ON THE SUMMATION OF RATIONAL FUNCTIONS*

S. A. ABRAMOV

Moscow

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AN algorithm is given for solving the following problem: let $F(x_1, \dots, x_n)$ be a rational function of the variables x_i with rational (real or complex) coefficients; to see if there exists a rational function $G(v, w, x_2, \dots, x_n)$ with coefficients from the same field, such that $\sum_{x_1=v}^{w} F(x_1, \dots, x_n) = G(v, w, x_2, \dots, x_n)$ for all integral values of $v \leqslant w$. If G exists to obtain it. Realization of the algorithm in the LISP language is discussed.

The summation of rational functions is often required during the solution of combinatorial problems. An algorithm for such summation will be described, together with its realization in the LISP algorithmic language [1] for the BESM-6 computer.

Some elementary results from algebra and the theory of finite differences, see e.g. [2, 3], will be used.

§ 1

Let P be a field, $P[x_1, \dots, x_n]$ a ring of polynomials, and $P < x_1, \dots, x_n >$ the field of rational functions of the variables $x_1, \dots, x_n$ with coefficients from P. Given $f(x), g(x) \in P[x]$, we write GCD $(f(x), g(x))$ for the greatest common divisor of $f(x)$ and $g(x)$ in $P[x]$, and deg $f(x)$ for the degree of $f(x)$.

Our only interest is in fields of characteristic 0. The integer ring can be imbedded in a natural way in such fields. The images of this imbedding, ordered in the natural way, will be called integers. If P is such a field, the following problem may be stated: let $F(x) \in P\langle x\rangle$; to discover if a rational function $H(w, v) \in P\langle v, w\rangle$ exists, such that

$$\sum_{x=v}^{w} F(x) = H(v, w) \tag{1}$$

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for all integral values of $v \leqslant w$.

The problem (1) is equivalent to the following: to discover if a rational function $G(x) \in P\langle x\rangle$, exists, satisfying the elementary finite-difference equation

$$G(x+1) - G(x) = F(x). \tag{2}$$

Equation (2) is well known to have a solution when $F(x)$ is a polynomial. We shall be ignoring this case and assume below that the degree of the denominator of $F(x)$ (which is assumed irreducible) is not less than 1. Notice that, if $F(x)$ is a polynomial, the problem reduces immediately to the solution of equation (6) with $g_1(x) = 1$.

An algorithm will be given, applicable to any rational function $F(x)$ and showing that a $G(x)$ satisfying (2) can be found, and finding such a $G(x)$ when it exists, provided that the field P comes within

Definition 1

A field P of characteristic 0 will be termed canonical if an algorithm exists, whereby all the integer-valued roots of the equation $p(x) = 0$, where the polynomial $p(x) \in P[x]$, can be found.

The fields of rational, real, and complex numbers, are canonical. In addition, it is easily shown that

Proposition 1

Let P be a canonical field. Then $P\langle x\rangle$ is also a canonical field.

Let $F(x)$ be a rational function and $s(x)$ the denominator of the irreducible form of $F(x)$. The statement of problem (1) compels us to assume that, for $H(v, w)$ to exist, the equation $s(x) = 0$ must have no integer-valued roots. Since the coefficients lie in a canonical field, this condition can easily be checked. Our algorithm enables a rather more general problem to be solved: to investigate the existence of an $H(v, w)$ satisfying (1) for all integer-valued values of v and w such that $w \geqslant v > x_0$, where x_0 is the maximum integer-valued root of $s(x) = 0$; if $H(v, w)$ exists, find it.

Henceforth, P is assumed to be a canonical field.

Definition 2

Let $f(x) \in P[x]$ and $\deg f(x) > 0$. The dispersion of the polynomial $f(x)$ (call it dis $f(x)$) will be defined as the maximum of the integers α for which

(3) $\deg \mathrm{GCD}(f(x), f(x+\alpha)) \geqslant 1.$

(The set of such numbers is bounded, since an expansion into irreducible factors is unique in $P[x]$.)

Proposition 2

Let $f(x) \in P[x]$ and $\deg f(x) > 0$. The dis $f(x)$ can be evaluated.

Let us describe the way in which all the integers satisfying (3) can be evaluated. Form the polynomial $f_1(x) = f(x+h) \in P\langle h\rangle[x]$ and use the fact that, given any concrete integer h, *GCD* $(f(x), f(x+h))$ in $P[x]$ can be found from Euclid's algorithm.

Divide $f_1(x)$ by $f(x)$ in $P\langle h\rangle[x]$ with remainder

(4) $f_1(x) = a(x)f(x) + b(x), \qquad \deg b(x) < \deg f(x).$

Since P is a canonical field, all the integers $h_0, \ldots, h_i$ can be found, such that, when substituted in (4), the leading coefficient of $b(x)$ vanishes.

Pick out by inspection those of $h_0, \ldots, h_i$ for which $f(x)$ and $f(x+h)$ have a non-trivial GCD. For all $h \neq h_0, \ldots, h_i$ the first division with a remainder is given by the expression resulting from substitution of h in (4). All the h for which $b(x)$ is zero in $P[x]$ can be found.

The process is continued thus until the remainder becomes zero in $P\langle h\rangle[x]$. After this, it remains to select the maximum element from a finite set of numbers.

Note. Another way of finding dis $f(x)$ is to note that it is the maximum integer-valued root of the resultant of polynomials $f(x)$ and $f(x+h)$, considered as elements of the ring $P[h][x]$.

Proposition 3

Let $F_1(x), F_2(x) \in P\langle x\rangle$, where $F_1(x+1) - F_1(x) = F_2(x)$ and $F_1(x)$, $F_2(x)$ have irreducible forms $f_{11}(x) / f_{12}(x)$ and $f_{21}(x) / f_{22}(x)$ $(f_{ij}(x) \in P[x])$ respectively. Then $\operatorname{dis} f_{12}(x) = \operatorname{dis} f_{22}(x) - 1$.

Split $F_1(x)$ into a sum of partial fractions:

$$F_1(x)=\sum_{i=1}^{\lambda}\frac{q_i(x)}{p_i(x)^{\varkappa(i)}}+g(x)$$

(here, $p_i(x)$, $i = 1, \ldots, \lambda$, are irreducible in $P[x]$, $q_i(x)$, $g(x)\in P[x]$ and $\deg q_i(x) < \deg p_i(x)^{\varkappa(i)}$). Let

(5) $$\operatorname{dis} f_{12}(x) = \alpha,\ p_1(x+\alpha) = p_2(x).$$

Noting that the decomposition into a sum of partial fractions is unique and that the substitution $x \to x + 1$ transforms an irreducible into an irreducible polynomial, the decomposition of $F_2(x)$ is found to be

$$F_2(x)=g(x+1)-g(x)+\frac{q_1(x+1)}{p_1(x+1)^{\varkappa(1)}}+\frac{q_2(x+1)}{p_1(x+\alpha+1)^{\varkappa(2)}}-\frac{q_1(x)}{p_1(x)^{\varkappa(1)}}-$$

$$-\frac{q_2(x)}{p_1(x+\alpha)^{\varkappa(2)}}+\sum_{i=3}^{\lambda}\left(\frac{q_i(x+1)}{p_i(x+1)^{\varkappa(i)}}-\frac{q_i(x)}{p_i(x)^{\varkappa(i)}}\right).$$

Condition (5) shows that no partial fractions, having powers of $p_1(x + \alpha + 1)$ and $p_1(x)$, are to be found under the summation sign on the right-hand side of the last equation, whence the proposition follows (it is of no consequence here whether P is canonical or not).

If, for some element $F(x) \in P\langle x\rangle$, there exists $G(x)$ such that $G(x+1)-G(x) = F(x)$, the denominator of the irreducible fraction equivalent to $G(x)$ can easily be evaluated. Let $F(x)$ and $G(x)$ have irreducible forms $f_1(x)/f_2(x)$ and $g_1(x)/g_2(x)$ respectively and $\operatorname{dis} f_2(x) = \alpha$.

Then,

$$\sum_{i=0}^{\alpha-1}\frac{f_1(x+i)}{f_2(x+i)}=\frac{g_1(x+\alpha)g_2(x)-g_1(x)g_2(x+\alpha)}{g_2(x+\alpha)g_2(x)}.$$

Recalling that $g_1(x)/g_2(x)$ is irreducible, and that $\operatorname{GCD}(g_2(x+\alpha), g_2(x)) = 1$ (see Proposition 3), it may be seen that, if the irreducible fraction

$$\frac{s(x)}{t(x)}=\sum_{i=0}^{\alpha-1}\frac{f_1(x+i)}{f_2(x+i)},$$

can be evaluated, then $t(x) = g_2(x+\alpha)g_2(x)$ up to invertible factors. We can further write: $t(x+\alpha) = g_2(x+2\alpha)g_2(x+\alpha)$, $t(x)/t(x+\alpha) = g_2(x)/g_2(x+2\alpha)$. The fraction on the right-hand side of the last equation is irreducible, while the fraction on the left is known; $g_2(x)$ can be found.

In view of this, the problem amounts to the following: given the polynomials $p(x),\ g_2(x) \in P[x]$ see if a polynomial $g_1(x) \in P[x]$ can be constructed such that

$$p(x) = g_1(x+1)g_2(x) - g_2(x+1)g_1(x), \tag{6}$$

and if so, construct it.

Assuming that $g_1(x)$ exists and that

$$\begin{aligned} g_1(x) &= a_n x^n + \ldots + a_0, \\ g_2(x) &= b_m x^m + \ldots + b_0, \end{aligned} \tag{7}$$

the coefficient of the $(m+n-1)$-th power in the polynomial on the right-hand side of (6) can be evaluated (the coefficient of the $(m+n)$-th power is known to be zero). This coefficient is equal to $(n-m)a_n b_m$. We thus have one of the two equations: $n = m$ or $n = deg\ p(x) - m + 1$.

Once the degree of $g_1(x)$ is known, the method of undetermined coefficients can be used to solve (6); this leads us to investigate the existence of, and possibly to find, the solution of a system of linear equations of order $\deg g_1(x)$, which, if it is compatible, has the rank $\deg g_1(x) - 1$ (this follows from the fact that the solution of (6) can be found in our case up to an arbitrary added term from the field of coefficients). It is not necessary to write down this system; a recursive procedure may be used for finding the solution of (6) in the form of a polynomial whose degree is not higher than n. Let $m \neq n$. Putting $g_1 = a_n x^n + g_1'$, $\deg g_1' = n - 1$, we get $g_1'(x+1)g_2(x) - g_1'(x)g_2(x+1) + a_n(x+1)^n g_2(x) - a_n x^n g_2(x+1) = p(x)$. The degree of the polynomial $g_2(x)g_1'(x+1) - g_2(x+1)g_1'(x)$ is not higher than $m+n-2$. The degree of the polynomial $a_n x^n g_2(x+1) + a_n(x+1)^n g_2(x)$ is not higher than $m+n-1$, where the coefficient of the $(m+n-1)$-th power is $(n-m)a_n b_m$, whence a_n can be found if $\deg p(x) \leqslant m+n-1$; otherwise, no solution exists.

Now let $m = n$. Then, using the notation of (7), the coefficient of the leading $(n+m-2)$-th power on the right-hand side of (6) is equal to $b_{m-1}a_n - b_m a_{n-1}$, where $b_m \neq 0$. If the degree of $p(x)$ is higher than $n+m-2$, no solution exists.

The rule for multiplying polynomials, and the above remark concerning the rank of the system of equations for the coefficients, give grounds for asserting that a_n can be fixed arbitrarily; in particular, it can be put equal to zero.

In either case, therefore, we either verify that no solution exists, or we transform from (6) to the equation

$$p'(x) = g_2(x)g_1'(x+1) - g_2(x+1)g_1'(x), \qquad \deg g_1'(x) = \deg g_1(x) - 1.$$

If the solution of (6) is sought as a polynomial of zero degree, the investigation is trivial.

This ends our description of the algorithm.

Notice that the isomorphism that exists between $P\langle x_1, \ldots, x_n\rangle$ and $P\langle x_1, \ldots, x_{n-1}\rangle\langle x_n\rangle$ and the Proposition 1, enable the algorithm to be applied to rational functions of several variables with coefficients from a canonical field, in the problem of solving the equation

$$G(x_1, \ldots, x_n + 1) - G(x_1, \ldots, x_n) = F(x_1, \ldots, x_n).$$

§ 2

The algorithm described (for the problem with several variables) was realized by the author as a program in the LISP language. The field of rational numbers was taken as the field of coefficients.

Some results obtained by using the program are as follows.

1. Evaluate

$$\sum_{i=1}^{n} i_3.$$

Answer: $(n^4 + 2n^3 + n^2) / 4$. Computing time 8 sec.

2. Evaluate

$$\sum_{i=1}^{n} \frac{6i+3}{4i^4 + 8i^3 + 8i^2 + 4i + 3}.$$

Answer: $(n^2 + 2n) / (2n^2 + 4n + 3)$. Computing time 15 sec.

3. Evaluate

$$\sum_{i=1}^{n} \frac{1}{i^2 + n^2 - 3i + 3n - 2in + 2}.$$

Answer: $n/(n + 1)$. Computing time 20 sec.

4. Evaluate

$$\sum_{i=1}^{n} \frac{1}{i^2}.$$

Answer: this expression is not a rational function of n.
Computing time 5 sec.

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