\_c_1U_{[1,3]} - \frac{4}{3}\_c_1U_{[2,3]}\Big) + x^2\Big(\frac{1}{4}\_c_1U_{[1,4]} - \frac{5}{12}\_c_1U_{[1,3]} + \frac{1}{3}\_c_1U_{[2,3]} - \frac{1}{8}\_c_1U_{[0,4]} - \frac{1}{2}\_c_1U_{[2,4]} + \frac{1}{8}\_c_1\Big) + O(x^3) + \ln(x)(\_c_1 + O(x^3)), \_c_2 + O(x^3)\Big]$$

The answer shows that, in order to obtain a 2-truncation as a continuation of the invariant truncation, it is necessary to specify $U_{[0,4]}$, $U_{[1,3]}$, $U_{[1,4]}$, $U_{[2,3]}$, $U_{[2,4]}$, i.e, coefficients of the equation multiplying x^4, $x^3\theta$, $x^4\theta$, $x^3\theta^2$, $x^4\theta^2$, respectively.

3. Equation

```
> eq3:=(1+x^2+O(x^3))*theta(y(x),x,3)+(4-x+(1/2)*x^2+O(x^3))*theta(y(x),x,2)+
      (4-2*x+x^2+O(x^3))*theta(y(x),x,1)+O(x^3)*y(x);
```

$$eq3 := (1 + x^2 + O(x^3))\theta(y(x), x, 3) + \Big(4 - x + \frac{1}{2}x^2 + O(x^3)\Big)\theta(y(x), x, 2) + (4 - 2x + x^2 + O(x^3))\theta(y(x), x, 1) + O(x^3)y(x)$$

```
> RegularSolution(eq3, y(x));
```

$$\left[\frac{\frac{21\_c_1}{16} + \frac{\_c_2}{2}}{x^2} + \frac{\_c_1}{x} + \_c_3 + O(x) + \ln(x)\Big(\frac{1}{2}\frac{\_c_1}{x^2} + \_c_2 + O(x)\Big) + \ln(x)^2\Big(\frac{1}{2}\_c_1 + O(x^3)\Big),\right.$$

$$\left.\frac{1}{2}\frac{\_c_2}{x^2} + \_c_3 + O(x) + \ln(x)(\_c_2 + O(x^3)), \_c_3 + O(x^3)\right]$$

In this case, we have three different truncations of the regular solution, the corresponding Laurent series are truncated to different degrees, and the logarithm enters up to degree 2; i.e., the Laurent solutions were found for three equations: S_0, S_1, and S_2.

4. Truncation of the equation from Example 1:

```
> eq4:=(-1+x+O(x^3))*theta(y(x),x,2)+(-1-x-(3/2)*x^2+O(x^3))*theta(y(x),x,1)+
        (3/4+(1/4)*x+(3/4)*x^2+O(x^3))*y(x);
```

$$eq4 := (-1 + x + O(x^3))\theta(y(x), x, 2) + \left(-1 - x - \frac{3}{2}x^2 + O(x^3)\right)\theta(y(x), x, 1) + \left(\frac{3}{4} + \frac{1}{4}x + \frac{3}{4}x^2 + O(x^3)\right)y(x)$$

```
> RegularSolution(eq4, y(x));
```

$$\left[\sqrt{x}\left(-\frac{2\_c_1}{x^2} + \frac{8\_c_1}{x} + \_c_2 + O(x) + \ln(x)(\_c_1 + O(x^3))\right), \sqrt{x}(\_c_2 + O(x^3))\right]$$

In this case, we obtained a regular solution with a noninteger λ in the factor x^{λ}.

5. Equation

```
> eq5:=(1+O(x^2))*theta(y(x),x,3)+(1+2*x+O(x^2))*theta(y(x),x,2)+
        (2+x+O(x^2))*theta(y(x),x,1)+(2-x+O(x^2))*y(x);
```

$$eq5 := (1 + O(x^2))\theta(y(x), x, 3) + (1 + 2x + O(x^2))\theta(y(x), x, 2) + (2 + x + O(x^2))\theta(y(x), x, 1) + (2 - x + O(x^2))y(x)$$

```
> RegularSolution(eq5, y(x));
```

$$\left[\frac{\_c_1}{x} + O(x) + x^{\mathrm{RootOf}(\_Z^2+2, index=1)}\left(\_c_2 - \frac{1}{54}x(20 + 23\mathrm{RootOf}(\_Z^2 + 2, index = 1))\_c_2 + O(x^2)\right) + x^{\mathrm{RootOf}(\_Z^2+2, index=2)}\left(\_c_3 - \frac{1}{54}x(20 + 23\mathrm{RootOf}(\_Z^2 + 2, index = 2))\_c_3 + O(x^2)\right)\right]$$

In this case, the indicial polynomial, $u_0(n) = (n+1)(n^2+2)$, has three nonequivalent roots: $\Lambda = \{-1, \sqrt{-2}, -\sqrt{-2}\}$, where $\sqrt{-2}$ and $-\sqrt{-2}$ are represented by the constructions $\mathrm{RootOf}(\_Z^2 + 2, \mathrm{index} = 1)$ and $\mathrm{RootOf}(\_Z^2 + 2, \mathrm{index} = 2)$.

6. An equation specified via the differentiation operator D rather than the operator θ:

```
> eq6:=(-x+x^2+x^3+O(x^4))*(diff(y(x),x,x))+
>       (-3+x+2*x^2+O(x^3))*(diff(y(x), x))+O(x^3)*y(x)
```

$$eq6 := (-x + x^2 + x^3 + O(x^4))\left(\frac{d^2}{dx^2}y(x)\right) + (-3 + x + 2x^2 + O(x^3))\left(\frac{d}{dx}y(x)\right) + O(x^3)y(x)$$

```
> RegularSolution(eq6, y(x));
```

$$\left[-\frac{\_c_1}{x^2} + \frac{4\_c_1}{x} + \_c_2 + O(x) + \ln(x)(\_c_1 + O(x^4)), \_c_2 + O(x^4)\right]$$

As a result of the transition to θ in the equation, we obtain an equation with truncations of the coefficients, similar to the equation in paragraph 2. Therefore, the calculation results are identical.

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