UDC 004.4

Laurent solutions of linear ordinary differential equations with coefficients in the form of truncated power series

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Linear ordinary differential equations with formal power series coefficients represented in a truncated form are considered. We discuss the information on solutions belonging to the field of Laurent formal series which can be obtained from this representation of a given equation. We emphasize that we are interested in such information about solutions which is invariant with respect to possible prolongations of the truncated series representing the coefficients of the equation.

Key words and phrases: symbolic computation, computer algebra, differential equations, infinite power series, truncated power series, computer algebra systems.

УДК 004.4

Лорановы решения линейных обыкновенных дифференциальных уравнений с коэффициентами в виде усеченных степенных рядов

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Рассматриваются линейные обыкновенные дифференциальные уравнения с коэффициентами в виде усеченных формальных степенных рядов. Обсуждается вопрос о том, что можно узнать из заданного таким образом уравнения о его решениях в поле формальных рядов Лорана. При этом нас интересует информация об этих решениях, инвариантная относительно возможных продолжений тех усеченных рядов, которыми представлены коэффициенты уравнения.

Ключевые слова: символьные вычисления, компьютерная алгебра, дифференциальные уравнения, бесконечные степенные ряды, усеченные степенные ряды, системы компьютерной алгебры.

1. Introduction

Power and Laurent series play a very important role in the theory and application of differential equations. In particular, the coefficients of a linear ordinary differential equation are often represented by series, and the problem can be to find such solutions to this equation which are series of some fixed kind.

Below, the formal power series will be used as coefficients of a given equation, and, the coefficients of these series themselves, will be elements of a given differential field K of characteristic 0. The solutions we are interested in, will belong to the field of formal

Laurent series over K. Such solutions we will call *Laurent solutions.* We will not be interested in the questions of convergence of series.

The algorithmic aspect of problems of this kind involves consideration of the question of the representation of infinite series, in particular, of series, playing the role of the coefficients of the equation. In the works [1–3] the algorithmic representation was considered: the series $\sum a_n x^n$ is represented by an algorithm that calculates a_n by a given n. It was found that some problems related to the solutions of the equations given in this way are algorithmically unsolvable, but, at the same time, the other part is successfully solvable. In particular, the problem of finding Laurent solutions turns out to be solvable. (In the mentioned papers, not only individual scalar equations were discussed, but also systems of equations.)

A number of problems of constructing solutions were also considered, on the assumption that series playing the role of coefficients of a given equation or system are presented in an "approximate", namely, in *truncated* form. For example, in [4] it is established what a truncation of the coefficients of the system is enough to calculate a given number of initial terms of the series that are included in the exponential-logarithmic solutions of the system. In [1] this problem is considered for constructing truncated Laurent solutions. In the present paper, for such solutions we indicate how many coefficients of a built Laurent solution can be trusted regardless of those coefficients of the original series, which disappeared during the truncation. We are interested in information on the solutions of the equation, invariant with respect to possible prolongations of the initially given truncated series representing the coefficients of the equation. The algorithm proposed by us obtains the maximal possible number of terms of Laurent solutions, which is guaranteed in this sense.

Details of the problem statement, see below in Section 2. The proposed algorithm is implemented in the Maple [5]. The implementation and some experiments are described in Section 7.

2. Formulation of the problem

First, introduce some notation. Let K be a field of characteristic 0 and $K[x]$ the ring of polynomials with coefficients in K. We denote by $K[[x]]$ the ring of formal power series with coefficients in K and $K((x))$ its quotient field; the elements of $K((x))$ are *Laurent series.* For a nonzero element $a(x) = \sum a_i x^i$ of $K((x))$ the *valuation* $\operatorname{val} a(x)$ is defined by $\operatorname{val} a(x) = \min\{i \mid a_i \neq 0\}$. By convention $\operatorname{val} 0 = \infty$. Let $l \in \mathbb{Z} \cup \{-\infty\}$, the *l-truncation* $a^{\langle l \rangle}(x)$ is obtained by vanishing all the coefficients of the terms of degree larger than l in the series $a(x)$; if $l = -\infty$ then $a^{\langle l \rangle}(x) = 0$.

Further (K, D) is a differential field of characteristic 0. We will consider differential operators and equations written using the notation $\theta = xD$. We assume that in the original operator

$$L = \sum_{i=0}^{r} a_i(x)\theta^{\,i} \in K[x][\theta], \tag{1}$$

coefficients are

$$a_i(x) = \sum_{j=0}^{t_i} a_{ij}x^j, \;\; t_i \geq \deg a_i(x), \;\text{ for } i = 0, 1, \ldots, r$$

(if $t_i > d_i = \deg a_i$ then $a_i = 0$ for $i = d_i + 1, d_i + 2, \ldots, t_i$). It is assumed that the constant term of at least one of the polynomials $a_0(x), \ldots, a_r(x)$ is non-zero.

Definition 1 *Let L have the form* (1)*, the polynomial $a_r(x)$ (*the leading coefficient *of the differential operator L) is assumed to be nonzero.* A prolongation *of the operator*

L we will call any operator $\tilde{L} = \sum_{i=0}^{r} b_i(x)\theta^{\,i} \in K[[x]][\theta]$ for which

$$b_i(x) - a_i(x) = O(x^{t_i+1})$$

(i.e. $\operatorname{val}(b_i(x) - a_i(x)) > t_i$*), $i = 0, 1, \ldots, r$.*

An algorithm will be proposed whose input is an operator $L \in K[x][\theta]$ and non-negative integers $t_0, t_1, \ldots, t_r$. As a result of applying the algorithm, two finite sets of integers becomes known

$$W = \{v_1, \ldots, v_k\}, \quad M = \{m_1, \ldots, m_k\}$$

with the following properties:

- For each $v_i \in W$ there is a solution $y(x)$, $\operatorname{val} y(x) = v_i$, of the operator L.
- If $\tilde{y}(x)$ is a solution of some prolongation $\tilde{L}$ of L and $\operatorname{val} \tilde{y}(x) = v_i$, $1 \le i \le k$, then there is a solution $y(x)$ of the operator L, for which $\tilde{y}(x) - y(x) = O(x^{m_i})$. The latter equality can be rewritten as

$$\tilde{y}(x) = y^{\langle m_i - 1\rangle}(x) + O(x^{m_i}). \qquad (2)$$

- If $y(x)$ is a solution of L such that $\operatorname{val} y(x) = v_i$, $1 \le i \le k$, then for any prolongation $\tilde{L}$ of the operator L there exists its solution $\tilde{y}(x)$, for which (2) holds.
- The values of m_i are the largest of the possible values associated with L in this way.

The sets W, M contain all the values v_i, m_i, for which these conditions are satisfied.

If we know the space of Laurent solutions of an operator L (this operator has polynomial coefficients; how to find its Laurent solutions see [6, 7]) and the sets W, M, then we have, thereby, a complete list of valuations of solutions and formulas of the form (2) invariant with respect to the prolongations of the operator L.

Remark 1 This task is directly related to the representation of infinite series, addressed in the introduction: the differential equation is given as

$$\left(a_r(x) + O(x^{t_r+1})\right)\theta^{\,r} y + \cdots + \left(a_1(x) + O(x^{t_1+1})\right)\theta y + \left(a_0(x) + O(x^{t_0+1})\right) y = 0, \quad (3)$$

$t_i \ge \deg a_i(x)$, $i = 0, 1, \ldots, r$. We associate the operator (1) with it, as well as the set of numbers $t_r, \ldots, t_0$, and solve the problem of finding $W = \{v_1, \ldots, v_k\}$, the set $M = \{m_1, \ldots, m_k\}$ and the formulas (2).

A prolongation of the operator (1) will be called also a *prolongation of the equation* (3).

The presentation of the results of the algorithm is discussed in Section 7.

3. Sequences of coefficients of Laurent solutions

Let σ denote the shift operator: $\sigma c_n = c_{n+1}$ for any sequence (c_n). The mapping

$$x \to \sigma^{-1}, \quad \theta \to n$$

transforms an original differential equation

$$\sum_{i=0}^{r} a_i(x)\theta^{\,i} y = 0 \qquad (4)$$

into the *induced* recurrent equation $u_0(n)c_n + u_{-1}(n)\sigma^{-1}c_n + \cdots = 0$ or equivalently

$$u_0(n)c_n + u_{-1}(n)c_{n-1} + \cdots = 0, \qquad (5)$$

where

- $(c_n)_{-\infty<n<\infty}$ is an unknown sequence such that $c_n = 0$ for all negative integers n with $|n|$ large enough.
- $u_0(n), u_{-1}(n), \dots \in K[n]$, each of the polynomials is of degree less than or equal to r.
- $u_0(n)$ is a non-zero polynomial, it is called the *leading* coefficient of the equation (5). Note that, by our supposition, the constant term of at least one of the polynomials $a_0(x), \dots, a_r(x)$ is non-zero. This implies not only that $u_0(n)$ is a non-zero polynomial, but also that it does not depend on a prolongation of the original operator L.

An equation of the form (4) has a Laurent solution $y(x) = c_v x^v + c_{v+1}x^{v+1} + \dots$ iff the double-sided sequence

$$\dots, 0,\ 0,\ c_v,\ c_{v+1},\ \dots$$

of coefficients of $y(x)$ satisfies the induced recurrent equation of the form (5), i.e.,

$$\begin{aligned}
&u_0(v)c_v = 0,\\
&u_0(v+1)c_{v+1} + u_{-1}(v+1)c_v = 0,\\
&u_0(v+2)c_{v+2} + u_{-1}(v+2)c_{v+1} + u_{-2}(v+2)c_v = 0,\\
&\dots
\end{aligned}$$

(the proof is given in [8]).

The leading coefficient $u_0(n)$ can be considered as a kind of the *indicial polynomial* of the original differential equation (4). The set of the integer roots of $u_0(n)$ is finite and contains all possible valuations of Laurent solutions of (4).

4. Additional relations for computed coefficients of the series

If $u_0(n) \neq 0$ for some integer n then (5) allows to find c_n by $c_{n-1}, c_{n-2}, \dots$. If $u_0(n) = 0$ then at such a moment we declare (possibly temporarily) c_n as an *undetermined* coefficient that is included in the lined up solution. It turns out that the preceding values of $c_{n-1}, c_{n-2}, \dots$, which we would like to look at as finally found, should satisfy the relation

$$u_{-1}(n)c_{n-1} + u_{-2}(n)c_{n-2} + \dots = 0. \tag{6}$$

Such relation has only finite number of non zero terms and will probably eliminate some of the previously undetermined coefficients. Only after incrementing the value of n does this value exceed the largest integer root of the equation $u_0(n) = 0$, there is a guarantee that both new undetermined coefficients and relations of the form (6) will not arise already.

5. Foundation of the algorithm

If the polynomial $u_0(n)$ does not have integer roots then none of the prolongations of the original differential equation has solutions in $K((x))$. The algorithm reports this and stops.

If there are integer roots $\alpha_1 < \dots < \alpha_s$ then for L and its prolongations, only for α_s, the existence of a Laurent solution with such a value is guaranteed. We need to deal with $\alpha_1, \dots, \alpha_{s-1}$. For each of these roots, there are three possibilities:

(a) Laurent solutions exist for all prolongations;
(b) Laurent solutions exist for some, but not for all prolongations; — depending on the specific prolongation;
(c) Laurent solutions do not exist for any prolongation.

By adding symbolic coefficients (not originally specified), it is possible to determine for which α_i we are considering which of these three possibilities takes place. If it is (a), then we find the corresponding m. If it is (b) or (c), then we remove α_i from consideration.

As a result, the set W is determined: $W = \{v_1, \dots, v_k\}$. For each of these valuations v_i, the corresponding m_i will be determined as well. So, we have the additional set $M = \{m_1, \dots, m_k\}$ of integers, for them, the corresponding relations of the form (2) are satisfied.

If we consider solutions that have a valuation α_l, $l < s$, then, using the induced recurrence equation, it is necessary to advance in constructing a solution at least to x^{α_s}, even if already before the moment, in the coefficients of the lined up series appeared symbolic *unspecified coefficients* of the prolongation. Using the induced recurrence equation with n be equal to one of the integer roots of its largest coefficient, gives a linear relation between the already obtained coefficients of the series, symbolic or belonging to K — see Section 4. If one reaches the largest integer root and at the same time found out what are these relations somehow limit the choice of a prolongation (prevents arbitrary choice), then we exclude from consideration solutions of valuation α_l. If the relation does not limit the choice of a prolongation (for example, all the relations only express the existing indefinite coefficients in terms of the added unspecified coefficients), then choose from the constructed segment its initial terms in accordance with the formula (2) with $v_i = \alpha_l$.

Now we can construct a basis for the space of truncated Laurent solutions of valuation greater than or equal to v_1 in the form of a finite set segments of series, which is represented as the union of k subsets; the ith subset, $1 \le i \le k$, consists of truncated solutions, of valuation v_i, and the truncated series included in some fixed subset are linearly independent over the field of constants.

6. Steps of the algorithm

Let's follow the steps of the proposed algorithm using the operator $L = -\theta^2 - 2\,\theta$ with $t_0 = t_1 = t_2 = 0$ as an example. The construction of the induced recurrent equation allows obtaining its indicial polynomial $u_0(n) = -n^2 - 2n$, the set $\{-2, 0\}$ of its integer roots contains all the valuations of the Laurent solutions of the equation $L(y) = 0$. Polynomials $u_i(n)$, $i = -1, -2, \dots$ involve symbolic unspecified coefficients, which take place in the prolongation of the operator L.

We compute the solution coefficients for the possible valuation -2, starting from c_{-2}.

- $n = -2$: $u_0(-2)\,c_{-2} = 0 \cdot c_{-2} = 0$, the coefficient c_{-2} remains undetermined.
- $n = -1$: $u_0(-1)\,c_{-1} + u_{-1}(-1)\,c_{-2} = c_{-1} + u_{-1}(-2)\,c_{-2} = 0$. It gives $c_{-1} = -u_{-1}(-1)\,c_{-2}$.
- $n = 0$: $u_0(0)\,c_0 + u_{-1}(0)\,c_{-1} + u_{-2}(0)\,c_{-2} = 0\,c_0 + u_{-1}(0)\,c_{-1} + u_{-2}(0)\,c_{-2} = -u_{-1}(0)\,u_{-1}(-1)\,c_{-2} + u_{-2}(0)\,c_{-2} = (-u_{-1}(0)\,u_{-1}(-1) + u_{-2}(0))\,c_{-2} = 0$. The coefficient c_0 remains undetermined, and it appears that if $-u_{-1}(0)\,u_{-1}(-1) + u_{-2}(0) \neq 0$ then $c_{-2} = 0$. As soon as $u_{-1}(n)$ and $u_{-1}(n)$ depends on unspecified coefficients, the case (b) is faced here, i.e. Laurent solutions with the valuation -2 exist only if $-u_{-1}(0)\,u_{-1}(-1) + u_{-2}(0) = 0$. This valuation is removed.

We compute the solution coefficients for the possible valuation 0, starting from c_0.

- $n = 0$: $u_0(0)\,c_0 = 0 \cdot c_0 = 0$, the coefficient c_0 remains undetermined.
- $n = 1$: $u_0(1)\,c_1 + u_{-1}(1)\,c_0 = -3\,c_1 + u_{-1}(1)\,c_0 = 0$. It gives $c_1 = \frac{u_{-1}(1)\,c_0}{3}$ with $u_{-1}(1)$ and hence c_1 being depended on the unspecified coefficient. $n = 0$ corresponding to the maximal possible valuation is passed, so no further computation is needed.

Thus, it is obtained that $W = \{0\}$, $m_1 = 1$ for the equation $L(y) = 0$. Any prolongation of the equation has the following Laurent solution:

$$y(x) = C + O(x),$$

where C is an arbitrary constant.

Consider the following prolongation of the operator L as one more example:

$$\tilde{L} = (-1 + x + x^2)\,\theta^2 - 2\,\theta,$$

$t_0 = 3$, $t_1 = t_2 = 2$. The construction of the induced recurrent equation for $\tilde{L}$ gives

$$u_0(n) = -n^2 - 2n,\ u_{-1}(n) = (n-1)^2,\ u_{-2}(n) = (n-2)^2.$$

All the subsequent $u_i(n)$ involve symbolic unspecified coefficients already, with $u_{-3}(n) = a(n-3)^2 + b(n-3)$, where a and b are such unspecified coefficients.

The computation of the solution coefficients for the possible valuation -2 is similar to the computation for L, but for $n = 0$ we have $-u_{-1}(0)\,u_{-1}(-1) + u_{-2}(0) = -(0-1)^2(-1-1)^2 + (0-2)^2 = 0$, which means that the coefficient c_{-2} remains undetermined, and correspondingly Laurent solutions with the valuation -2 exist for any prolongation of $\tilde{L}$. The case (a) is faced here. It follows that $c_{-1} = -u_{-1}(-1)\,c_{-2} = -(-1-1)^2\,c_{-2} = -4\,c_{-2}$. The further computation is not needed, since $n = 0$ corresponding to the maximal possible valuation is passed, with the expressions for the subsequent solution coefficients being depended on the unspecified coefficient. So, any prolongation of the equation $\tilde{L}(y) = 0$ has the following Laurent solution with the valuation -2:

$$\frac{C_1}{x^2} - \frac{4C_1}{x} + C_2 + O(x),$$

where C_1, C_2 are arbitrary constants.

We compute the solution coefficients for the second possible valuation 0, starting from c_0.

- $n = 0$: $u_0(0)\,c_0 = 0 \cdot c_0 = 0$, the coefficient c_0 remains undetermined.
- $n = 1$: $u_0(1)\,c_1 + u_{-1}(1)\,c_0 = -3\,c_1 + 0\,c_0 = 0$. It gives $c_1 = 0$.
- $n = 2$: $u_0(2)\,c_2 + u_{-1}(2)\,c_1 + u_{-2}(2)\,c_0 = -8\,c_2 + 1\,c_1 + 0\,c_0 = 0$. It gives $c_2 = 0$.
- $n = 3$: $u_0(3)\,c_3 + u_{-1}(3)\,c_2 + u_{-2}(3)\,c_1 + u_{-3}(3)\,c_0 = -15\,c_3 + 4\,c_2 + 1\,c_1 + 0\,c_0 = 0$. It gives $c_3 = 0$. Note, that c_3 is happened to be computed in this case in spite of the fact that $u_{-3}(n)$ involves unspecified coefficients already. It happens since $u_{-3}(3) = 0$ for any values of the unspecified coefficients.

The further computation is not needed, since $n = 0$ corresponding to the maximal possible valuation is passed, with the expressions for the subsequent solution coefficients being depended on the unspecified coefficient. So, any prolongation of the equation $\tilde{L}(y) = 0$ has the following Laurent solution with the valuation 0:

$$C + O(x^4).$$

Thus, it is obtained that $W = \{-2, 0\}$, $m_1 = 1$, $m_2 = 4$ for the equation $\tilde{L}(y) = 0$. If we consider another prolongation of the operator L

$$\tilde{\tilde{L}} = (-1 + x + x^2)\,\theta^2 + (-2 + x^2)\,\theta,$$

$t_0 = 4$, $t_1 = t_2 = 2$, then the similar computation gives that $W = \{0\}$, $m_1 = 4$ for the equation $\tilde{\tilde{L}}(y) = 0$. Any prolongation of the equation has the following Laurent solution: $C + O(x^4)$. The case (c) is faced here, which leads to removing the possible valuation -2.

Proposition 1 *Let the values $v_1, \dots, v_k$, $m_1, \dots, m_k$ be found by the proposed algorithm for an equation $L(y) = 0$, where L has the form (1) and $t_i \geq \deg a_i$, $i = 0, 1, \dots, r$. Let m be such a positive integer that $m > m_i$ for some $1 \leq i \leq k$. Then for $L(y) = 0$ there exists a prolongation $\tilde{L}(y) = 0$ such that for some of its solutions $\tilde{y}(x) \in K((x))$, $\operatorname{val} \tilde{y}(x) = v_i$, the equality $\tilde{y}(x) = y^{\langle m-1 \rangle}(x) + O(x^m)$ does not hold for any solution $y(x) \in K((x))$, $\operatorname{val} y(x) = v_i$ of the equation $L(y) = 0$.*

7. Implementation and usage examples

The algorithm is implemented as the procedure `LaurentSolution` in Maple environment. The procedure takes a differential equation as its first parameter. The application of θ^k to the unknown function $y(x)$ is specified as `theta(y(x),x,k)`. The truncated coefficients of the equation are given as $a_i(x) + O(x^{t_i+1})$, where $a_i(x)$ is a polynomial of the degree not greater than t_i. The unknown function is given as the second parameter of the procedure.

The procedure returns the list of the truncated Laurent solutions, which correspond to valuations $v_i \in W$. Each element of the list is represented as

$$c_{v_i} x^{v_i} + c_{v_i+1} x^{v_i+1} + \dots + c_{m-1} x^{m_i-1} + O(x^{m_i}),$$

where $v_i \in W$ is a valuation for which it is guaranteed that there exists Laurent solution for any prolongation of the given equation; m_i has the same meaning. Each c_j is the computed coefficient of Laurent solution, the coefficient might be a linear combination of the arbitrary constants of the form $\_c_k$.

Now consider two equations

$$\sin(x)\, \theta y - x \cos(x)\, y = 0 \tag{7}$$

and

$$(\exp(x) - 1)\, \theta y - x \exp(x)\, y = 0. \tag{8}$$

Both the equations are represented as

$$(x + O(x^2))\, \theta y - (x + O(x^2))\, y = 0. \tag{9}$$

We apply the implemented procedure to (9):

```
> eq := (x+O(x^2))*(theta(y(x),x,1))-(x+O(x^2))*y(x);
```

$$eq := \left(x + O(x^2)\right) \theta(y(x), x, 1) - \left(x + O(x^2)\right) y(x)$$

```
> LaurentSolution(eq, y(x));
```

$$[x\_c_1 + O(x^2)]$$

The answer means that $W = \{1\}$, $m_1 = 2$.

Consider the prolongation of the equation (9) in accordance with (7). It gives the prolongation of the truncated solution up to the degree x^2 that corresponds to the series expansion of the function $\sin(x)$, the function is the solution of (7):

```
> eq1 := (x+O(x^3))*(theta(y(x),x,1))-(x-x^3/2+O(x^4))*y(x);
```

$$eq1 := \left(x + O(x^3)\right) \theta(y(x), x, 1) - \left(x - \frac{x^3}{2} + O(x^4)\right) y(x)$$

```
> LaurentSolution(eq1, y(x));
```

$$[x\_c_1 + O(x^3)]$$

The answer also means that $W = \{1\}$, $m_1 = 3$.

Now consider the prolongation of the equation (9) in accordance with (8). It gives the prolongation of the truncated solution up to the degree x^2 that corresponds to the series expansion of the function $\exp(x) - 1$, the function is the solution of (8).

```
> eq2 := (x+x^2/2+O(x^3))*theta(y(x),x,1)-(x+x^2+x^3/2+O(x^4))*y(x);
```

$$eq2 := \left(x + \frac{x^2}{2} + O(x^3)\right)\theta(y(x), x, 1) - \left(x + x^2 + \frac{x^3}{2} + O(x^4)\right) y(x)$$

```
> LaurentSolution(eq2, y(x));
```

$$\left[x\_c_1 + \frac{x^2\_c_1}{2} + O(x^3)\right]$$

The answer also means that $W = \{1\}$, $m_1 = 3$.

All the solutions have only one valuation for which there is Laurent solution for any prolongation of the equation.

Finally consider one more example of the procedure application to the equation that corresponds to the operator $\tilde{L}$ from Section 6.

```
> eq3 := (-1+x+x^2+O(x^3))*theta(y(x),x,2)+(-2+O(x^3)*theta(y(x),x,1)+
         O(x^4)*y(x);
```

$$eq3 := \left(-1 + x + x^2 + O(x^3)\right)\theta(y(x), x, 2) + \left(-2 + O(x^3)\right)\theta(y(x), x, 1) + O(x^4)y(x)$$

```
> LaurentSolution(eq3, y(x));
```

$$\left[\frac{\_c_1}{x^2} - \frac{4\_c_1}{x} + \_c_2 + O(x), \_c_1 + O(x^4)\right]$$

The answer means that $W = \{-2, 0\}$, $m_1 = 1$, $m_2 = 4$.

Acknowledgments

The work is partially supported by RFBR grant No 19-01-00032.

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