Continuation of Holomorphic Solutions of Linear Difference Equations with Polynomial Coefficients

S. A. Abramov and M. A. Barkatou

Computing Center of RAS, Limoges University

We consider a linear difference operator

$$L = a_d(z)E^d + \cdots + a_1(z)E + a_0(z), \tag{1}$$

and the corresponding linear difference equation $Ly = 0$. Here the coefficients $a_i(z)$ are polynomial over $\mathbb{C}$ and E is the "shift operator" acting on functions of the complex variable z as $Ey(z) = y(z+1)$. The difference operator L is an element of the non-commutative ring $\mathbb{C}[z, E]$. We shall call a_d the leading coefficient and a_0 the trailing coefficient of L and we shall suppose that $a_0 a_d \neq 0$. We shall call the *singularities* of both L and $Ly = 0$ the zeros of the polynomials $a_0(z)$ and $a_d(z-d)$. A point $p \in \mathbb{C}$ will be said to be *congruent* to the singularities of L if $a_0(p+\nu) = 0$ or $a_d(p-\nu) = 0$ for some non-negative integer ν.

Equation $Ly = 0$ can be used as a tool to define a sequence or a function. If we know the value $y(z)$ at every point z of a given strip $\lambda \leq \operatorname{Re} z < \lambda + d$ we can find the value of $y(z)$ in the strip $\lambda - 1 \leq \operatorname{Re} z < \lambda$, and hence in the strip $\lambda - 2 \leq \operatorname{Re} z < \lambda - 1$, and so on. We can continue $y(z)$ indefinitely to the left except at the points that are congruent to the zeros of the trailing coefficient a_0. Similarly, we can continue $y(z)$ to the rignt except at the points that are congruent to the zeros of the leading coefficient a_d of our equation. Thus, the points that are congruent to the singularities of L may present obstacles to continuing solutions of $Ly = 0$. Sometimes it is possible to avoid singularities, related to the leading or trailing coefficient of L. This applies, in particular, to the class of *desingularizable* difference operators [1]. Recall that an operator L

Russian Academy of Sciences,
Dorodnicyn Computing Centre,
Vavilova 40, 119991, Moscow GSP-1, Russia
Phone: 7 095 135 1620
Fax: 7 095 135 6159
E-mail: abramov@ccas.ru
Université de Limoges, Faculté des Sciences
Laboratoire d'Arithmétique, de Calcul formel et d'Optimisation
123, avenue Albert Thomas
F-87060 LIMOGES cedex – France
Phone: +33 (5) 55-45-73-32
Fax: +33 (5) 55-45-73-22
E-mail: moulay.barkatou@unilim.fr

is called to be desingularizable w.r.t. the trailing (resp. leading) coefficient if there exists an operator $R \in \mathbb{C}(z)[E]$ such that $T = R \circ L \in \mathbb{C}[z, E]$ and the trailing (resp. leading) coefficient of T is 1. For example, the operator $E - z$ is not desingularisable, while the operator $(z - 10)E - z$ is desingularizable since it right-divides $(E - 1)^{11}$.

One can prove that if L is desingularizable w.r.t. the trailing coefficient then any solution of L, defined and holomorphic on the half-plane $\operatorname{Re} z > A$ (here A is a real number large enough), can be continued to a holomorphic solution on $\mathbb{C}$. Our work aims to prove that the converse is also true. More precisely, we prove the following theorem

Theorem 1 *An operator L is desingularizable w.r.t. the trailing coefficient iff any solution $F(z)$ of $Ly = 0$ which is defined and holomorphic on a half-plane of the form $\operatorname{Re} z > A$, can be continued to a holomorphic solution on $\mathbb{C}$.*

The corresponding statement is valid, mutatis mutandis, for operators L that are desingularizable w.r.t. their leading coefficient.

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References

[1] Abramov S.A, van Hoeij M. Desingularization of linear difference operators with polynomial coefficients// *Proc. ISSAC'99*. Vancouver. 1999. P. 269–275.