# Subanalytic Solutions of Linear Difference Equations and Multidimensional Hypergeometric Sequences


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A doubly infinite complex sequence $(c_n)$, $n \in \mathbb{Z}$, is a *sequential* solution of a difference equation of the form

$$a_d(z)y(z+d) + \cdots + a_1(z)y(z+1) + a_0(z)y(z) = 0, \qquad (1)$$

$a_0(z), \ldots a_d(z) \in \mathbb{C}[z]$, $a_0(z), a_d(z) \in \mathbb{C}[z] \setminus \{0\}$, if

$$a_d(n)c_{n+d} + \cdots + a_1(n)c_{n+1} + a_0(n)c_n = 0,$$

for all $n \in \mathbb{Z}$.

A sequential solution $(c_n)$ of (1) is *subanalytic* if equation (1) has a solution in the form of a single-valued analytic funcion $f : \mathbb{C} \to \mathbb{C}$ such that $c_n = f(n)$ for all $n \in \mathbb{Z}$.

We show that the dimension of the $\mathbb{C}$–linear space of all sequential solutions of (1) is always at least $d$, and that for any integer $m \geq d$ there exists an equation of the form (1) of order $d$ such that this dimension is equal to $m$. However the space of subanalytic solutions of an equation (1) of order $d$ has always dimension $d$.

If $d = 1$, then a sequential solution of (1) is a *hypergeometric* sequence. We also consider $s$–dimensional ($s \geq 1$) hypergeometric sequences, *i.e.*, sequential, *resp.*, subanalytic solutions of consistent systems of first-order difference equations for a single unknown function:

$$f_i(z_1, \ldots, z_s)y(z_1, \ldots, z_{i-1}, z_i + 1, z_{i+1}, \ldots, z_s) = g_i(z_1, \ldots, z_s)y(z_1, \ldots, z_s), \qquad (2)$$

where $(z_1, \ldots, z_s) \in \mathbb{C}^s$, and $f_i, g_i$ are non-zero polynomials which are relatively prime for each $i \in \{1, 2, \ldots, s\}$, and satisfy

$$\begin{aligned} &\frac{g_i(z_1,\ldots,z_s)}{f_i(z_1,\ldots,z_s)} \frac{g_j(z_1,\ldots,z_{i-1},z_i+1,z_{i+1},\ldots,z_s)}{f_j(z_1,\ldots,z_{i-1},z_i+1,z_{i+1},\ldots,z_s)} \\ &= \frac{g_j(z_1,\ldots,z_s)}{f_j(z_1,\ldots,z_s)} \frac{g_i(z_1,\ldots,z_{j-1},z_j+1,z_{j+1},\ldots,z_s)}{f_i(z_1,\ldots,z_{j-1},z_j+1,z_{j+1},\ldots,z_s)} \end{aligned}$$

for all $i, j \in \{1, 2, \ldots, s\}$. We show that the dimension of the space of subanalytic solutions is always at most 1, and that this dimension may be equal to 0 for some systems (although the dimension of the space of all sequential solutions is always positive).

Subanalytic solutions have applications in computer algebra. We show that some implementations of certain well-known summation algorithms (Gosper, Zeilberger, Accurate Summation) in existing computer algebra systems work correctly when the input sequence is a subanalytic solution of an equation or a system, but can give incorrect result for some sequential solutions.