THE RATIONAL COMPONENT OF THE SOLUTION OF A FIRST-ORDER LINEAR RECURRENCE RELATION WITH A RATIONAL RIGHT SIDE*

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THE RELATION $F(x+1)+aF(x)=R(x)$, is considered, where $R(x)$ is a rational function. A method of separating the rational component (term) of the solution of this relation is presented.

In [1, 2] the problems of the solution of linear recurrence relations with rational right sides in the field of rational functions were studied. In the case where the relation does not have a rational solution it may be desirable to separate from the solution of the relation some rational component, in order that another component may satisfy a relation that is (in some sense) simpler. In this paper a method is proposed of separating from the solution of a first-order relation, a rational term, in order that the second term may satisfy a relation with the previous left side, but with a modified right side. The new right side is a rational function with a denominator possibly of lower degree.

1. Consider the relation

$$F(x+1)+aF(x)=R(x). \tag{1}$$

Here $R(x)$ is a known rational function. We suppose that the coefficients belong to a suitable field P (here and below we use the definitions and notation of [2]). Therefore, $a\in P,\ a\neq 0,\ R(x)\in P\langle x\rangle$.

We search for a function $T(x)\in P\langle x\rangle$ such that $S(x)=R(x)-T(x+1)-aT(x)$ has in expanded form a denominator of the least possible degree. We call the function $S(x)$ the bound of the function $R(x)$. We say of $T(x)$ that it realizes the bound $S(x)$. As we shall see, the bound and the function realizing it are not, in general, uniquely defined.

We can remove from $R(x)$ the integral part $E(x)$. The relation $F(x+1)+aF(x)=E(x)$ can be solved in polynomial form. Moreover, if $F(x)$ is a proper fraction (that is, the numerator $F(x)$ is of lower degree than the denominator), then $F(x+1)+aF(x)$ is also a proper fraction. This fact enables us to regard $R(x)$, $S(x)$ and $T(x)$ as proper fractions.

2. The simplest case is when the rational functions considered have the expansion

$$\frac{q_m(x)}{p(x+m)^\alpha}+\frac{q_{m-1}(x)}{p(x+m-1)^\alpha}+\ldots+\frac{q_0(x)}{p(x)^\alpha}, \tag{2}$$

where $p(x)$ is irreducible in $P[x]$, $\deg q_i(x)<\deg p(x)$, $i=0, 1, \ldots, m$, α is a natural number, and m is a non-negative integer.

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The following proposition is quite obvious.

Proposition 1

Let $F(x)$ be expanded in the sum (2), then $F(x+1)+aF(x)$ is expanded in a sum of the same form:

$$\frac{q^*_{m+1}(x)}{p(x+m+1)^\alpha}+\frac{q_m^*(x)}{p(x+m)^\alpha}+\ldots+\frac{q_0^*(x)}{p(x)^\alpha},$$

where $q^*_{m+1}(x)=q_m(x+1)$, $q_0^*(x)=aq_0(x)$. Moreover, if relation (1) has a rational solution and $R(x)$ is of the form (2), then the resulting rational will be a function of the same form.

From Proposition 1 we can deduce a corollary the content of which forms the following proposition.

Proposition 2

Let $R(x)$ be expanded in the sum (2). Let relation (1) not have a rational solution. Let $S(x)$ be the bound of $R(x)$, and let $T(x)$ realize $S(x)$. Then $S(s)$ and $T(x)$ are rational functions of the form (2).

We now present a method of solving the problem posed for the simple case discussed.

Proposition 3

Let $R(x)$ be expanded in the sum (2). Then either relation (1) has a rational solution, or for any integer i a bound of the rational function $R(x)$ can be constructed in the form

$$\frac{v(x)}{p(x+i)^\alpha},\qquad v(x)\in P[x],\qquad \deg v(x)<\deg p(x).$$

Proof. We first suppose that relation (1) has a rational solution $F(x)$. Then by Proposition 1,

$$F(x)=T_1(x)+F_1(x),$$

where

$$T_1(x)=\frac{q_m(x-1)}{p(x+m-1)^\alpha},$$

$$F_1(x+1)+aF_1(x)=R(x)=R_1(x)-\frac{q_m(x)}{p(x+m)^\alpha}-a\frac{q_m(x-1)}{p(x+m-1)^\alpha}.$$

We obtain either $R_1(x)=0$, or $\operatorname{Dis} R_1(x)<\operatorname{Dis} R(x)$. If the first is true, then $F(x)=T_1(x)$ satisfies (1), otherwise we can continue the process and construct $T_2(x)$, $R_2(x)$, $\ldots$, $T_k(x)$, $R_k(x)$, where $R_k(x)=0$. Then the solution of relation (2) will obviously be

$$T_1(x)+\ldots+T_k(x). \tag{3}$$

If relation (1) does not have a rational solution, then, carrying out the process described, we finally arrive at a position where $R_k(x)=\dfrac{u(x)}{p(x+j)^\alpha}$, j is some integer. Then, by Proposition 2, $R_k(x)$ is the bound of $R(x)$ and the sum (3) realizes $R_k(x)$.

Continuing the process, we obtain $R_{k+1}(x),\ R_{k+2}(x),\ \ldots$, which are of the form

$$\frac{u_1(x)}{p(x+j-1)^{\alpha}},\ \frac{u_2(x)}{p(x+j-2)^{\alpha}},\ldots.$$

If in the relation $F_k(x+1)+aF_k(x)=R_k(x)$ we substitute $F_k(x)=T_{k+1}(x+F_{k+1}(x)$, where

$$T_{k+1}(x)=\frac{(1/a)\,u(x)}{p(x+j)^{\alpha}},$$

we obtain

$$F_{k+1}(x+1)+aF_{k+1}(x)=R_{k+1}(x)=\frac{(1/a)\,u(x+1)}{p(x+j+1)^{\alpha}}.$$

Therefore, we have described a procedure whose execution leads to the construction of a rational solution of relation (1) with right side of the form (2) in the case where a rational solution exists. If a rational solution does not exist, then the proposed procedure permits us, for any integer t, to construct a bound of the function $R(x)$ in the form $\frac{v(x)}{p(x+i)^{\alpha}}$, and also to construct a rational solution realizing this bound.

3. We pass to the general case for $R(x)$. In the first place, we agree to call the simplest fractions

$$\frac{b(x)}{q(x)^{\alpha}}\ \text{and}\ \frac{c(x)}{q(x+i)^{\alpha}}\quad (q(x)\ \text{is irreducible in}\ P[x])$$

neighbours. Accordingly, we will speak of neighbouring elements of the expansion of a given rational function in a sum of simple elements.

Proposition 4

Neighbouring elements do not occur in the expansion in a sum of simple elements of the bound of the rational function $R(x)$. Every element of the expansion of the bound is the bound of the sum of neighbouring elements of the expansion of $R(x)$. If the expansion of the bound of a rational function $R(x)$ contains two elements whose denominators are $p(x)^{\alpha}$ and $p(x+k)^{\beta}$, then $k = 0$.

Proof. The statement follows directly from Propositions 2 and 3.

Proposition 5

Let $T^*(x)\in P\langle x\rangle$. Let $R^*(x)=R(x)-T^*(x+1)-aT^*(x)$. Let $S(x)$ be the bound of $R^*(x)$; then $S(x)$ is the bound of $R(x)$.

Proof. We have for some $T(x)\in P\langle x\rangle$

$$S(x)=R^*(x)-T(x+1)-aT(x)=R(x)-[T(x+1)+T^*(x+1)]-a[T(x)+T^*(x)],$$

where the degree of the denominator of $S(x)$ is the least possible. On the other hand, for every $S_1(x)\in P\langle x\rangle$ such that

$$S_1(x)=R(x)-T_1(x+1)-aT_1(x)$$

for some $T_1(x)\in P\langle x\rangle$, we have

$$S_1(x)=R^*(x)+T^*(x+1)+aT^*(x)-T(x+1)-aT(x)$$
$$=R^*(x)-[T(x+1)-T^*(x+1)]-a[T(x)-T^*(x)].$$

Therefore, the degree of the denominator of $S_1(x)$ cannot be less than the degree of the denominator of $S(x)$.

A corollary of Proposition 5 is Proposition 6.

Proposition 6

Let Dis $R(x)=0$. Then $R(x)$ is the bound of $R(x)$.

Proposition 7

Let $T(x)\in P\langle x\rangle$ and $S(x)=R(x)-T(x+1)-aT(x)$. Let Dis $S(x)=0$. Then $S(x)$ is the bound of $R(x)$.

Proof. The statement is a corollary of Propositions 5, 6.

Proposition 8

Let Dis $R(x)>0$. Then we can construct $U(x)\in P\langle x\rangle$ such that either $R(x)-U(x+1)-aU(x)=0$, or Dis $[R(x)-U(x+1)-aU(x)]<$ Dis $R(x)$.

Proof. Let

$$R(x)=\frac{r_1(x)}{r_2(x)},\quad r_1(x),\ r_2(x)\in P[x],\qquad \mathrm{GCD}(r_1(x),r_2(x))=1.$$

Let Dis $R(x)=h$. We can find $t(x)=\mathrm{GCD}(r_2(x),r_2(x+h))$. Applying Euclid's algorithm several times and performing the division of polynomials, we can represent $r_2(x)$ in the form $v(x)w(x)$, and thus the polynomial $v(x)$ is constructed from the next one. Let the expansion of $t(x)$ in a product of irreducibles be of the form $t_1(x)^{\alpha_1}t_2(x)^{\alpha_2}\dots t_n(x)^{\alpha_n}$, let $r_2(x)=t_1(x)^{\beta_1}t_2(x)^{\beta_2}\dots t_n(x)^{\beta_n}r_2^*(x)$ and the polynomial $r_2^*(x)$ be free from $t_i(x)$, $i=1,2,\dots,n$. Then $v(x)=t_1(x)^{\beta_1}\dots t_n(x)^{\beta_n}$.

The construction of the polynomial $v(x)$ does not require the expansion of $t(x)$ and $r_2(x)$ in a product of irreducibles. Indeed, the polynomials

$$v^{(0)}(x)=t(x),\qquad r^{(0)}(x)=r_2(x)/t(x),\qquad d^{(0)}(x)=\text{НОД}(r^{(0)}(x),v^{(0)}(x)),$$
$$v^{(i)}(x)=v^{(i-1)}(x)d^{(i-1)}(x),\qquad r^{(i)}(x)=r^{(i-1)}(x)/d^{(i-1)}(x),$$
$$d^{(i)}(x)=\mathrm{GCD}(r^{(i)}(x),v^{(i)}(x)),\qquad i=1,2,\dots.$$

can be constructed successivly. As soon as it is observed that $d^{(k)}(x)=1$, we can conclude that $v(x)=v^{(k)}(x)$. It is obvious that $v(x)$ and $w(x)=r_2(x)/v(x)$ are coprimes.

We can now represent $R(x)$ in the form

$$\frac{b(x)}{v(x)}+\frac{c(x)}{w(x)}, \qquad b(x),\ c(x)\in P[x],$$

which is done by the sequential determination of the remainders. Moreover,

$$U(x)=\frac{b(x-1)}{v(x-1)} \qquad R_1(x)=R(x)-\frac{b(x)}{v(x)}-a\frac{b(x-1)}{v(x-1)}=\frac{c(x)}{w(x)}-a\frac{b(x-1)}{v(x-1)}.$$

It is obvious from the method of constructing $v(x)$ that $U(x)$ satisfies the condition formulated.

We mention that the execution of the proposed procedure does not require the expansion of the polynomials into irreducible polynomials or the expansion of the fractions into simple fractions.

The method of calculating Dis $R(x)$ was discussed in [1, 2].

It is obvious from the above that the repeated application of the procedure leads to the construction of a rational solution of relation (1), if a rational solution exists, and leads to the construction of a bound of the rational function $R(x)$, if Eq. (1) does not have a rational solution.

4. In the case where it is known that relation (1) has a rational solution, the methods presented in [1, 2] ensure a more rapid determination of the solution than the method explained above. The methods presented in [1, 2], enable us to recognize whether a rational solution of relation (1) exists, but this recognition takes place in the process of constructing the supposed solution, and on finding that a rational solution does not exist, a great deal of effort may have been expended. Applying the method explained in this paper, we can finally verify that a rational solution does not exist, but as well as this we simplify the right side of the relation, which may be useful.

5. Consider the case $a=-1$. separately. Relation (1) is transformed into the simple finite-difference equation

$$F(x+1)-F(x)=R(x).$$

The problem of finding the solution of this equation is equivalent to finding the sum

$$\sum_{x=m}^{n} R(x). \tag{4}$$

The proposed method enables us to represent (4) in the form

$$A(m,n)+\sum_{x=m}^{n} S(x),$$

where $A(m,n)\in P\langle m,n\rangle$, $S(x)\in P\langle x\rangle$, and $S(x)$ has a denominator of the least possible degree. In this case the proposed method is an analog of the well-known Hermite method of finding the rational part of the integral of a rational function.

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CONTENTS

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