ISSN 0361-7688, Programming and Computer Software, 2008, Vol. 34, No. 2, pp. 64–68. © Pleiades Publishing, Ltd., 2008.
Original Russian Text © S.A. Abramov, A.A. Ryabenko, 2008, published in Programmirovanie, 2008, Vol. 34, No. 2.

On a Computer-Algebraic Technology

S. A. Abramov and A. A. Ryabenko

Dorodnicyn Computing Center, Russian Academy of Sciences, ul. Vavilova 40, Moscow, 119991 Russia

E-mail: abramov@ccas.ru, ryabenko@cs.msu.ru

Received June 2, 2007

Abstract—Modern computer algebra systems provide means for exact experimental calculations (including manipulations on formulas) that enable one to obtain a solution of a problem under examination for certain not very large initial data. The results can provide a basis for a conjecture concerning the general solution of the problem. A computer algebra system can also be helpful in verifying the conjecture.

DOI: 10.1134/S0361768808020023

1. INTRODUCTION

In this paper, we give a detailed description of the way used to obtain a mathematical result. This result is formulated in Section 7 in Theorem 1. The computer algebra system Maple [1] plays an important role in obtaining this result.

In particular, this paper demonstrates some capabilities of computer algebra systems that may be useful for theoretical studies.

2. EXPERIMENTS, CONJECTURES, AND PROVES

The viewpoint that mathematics is an experimental science deserves attention. The difference between mathematics and physics is that physical experiments cost millions of dollars while mathematical experiments cost only several dollars (see [2]).

The use of modern computer algebra systems for solving mathematical problems often opens possibilities for an experiment: using available mathematical theories and a computer algebra system, it turns out to be possible to obtain an exact solution of a problem for small values of the initial data, for example, for small values of parameters. If one is lucky, an examination of such solutions can lead to a conjecture about the solution in the case of arbitrary admissible initial data. Moreover, a computer algebra system can be helpful in verifying and, maybe, even proving the conjecture. This computer technology can be helpful in mathematical studies.

In this paper, we discuss an example of using this technology. We describe the use of Maple to first find and then prove a new formula for the integral of the Bessel function $J_n(z)$ for odd positive integers n. The same computer-algebraic technique is used to show that no similar formula for even positive integers holds.

3. PRELIMINARY OBSERVATION

The preliminary observation is that the Bessel function $J_1(z)$ satisfies the relation

$$\int J_1(z)dz = -J_1'(z) - \frac{1}{z}J_1(z) + C. \qquad (1)$$

Rewriting it in the form

$$\left(-J_1'(z) - \frac{1}{z}J_1(z)\right)' - J_1(z) = 0, \qquad (2)$$

we can prove it on the basis of the differential equation for $J_1(z)$

$$z^2J_1''(z) + zJ_1'(z) + (z^2-1)J_1(z) = 0. \qquad (3)$$

Indeed, removing the parentheses on the left-hand side of (2) and multiplying by $-z^2$, we obtain the left-hand side of (3). An attempt to obtain a similar result for $\int J_2(z)dz$ fails; the situation in the general case is even more unclear: For which positive integer n, can one obtain a representation for an antiderivative of $J_n(z)$ in the form of a linear combination of $J_n(z)$ and its derivatives over $\mathbb{C}(z)$? (Recall that $J_n(z)$ satisfies the equation $z^2J_n''(z) + zJ_n'(z) + (z^2 - n^2)J_n(z) = 0$.)

4. INTEGRATION BY MEANS OF DIFFERENTIATION

In [3], the following general problem was considered. Let

$$L = a_k(z)D^k + \ldots + a_1(z)D + a_0(z) \in \mathbb{C}(z)[D],$$

where $D = \frac{d}{dz}$ and $a_0(z), a_1(z), \ldots, a_k(z)$ are polynomials or rational functions. We want to know whether an operator $R \in \mathbb{C}(z)[D]$[1] exists such that

$$\int y dz = R(y) + C$$

for any function $y(z)$ for which L is the minimal annihilator operator in the ring $\mathbb{C}(z)[D]$. If it does exist (such an operator is called an integrating operator), we want to construct it.

It was proved in [3] that the operator R exists if and only if the equation

$$L^*(y) = 1$$

has a rational solution (that is, a solution belonging to the field of rational functions $\mathbb{C}(z)$). Here, L^* is the adjoint operator of L:

$$L^* = (-D)^k \circ a_k(z) + \ldots + (-D) \circ a_1(z) + a_0(z).$$

It was also proved that, if $r(z)$ is a rational solution to the equation $L^*(y) = 1$, then D divides the operator $1 - r(z)L$ on the left to produce the left quotient R:

$$D \circ R = 1 - r(z)L. \tag{4}$$

In [3], the corresponding algorithm was called the accurate integration algorithm. It could also be called the integration through differentiation algorithm.

Let us denote the minimal annihilator operator for $J_1(z)$ in $\mathbb{C}(z)[D]$ by L_1. We have

$$L_1 = z^2D^2 + zD + (z^2 - 1),$$
$$L_1^* = z^2D^2 + 3zD + z^2.$$

The equation $L_1^*(y) = 1$ has the unique rational solution
1 $\frac{1}{z^2}$, which can be verified using Maple(this issue is discussed in Section 5 in more detail).

Furthermore, we have

$$1 - \frac{1}{z^2}L_1 = -D^2 - \frac{1}{z}D + \frac{1}{z^2}$$
$$= -D^2 - D \circ \frac{1}{z} = D \circ \left(-D - \frac{1}{z}\right),$$

which implies $R = -D - \frac{1}{z}$.

The equality $\int f(z)\,dz = -f'(z) - \frac{1}{z}f(z) + C$ holds not only for $J_1(z)$ but also for any other solution $f(z)$ to the equation $L(y) = 0$, where $L = z^2D^2 + zD + (z^2 - 1)$. In the theory of special functions, the Bessel function of the second kind $Y_1(z)$ is considered along with $J_1(z)$. The function $Y_1(z)$ can be integrated using the formula

$$\int Y_1(z)dz = -Y_1'(z) - \frac{1}{z}Y_1(z) + C.$$

Below, we do not specifically discuss the Bessel functions of the second kind $Y_n(z)$; however, all the facts that will be established for $J_n(z)$ are also valid for $Y_n(z)$.

In the example considered above, the equation $L_1^*(z) = 1$ had a unique solution in $\mathbb{C}(z)$. It was shown in [3] that, if L is a minimal annihilator operator, the equation $L^*(y) = 1$ either has no rational solutions, or has a unique rational solution, or the set of its rational solutions has the form $\{r_0 + Ch | C \in \mathbb{C}\}$, where r_0 and h are fixed rational functions. In the latter case, all the antiderivatives can be obtained using the method under examination.

It turns out that no integrating operator can be constructed for $J_2(z)$. In this case,

$$L_2 = z^2D^2 + zD + (z^2 - 4),$$
$$L_2^* = z^2D^2 + 3zD + z^2 - 3,$$

and the equation $L_2^*(y) = 1$ has no rational solutions.

5. EXPERIMENT LEADS TO A CONJECTURE

The Bessel function of the first kind $J_n(z)$ satisfies the equation $L_n(y) = 0$, where $L_n = z^2D^2 + zD + (z^2 - n^2)$, $n = 1, 2, \ldots$. We know that $J_1(z)$ has an antiderivative that is a linear combination of $J_1'(z)$ and $J_1(z)$ over $\mathbb{C}(z)$. For $J_2(z)$, no such antiderivative exists. This is related to the fact that the equation $L_1^*(y) = 1$ has a rational solution, while $L_2^*(y) = 1$ has no such solutions.

It is easy to verify that

$$L_n^* = z^2D^2 + 3zD + z^2 = 1 - n^2, \quad n = 1, 2, \ldots.$$

Thus, we have the question: For what n does the equation $L_n^*(y) = 1$ have a solution?

[1] If K is a field and t is a variable, then $K(t)$ denotes the field of rational functions of t with the coefficients in K. Similarly, if A is a ring (maybe, a field), then $A(t)$ denotes the ring of polynomials in t with the coefficients in A. Similarly, $\mathbb{C}(z)[D]$ is the ring of linear differential operators whose coefficients are rational functions of z with complex coefficients. Respectively, $\mathbb{C}[z][D]$ or, which is the same, $\mathbb{C}[z, D]$ is the ring of linear differential operators whose coefficients are polynomials in z with complex coefficients.

Let us check the first several values of n using Maple:

```
> M := z^2*diff(y(z), z, z)+3*z*diff(y(z), z)+
> (z^2+1-n^2)*y(z)=1:
> DEtools[ratsols](eval(M,n = 1), y(z));
```

$$\left[[\,],\frac{1}{z^2}\right]$$

For the given nonhomogeneous differential equation, the procedure ratsol in the package DEtools produces a list consisting of two items. The first item contains the list consisting of the basis of rational solutions to the corresponding homogeneous equation; the second item contains a particular rational solution to the given nonhomogeneous equation. We see that, for $n = 1$, the homogeneous equation has no rational solutions, while $\frac{1}{z^2}$ is a particular solution to the nonhomogeneous equation. For $n = 2$, there are no rational solutions:

```
> DEtools[ratsols](eval(M, n = 2), y(z));
```

$$[[\,]]$$

For $n = 1, 2$, these results were mentioned in Section 4. Continuing the calculations, we obtain

```
> DEtools[ratsols](eval(M, n = 3), y(z));
```

$$\left[[\,],\frac{8+z^2}{z^4}\right]$$

```
> DEtools[ratsols](eval(M, n = 4), y(z));
```

$$[[\,]]$$

```
> DEtools[ratsols](eval(M, n = 5), y(z));
```

$$\left[[\,],\frac{384+23z^2+z^4}{z^6}\right]$$

```
> DEtools[ratsols](eval(M, n = 6), y(z));
```

$$[[\,]]$$

```
> DEtools[ratsols](eval(M, n = 7), y(z));
```

$$\left[[\,],\frac{46080+1920z^2+48z^4+z^6}{z^8}\right]$$

```
> DEtools[ratsols](eval(M, n = 8), y(z));
```

$$[[\,]]$$

The first eight values of n suggest the following conjecture: The equation $L_n^*(y) = 1$ has a rational solution for odd n and has no such solutions for even n.

6. PROOF OF THE CONJECTURE

Unfortunately, Maple cannot directly find out for which n the equation $L_n^*(y) = 1$ has a rational solution. It is easily seen from the form of $L_n^*(y) = 1$ that this equation has no polynomial solutions for any n (if $p(z) \in \mathbb{C}(z)\backslash\{0\}$ and $d = \deg p(z)$, then $L_n^*(p(z))$ is a polynomial of degree $d + 2$); if there exists a solution in $\mathbb{C}(z)\backslash\mathbb{C}[z]$, then its denominator is z^m, where $m \in \mathbb{N}$ (0 is the only singular point of the operator L_n^*).

Let us expand the solution to $L_n^*(y) = 1$ into a series at the point $z = \infty$ (here, n is a parameter):

```
> Ser := Slode[FPseries](M, y(z), v(k),
> z=infinity);
```

$$FPSstruct\left(\frac{1}{z^2}+\sum_{k=3}^{\infty}\frac{v(k)}{z^k},\; v(k)+(k^2+9-n^2-6k)v(k-2)\right).$$

The procedure FPseries in the package Slode constructs a solution to the given differential equation with polynomial coefficients in the form of a formal power series at the given point.[2] The solution is represented in the form of a special structure FPStruct. The first element of this structure is the solution in the form of a series for which the first several coefficients are calculated (here, the first three coefficients are found) and the remaining ones are denoted by $v(k)$. The second element of the structure is the recurrence that can be used to find any number of the series coefficients. By default, the minimal number of coefficients is calculated that are sufficient for finding all the other coefficients using the recurrence (the order of the recurrence and the integer roots of its leading coefficient are taken into account). Several more coefficients can be calculated; for example, all the coefficients of z^{-k} ($k \le 7$). To this end, we supply FPSeries with the additional parameter terms = 7:

```
> Ser := Slode[FPseries](M, y(z), v(k),
> z=infinity, terms=7);
```

$$FPSstruct\left(\frac{1}{z^2}+\frac{-1+n^2}{z^4}+\frac{(-9+n^2)(-1+n^2)}{z^6}+\sum_{k=8}^{\infty}\frac{v(k)}{z^k},\right.$$

[2] In Maple, up to version 11 inclusive, the procedure FPseries constructs solutions only to homogeneous equations. We used an extended version of Slode that can deal with nonhomogeneous equations having rational right-hand sides.

$$v(k) + (k^2 + 9 - n^2 - 6k)v(k-2)\Big).$$

We see that $v(k) = 0$ for odd k. Therefore, it is reasonable to exclude the odd values of k. Moreover, to facilitate the verification of the conjecture formulated in Section 5, we factorize the lowest term of the recurrence:

```
> S:=indets(Ser, 'Sum'(anything, any-
thing))[];
>   S=Sum(eval(op(1,S),   [v(k)=v(2*k),
k=2*k],
> k=4..infinity);
```

$$\sum_{k=8}^{\infty} \frac{v(k)}{z^k} = \sum_{k=4}^{\infty} \frac{v(2k)}{z^{2k}}.$$

```
> Ser := eval(op(2,Ser), %));
> map(factor,eval(op(2,Ser), k=2*k));
```

$$\frac{1}{z^2} + \frac{-1+n^2}{z^4} + \frac{(-9+n^2)(-1+n^2)}{z^6} + \sum_{k=4}^{\infty} \frac{v(2k)}{z^{2k}},$$

$$v(2k) - (n-3+2k)(n+3-2k)v(2k-2).$$

The resulting solution *Ser* can be a rational function with the denominator z^m ($m \in \mathbb{N}$) if and only if the lowest term of the recurrence for $v(k)$ vanishes for a certain integer $k_0 > 0$. Therefore, if n is odd, then a rational solution exists, and $k_0 = \frac{n+3}{2}$ in this case. If n is even, there are no rational solutions.

7. THE MAIN RESULT

For odd n, we have

```
>         1/z^2+(-1+n^2)/z^4+(-9+n^2)*(-
1+n^2)/z^6+s
> Sum(v(2*k)/z^(2*k),k=4..(n+1)/2),
> v(2*k)-(n^2-2*k-3)^2)*v(2*k-2);
```

$$\frac{1}{z^2} + \frac{-1+n^2}{z^4} + \frac{(-9+n^2)(-1+n^2)}{z^6} + \sum_{k=4}^{\frac{n+1}{2}} \frac{v(2k)}{z^{2k}},$$

$$v(2k) - (n^2 - (-3+2k)^2)v(2k-2).$$

This rational solution can be written in the form

$$r_n = \sum_{k=1}^{\frac{n+1}{2}} \frac{\prod_{j=1}^{k-1}(n^2-(2j-1)^2)}{z^{2k}}.$$

Now, using (4), we find the integrating operator for $J_n(z)$ for odd n. It is easily verified that, if $r(z) \in \mathbb{C}(z)$, then

$$1 - rL_n = D \circ (-rz^2D - rz + r'z^2) + f, \qquad (5)$$

where $f \in \mathbb{C}(z)$. If n is odd and r is the solution r_n to the equation $L_n^*(y) = 1$, then $f = 0$; otherwise, the right-hand side of (5) would not be divisible by D on the left.

Therefore, for odd n, we have

$$\int J_n(z)dz$$
$$= -r_n(z)z^2J'_n(z) + (r'_n(z)z^2 - r_n(z)z)J_n(z) + C.$$

The result obtained using the computer algebra system can be formulated in the form of the following theorem.

Theorem. *For odd positive integers n, we have*

$$\int J_n(z)dz = -\sum_{k=0}^{\frac{n-1}{2}} \frac{\prod_{j=1}^{k}(n^2-(2j-1)^2)}{z^{2k}} J'_n(z)$$
$$+ \sum_{k=1}^{\frac{n+1}{2}} \frac{(1-2k)\prod_{j=1}^{k-1}(n^2-(2j-1)^2)}{z^{2k-1}} J_n(z) + C.$$

For even positive integers n, no antiderivative of $J_n(z)$ can be represented as a linear combination over $\mathbb{C}(z)$ of $J_n(z)$ and its derivatives.

It is not difficult to rewrite the proof of this theorem without describing the experiments and without mentioning Maple.

The formula given in the Theorem is not included in the available reference books on special functions; in particular, it is not included in [4].

To integrate $J_n(z)$ for odd positive integers n, Maple uses the following formulas presented in [4]:

$$\int J_1(z)dz = -J_0(z) + C,$$

$$\int J_n(z)dz = -J_0(z) - 2\sum_{k=0}^{\frac{n-1}{2}} J_{2k}(z) + C, \quad n = 3, 5, \ldots;$$

```
> int(BesselJ(1, z), z);
```

–*BesselJ*(0, *z*)

```
> int(BesselJ(3, z), z);
```

1 – *BesselJ*(0, *z*) – 2*BesselJ*(2, *z*)

and so on.

ACKNOWLEDGEMENTS

This work was supported in part by the Russian Foundation for Basic Research, project no. 07-01-00482a.

REFERENCES

1. Monagan, M.V., Geddes, K.O., Heal, K.M., Labahn, G., Vorkoetter, S.M., McCarron, J., and DeMarco, P., *Maple 11: Introductory Programming Guide*, Waterloo, Ontario, Canada: Waterloo Maple Inc., 2007.
2. Arnold, V.I., *Chto takoe matematika?* (What Is Mathematics?) Moscow: International Center for Continuous Mathematical Education, 2002.
3. Abramov, S.A. and Hoeij, M., Integration of Solutions of Linear Functional Equations, *Integral Transformations Special Functions*, 1999, vol. 8, no. 1-2, pp. 3–12.
4. Abramowitz, M. and Stegun, I.A., *Handbook of Mathematical Functions*, New York: Dover, 1965. Translated under the title *Spravochnik po spetsiak'nym funktsiyam s formulami, grafikami i matematicheskimi tablitsami*, Moscow: Nauka, 1979.

SPELL: 1. Maple(this