On d'Alembert substitution

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Abstract

Let some homogeneous linear ordinary differential equation with coefficients in a differential field F be given. If we know a nonzero solution φ, then the order of the equation can be reduced by d'Alembert substitution $y = \varphi \int v\,dx$, where v is a new unknown function. In the situation when $\varphi \in F$, after d'Alembert substitution an equation with coefficients in F arises again. Let the obtained equation have a nonzero solution $\psi \in F$, then it is possible to reduce the order of the equation again and so on, until an equation without nonzero solutions in F is obtained.

If we can find solutions not only in F but in some larger set L as well (L can be a field or a linear space), then we can build up a certain subspace M (d'Alembertian subspace) of the space of all solutions of the original equation. Thus if we have algorithms A_F and A_L to search for the solutions in F and L, then by incorporating d'Alembert substitution we can design a more general algorithm (in case $L = F$ we will obtain a more general algorithm than A_F). We would like, certainly, to know the kind of solutions that can be found by the new algorithm. The construction of the subspace M will be described in the paper.

Additionally we propose an algorithm which answers if an expression $f_1 \int f_2 \ldots f_{k-1} \int f_k\,dx \ldots dx$ with one or other choice of the primitive function of the integrand is a solution of the original equation. If the answer is affirmative then the algorithm rewrites given expression in the form of one of the same structure, but the new expression gives a solution for any choice of its primitive functions of the integrands.

We consider a similar problem for the case when $f_1, f_2, \ldots$ need not be in F, but $f_1'/f_1, f_2'/f_2, \ldots \in F$.

Together with differential equations difference equations will be considered.

1 D'Alembertian solutions

Computer algebra algorithms that find exact solutions of homogeneous linear ordinary differential equations have as input, in general, an equation with coefficients in some fixed differential field F:

$$a_n y^{(n)} + \cdots + a_0 y = 0 \quad (a_n, \cdots, a_0 \in F). \tag{1}$$

These algorithms return a basis of the space of solutions which are in some fixed set L. The set L can be a field or a linear space. Henceforth we will assume that F and L are closed under differentiation, that $F \subset L$ and, additionally, that $a \in F,\ b \in L$ imply that $ab \in L$. It is possible that $L = F$.

Examples of such algorithms include those which for equations with rational coefficients find a basis of the space of algebraic solutions, or a basis of the space of solutions which are linear combinations of functions with rational logarithmic derivative [Bron92].

Let A_L be an algorithm to search for solutions in L. Often together with A_L a simpler algorithm A_F which finds solutions in F is known. For example, a fast algorithm to search for rational solutions of equations with rational function coefficients is known [Abr89a, Abr89b, Abr&Kva91]. In this case one tries first to find solutions belonging to F, and to reduce the order of the equation while keeping the coefficients in F. If a nonzero solution $\varphi \in F$ of (1) is known then the substitution

$$y = \varphi \int u\, dx \tag{2}$$

where u is a new unknown function transforms (1) to an $(n-1)$-order equation with coefficients in F. The substitution (2) we will call *d'Alembert substitution*, connected with the solution φ (the order reducing technique based on this substitution was essentially known to and used by d'Alembert).

Suppose that the following transformations have been applied to some equation of the form (1). The d'Alembert substitution connected with some solution $\eta_1 \in F$ of the equation has been applied to reduce the order of the equation; then the order of the new equation has been reduced using its solution $\eta_2 \in F$ and so on, until the last d'Alembert substitution connected with $\eta_r \in F$ produces the equation

$$b_{n-r} w^{(n-r)} + b_{n-r-1} w^{(n-r-1)} + \cdots + b_0 w = 0 \tag{3}$$

(w is an unknown function) which has no nonzero solution in F. (If we start with a basis $\varphi_1, \ldots, \varphi_t$ of the space of solutions belonging to F, then on the first step of reduction of order we can use $\eta_1 = \varphi_1$, on the second step $\eta_2 = (\varphi_2/\varphi_1)'$, on the third one $\eta_3 = ((\varphi_3/\varphi_1)'/(\varphi_2/\varphi_1)')'$ and so on. When $\varphi_1, \ldots, \varphi_t$ are used up we can find for the new equation a basis of the space of solutions belonging to F, and so on.) Let $\tau_1, \ldots, \tau_s,\ s \geq 0$ be a basis of the space of solutions of the equation (3) which are in L. Then the result of all this is that some subspace M of the space of all solutions of the original equation $Ay = 0$ has been found. This subspace is generated by

$$\eta_1, \eta_1 \int \eta_2\, dx\,, \ldots, \eta_1 \int \eta_2 \ldots \int \eta_r\, dx \ldots dx\,,$$

$$\eta_1 \int \eta_2 \ldots \int \eta_r \int \tau_1 \, dx \; dx \ldots dx \, , \ldots \tag{4}$$
$$\ldots , \eta_1 \int \eta_2 \ldots \int \eta_r \int \tau_s \, dx \; dx \ldots dx \, .$$

In these formulae each indefinite integral is understood in the sense of one unique primitive function of the integrands. But if we understand each indefinite integral as the set of all primitive functions of the integrand then one can describe the set M as the subspace generated by

$$\eta_1 \int \eta_2 \ldots \int \eta_r \int \tau \, dx \; dx \; \ldots \; dx \, , \tag{5}$$

where τ is a linear combination of $\tau_1, \ldots, \tau_s$ (different primitive functions and linear combinations give rise to different solutions of the original equation).

Let the equation (3) have no nonzero solution in L. Then only 0 can be used as τ in (5). An equivalent form for (5) in this case will be

$$C \, \eta_1 \int \eta_2 \ldots \int \eta_r \, dx \; \ldots \; dx \, ,$$

where C is an arbitrary constant.

At least two questions arise here:

1. Does M depend on the choice of $\eta_1, \ldots, \eta_r$? The original equation could have together with φ_1 a solution $\psi \in F$ linearly independent of φ_1. The order of the original equation could be reduced by ψ and so on.

2. If the answer to the previous question is affirmative, how can we characterize M without recourse to $\eta_1, \ldots, \eta_r$?

An answer to the first question was given in [Abr91]: M is independent of the choice of $\eta_1, \ldots, \eta_r$ if (3) has no nonzero solutions in F.

It was shown in [Abr91] also that M contains all solutions *placed above* L. A function $y(x)$ is placed above the linear space L if there exists an integer $k \geq 0$ such that $y^{(k)} \in L$ (we assume that $y^{(0)} = y$); the least k of this kind is called *the height* of the function y above L. However, M can contain solutions of other kind as well. We show below that this subspace contains all solutions of the form

$$f_1 \int f_2 \ldots \int f_k \, dx \; \ldots \; dx \tag{6}$$
$$(f_1, \ldots, f_{k-1} \in F, \;\; f_k \in L),$$

with some concrete choice of primitive functions of the integrands. It is possible here that f_1 is not a solution of the original equation, and so on. Note that a solution placed above L is of the form $1 \int 1 \ldots 1 \int f \, dx \; \ldots \; dx$ where $f \in L$. We will call solutions that can be represented in the form (6) with some concrete primitive functions of the integrands, *d'Alembertian solutions*.

If f_1 in (6) is a solution of the original equation, f_2 is a solution of that equation which is obtained from the original equation by d'Alembert substitution connected with f_1, and so on, then we say that the solution (6) is given in *normal form*. Since M does not depend on the choice of $\eta_1, \ldots, \eta_r$, every solution having normal form is in M.

So, we are going to show that M consists exactly of d'Alembertian solutions. For this purpose we will prove that each d'Alembertian solution can be represented in normal form.

Note that all the propositions stated here are valid also for homogeneous linear difference equations after derivatives are replaced with differences (we take $\Delta f(x) = f(x+1) - f(x)$), integrals with sums etc. The expression $\sum f(x)$ means the set of all functions $g(x)$ such that $\Delta g(x) = f(x)$, this indefinite sum can be concretized by the choice of one function with this property.

2 Equation for the derivative

Let an equation $Ay = 0$ of the form (1) be given. We transform it as follows. If $a_0 \neq 0$ then divide the equation by a_0 to obtain an equation of the form

$$h_n y^{(n)} + \ldots + h_1 y' + y = 0 \quad (h_i = a_i/a_0,\ i = 1, \ldots, n) \tag{7}$$

differentiate both sides of this equation and replace $y', y'', \ldots$ with $y, y', \ldots$ which gives an n-th order equation:

$$\begin{aligned} h_n y^{(n)} + (h_n' + h_{n-1}) y^{(n-1)} + \ldots + & \\ (h_2' + h_1) y' + (h_1' + 1) y &= 0. \end{aligned}$$

If originally $a_0 = 0$, then the equation has the form $a_n y^{(n)} + \cdots + a_1 y' = 0$, and we can directly replace $y', y'', \ldots$ with $y, y', \ldots$. This gives an equation of order $n - 1$:

$$a_n y^{(n-1)} + \cdots + a_2 y' + a_1 y = 0.$$

This transformation will be called *passage to the equation for the derivative*. The equation for the derivative will be denoted by $A^{[1]} y = 0$.

In [Abr91] the following lemma was proved.

Lemma 1 *Let $A\,y = 0$ be the equation (1) and ψ some solution of the equation $A^{[1]} y = 0$. Then $Ay = 0$ has a solution φ such that $\varphi' = \psi$. If the coefficient of y in the equation $A\,y = 0$ is 0 then*

$$\varphi = \int \psi \, dx$$

(any primitive function can be taken), otherwise the solution φ is unique:

$$\varphi = -\sum_{i=0}^{n-1} h_{i+1}\psi^{(i)}, \tag{8}$$

where $h_1, \ldots, h_n$ are determined by formula (7).

We remark that the right-hand side of (8) is a primitive function of ψ and can be rewritten as $\int \psi\, dx$; here the indefinite integral is this particular primitive function.

This lemma allows us to define the *transformation of an equation connected with a nonzero function f*. This transformation is a generalization of d'Alembert substitution. In the case when f is a solution of the equation under consideration, the transformation connected with f is simply d'Alembert substitution connected with f. Otherwise, the substitution $y = fu$ where u is a new unknown function, transforms an equation $Ay = 0$ of the form (1) to another equation $Bu = 0$, which we further transform to $B^{[1]}u = 0$. One can construct a solution of the form $f \int \psi\, dx$ of the equation $Ay = 0$ for a solution ψ of the transformed equation. If f is a solution of $Ay = 0$ then a primitive function can be chosen arbitrarily, otherwise the choice is unique, according to Lemma 1.

In the transformed equation we will use the same letter for the unknown function as in the original equation (the letter y).

In [Abr91] it has been shown that if an equation of the form (1) has a solution placed above L at positive height then the equation has to have a nonzero polynomial solution.

Here we prove the following lemma.

Lemma 2 *Let the equation $Ay = 0$ of the form (1) have a d'Alembertian solution $y \notin L$ of the form (6) with some concrete primitive functions of the integrands. Then there exists an $l < k$ such that*

1. *a special choice of primitive functions in*

$$f_1 \int f_2 \ldots \int f_l\, dx\, \ldots\, dx \tag{9}$$

   gives a solution of the original equation;

2. *all the functions*

$$f_{l-1} \int f_l\, dx\,,\ f_{l-2} \int f_{l-1} \int f_l\, dx\ dx\,, \\ \ldots, f_1 \int f_2 \ldots \int f_l\, dx\, \ldots\, dx \tag{10}$$

   are in F for this choice of primitive functions.

Proof. We can apply to $Ay = 0$ the transformation connected with f_1, then apply to the obtained equation the transformation connected with f_2, and so on. We claim that eventually we will obtain an equation of order $(n-1)$. Assume the contrary. After transformation connected with f_{k-1}, an equation with solution f_k will arise. Since the order of any intermediate equation is n, then, in the case when $f_1, f_2, \ldots$ are fixed, the original equation has a unique solution of the form (6). This solution may be obtained from f_k by using step by step formula (8) and multiplication by $f_{k-1}, f_{k-2}, \ldots, f_1$. But these operations do not take us out of L. Therefore the solution considered is in L, a contradiction.

Let an equation of order $(n-1)$ arise first as a result of the transformation connected with f_l. This means that f_l is a solution of the previous equation. But all earlier equations are of order n. Therefore there exists a unique solution of the form (9) of the original equation, and one can obtain it by using step by step formula (8) and multiplication by $f_{l-1}, f_{l-2}, \ldots, f_1$. Thus all the functions (10), including the solution (9) itself, are in F.

The author has to remark that the idea of search for "companion" solutions in F of general d'Alembertian solution was suggested (after reading a preliminary version of [Abr91]) by M.Petkovšek. More, Lemma 2 has been proved by M.Petkovšek for the case of nested indefinite hypergeometric sums (cf. [Pet92b]).

3 D'Alembertian subspace

We start with a technical lemma.

Lemma 3 *Let the expression (6) with $k \geq 3$, nonzero f_1, f_2, and some specific choice of primitive functions of the integrands, be a solution of the equation (1). Let a function h be such that $h' = f_2$. Then the given solution can be rewritten as*

$$(h\,f_1) \int (1/h)' \int (-h\,f_3) \int f_4 \ldots \ldots \int f_k\,dx\,\ldots\,dx\;dx\;dx\,. \tag{11}$$

Proof. For any v and nonzero u_1, u_2

$$((v/u_1)'/(u_2/u_1)')' = -\frac{u_2}{u_1}((v/u_2)'/(u_1/u_2)')' \tag{12}$$

(this equality was used in [Abr91]). One can take v being equal to the given solution of the form (6), $u_1 = f_1, u_2 = f_1 h$. Two successive transformations of the original equation connected with u_1 and $(u_2/u_1)'$ give an equation $D_1 y = 0$ which has a solution

$$f_3 \int f_4 \ldots \int f_k\,dx\,\ldots\,dx\,.$$

Therefore two successive transformations of the original equation, connected with u_2 and $(u_1/u_2)'$, give an equation $D_2 y = 0$ which has a solution

$$(-\frac{u_2}{u_1}) f_3 \int f_4 \ldots \int f_k \, dx \, \ldots \, dx \, .$$

Since $u_2/u_1 = h$ we see that the equation $D_2 y = 0$ has a solution of the form

$$(-h f_3) \int f_4 \ldots \int f_k \, dx \, \ldots \, dx \, .$$

Using the equality $(u_1/u_2)' = (1/h)'$ one can obtain now the form (11) for the given solution of the original equation.

This lemma can be easily generalized.

Lemma 4 *Let $k \geq 3,\ 1 < l < k$. Let the expression (6) with nonzero f_{l-1}, f_l and some specific choice of primitive functions of the integrands be a solution of the equation (1). Let a function h be such that $h' = f_l$. Then the given solution can be rewritten in the form*

$$\begin{aligned} f_1 \int \ldots \int f_{l-2} \int (h \, f_{l-1}) \int (1/h)' \int (-h \, f_{l+1}) \\ \int f_{l+2} \ldots \int f_k \, dx \, \ldots \, dx \; dx \; dx \; dx \; dx \, \ldots \, dx \end{aligned} \tag{13}$$

The proof is the same as for Lemma 3.

We remark that if $h \in F$ then, obviously, $h \, f_{l-1},\ (1/h)',\ -h \, f_{l+1} \in F$ also.

Now we are ready to prove that all d'Alembertian solutions are in M. It is enough to show that any solution (6) has a normal form.

We may suppose that $f_1, \ldots, f_k$ in (6) are nonzero.

The proof is by induction on k. For $k = 1$ the solution is in L and therefore has a normal form. For $k = 2$ either the solution is in L or f_1 is a solution of the original equation. Let $k \geq 3$ and assume that the solution (6) is not in L. By Lemma 2, the original equation has solution (9) such that all the functions (10) are in F. In particular, $\int f_l dx \in F$. Let $h \in F$ be such that $h' = f_l$, then Lemma 4 can be used. But according to (10), $\int f_{l-1} h \, dx = \int f_{l-1} \int f_l \, dx \; dx$ is in F. Therefore there exists a function $\tilde{h} \in F$ such that $\tilde{h}' = h \, f_{l-1}$ and Lemma 4 can be used again, and so on. Finally we will have expressed the given solution in the form

$$\psi \int g_1 \int \ldots g_{k-2} \int g_{k-1} \, dx \, \ldots \, dx \; dx \, , \tag{14}$$

where $g_1, \ldots, g_{k-2} \in F,\ g_{k-1} \in L$ and ψ is a solution of the original equation (1) belonging to F . We can execute d'Alembertian substitution connected with ψ. This transforms (1) into a new equation $By = 0$, which has solution $g_1 \int \ldots g_{k-2} \int g_{k-1}\, dx\, \ldots\, dx$. By inductive hypothesis this solution can be rewritten in the normal form

$$\xi_1 \int \ldots \xi_{p-1} \int \xi_p\, dx\, \ldots\, dx\,. \tag{15}$$

Then

$$\psi \int \xi_1 \ldots \xi_{p-1} \int \xi_p\, dx\, \ldots\, dx\,. \tag{16}$$

is a normal form for (14).

For the case when the given solution (6) is not in L, we describe briefly an algorithm to construct $g_1, \ldots, g_{k-1}$. Let $s_i = f_i \int \ldots \int f_l\, dx\, \ldots\, dx$, $i = 1, \ldots, l-1$, be the functions belonging to (10). Let $h_i = s_i/f_i$, $i = 1, \ldots, l-1$. Then we can compute $g_1, g_2, \ldots, g_{k-1}$ by the following algorithm:

1. for $i = k-1, k-2, \ldots, l$ do $g_i := f_{i+1}$;
2. for $i = l-1, l-2, \ldots, 1$ do
   $g_i := (1/h_i)'$; $\;g_{i+1} := -h_i g_{i+1}$

But we can see that

$$\begin{aligned} g_1 &= (1/h_1)' = -h_1'/h_1^2 = -s_2/h_1^2, \\ g_i &= -h_{i-1}(1/h_i)' = (h_{i-1}s_{i+1})/h_i^2, \\ &\quad i = 2, \ldots, l-1. \end{aligned}$$

Thus the following form of this algorithm

1. for $i = k-1, k-2, \ldots, l+1$ do $g_i := f_{i+1}$;
2. $g_l := -h_{l-1} f_{l+1}$;
3. for $i = l-1, l-2, \ldots, 2$ do $g_i := (h_{i-1}s_{i+1})/h_i^2$;
4. $g_1 := -s_2/h_1^2$

will be more efficient.

Thus we have proved the following theorem.

Theorem 1 *D'Alembertian subspace M of the space of all solutions of Eq.(1) contains all d'Alembertian solutions and only them.*

Let us consider the proof given above in more detail. It is obvious that any primitive function of the integrand (15) multiplied by ψ is a solution of the original equation. The same is true of all other primitive functions involved in (16). The unique choice of primitive functions gives normal form of (6), other choices give normal forms of other solutions. It means that having a d'Alembertian solution (6) not belonging to L, we obtain a subspace of M of dimension more than 1.

Thus we possess an algorithm to build up the subspace M. This algorithm uses the algorithms A_F and A_L. Furthermore, we possess an algorithm which without using A_F or A_L allows us to obtain some subspace of d'Alembert ian solutions provided that we have one such solution not belonging to L . The dimension of this space is more than 1.

Example. Let F and L be equal to the rational function field. One solution of the equation

$$x^3y''' - x^2y'' + 2xy' - 2y = 0 \tag{17}$$

is $y = x \ln x$. This solution can be written as

$$\int\int \frac{1}{x}\,dx\;dx\,, \tag{18}$$

if we consider $\int \frac{1}{x}\,dx\ = \ln x + 1,\ \ \int (\ln x + 1)\,dx\ = x \ln x$.

If we execute the transformation of Eq.(17) connected with function 1 we obtain $xy''' + 2y'' = 0$. The next transformation gives $xy'' + 2y' = 0$, the order of this equation is equal to 2. By Lemma 2 some special choice of a primitive functions of the integrand 1 gives a solution of the original equation. To find this solution we use (8) (here $h_0, \ldots, h_3$ are coefficients of the original equation, divided by -2; thus $h_1 = -x$). We obtain $-(-x)$, i.e. x. If we apply to (17) d'Alembert substitution connected with this solution we obtain

$$xy'' + 2y' = 0. \tag{19}$$

This equation has the solution of the form $g_1 \int g_2\,dx$. We have $f_1 = f_2 = 1,\ f_3 = 1/x$. We can construct g_1, g_2 step by step; $l = 2,\ h_1 = s_1 = x,\ h_2 = s_2 = 1$, and we execute $g_2 := -h_1 f_3;\ g_1 := -s_2/h_1^2$. Thus $g_1 = -1/x^2,\ g_2 = -1$.

So Eq.(19) has the solution

$$(-\frac{1}{x^2}) \int (-1)\,dx\,. \tag{20}$$

Since $-1/x^2$ is not a solution of Eq.(19), (20) is in the rational function field and $\int(-1)\,dx$ can be evaluated by (8). The evaluation gives $-x$. Thus instead of (18) we obtain

$$x \int \frac{1}{x}\,dx\,.$$

Any choice of a primitive function in the expression gives a solution of (17) (we have the space $\{C_1 x \ln x + C_2 x\}$).

Remark that when we are evaluating the external integral in (18) we can not choose a primitive function of the integrand arbitrarily. Let us return on the short time to the proof of the Theorem 1. We noted that if an $(n-1)$-order equation arises the first time after the transformation connected with f_l then f_l is a solution of the previous equation (i.e. of the equation which is the result of the transformation, connected with f_{l-1}, or, if $l = 1$, of the original equation). Together with f_l this previous equation, which will be denote as $K\,y = 0$, has, obviously, the solution

$$f_l \int f_{l+1} \dots \int f_k \, dx \, \dots \, dx \,, \tag{21}$$

where primitive functions are chosen in the same manner as in the solution (6). But if the equation $K\,y = 0$ has both these solutions then a primitive function connected with the external sign of the integral in (21) can be chosen arbitrarily. Any choice of a primitive function gives a solution of $K\,y = 0$. Therefore the expression of the form (6) which describes a not belonging to L solution of the original equation has the following property: a primitive function connected with the integral sign which is situated before can be chosen arbitrarily.

In above example a primitive function, connected with the internal integral sign, can be chosen arbitrarily. But the primitive function connected with the external internal sign has to be chosen by one unique way. The way provides a solution of Eq.(17).

4 Using solutions with logarithmic derivative in F

We denote by $\mathrm{Ld}(F)$ the set of all functions φ such that $\varphi'/\varphi \in F$. Let Eq.(1) have a solution $\eta \in \mathrm{Ld}(F)$. It is easy to see that d'Alembert substitution gives in this case an equation whose coefficients are in F. Thus if we possess an algorithm which allows for equations with coefficients in F to find solution in $\mathrm{Ld}(F)$, then we can construct an analogue of the d'Alembertian subspace. The new subspace consists of all solutions of the form (6), but $f_1, \dots, f_{k-1} \in \mathrm{Ld}(F)$, $f_k \in L$ now. An expression describing this space (like (5)) can be constructed by means of solutions from $\mathrm{Ld}(F)$ and d'Alembert substitutions. Since for any $f, f_1, f_2 \in \mathrm{Ld}(F)$ we have $f_1 f_2, f_1/f_2 \in \mathrm{Ld}(F)$, $f' \in \mathrm{Ld}(F) \bigcup \{0\}$, we can prove that the subspace does not depend on the choice of solutions from $\mathrm{Ld}(F)$, and the proof will be the same as the proof in [Abr91] for the case of solutions belonging to F.

For example, the space of solutions of the equation $y'' + 2y' - 3y = 0$, i.e. the space $\{C_1 e^x + C_2 e^{-3x}\}$ can be described both by $e^x \int e^{-4x} \int 0 \, dx \; dx$ and by $e^{-3x} \int e^{4x} \int 0 \, dx \; dx$ (we mentioned above that such expressions can be rewritten as $C\, e^x \int e^{-4x} \, dx$ and $C\, e^{-3x} \int e^{4x} \, dx$, where C is an arbitrary constant).

We can construct the normal form for any solution of the kind considered. Lemma 2 still holds if we suppose that either $L = F$ or $a \in \mathrm{Ld}(F)$, $b \in L$ imply $ab \in L$. The old proof remains valid due to the following. Let in (8)

$h_1, \ldots, h_n \in F$ and $\psi \in \mathrm{Ld}(F)$, then $\varphi \in \mathrm{Ld}(F)$. If $\psi \in L$, then $\varphi \in L$ because $\psi^{(k)} \in L$ for $k = 1, 2, \ldots$.

5 Connection with factorization of differential operators

It is known that any nonzero solution ψ of differential equation of the form (1) corresponds to the right-hand factor

$$\frac{d}{dx} - \frac{\psi'}{\psi} \tag{22}$$

of the operator A. Thus

$$A = P \circ \left(\frac{d}{dx} - \frac{\psi'}{\psi} \right) \tag{23}$$

where P is an $(n-1)$-order operator ([Schl, Sch89]). If we execute d'Alembert substitution connected with ψ, and obtain an equation $By = 0$, then $A = B \circ C$ where C is equal to

$$\frac{1}{\psi} \left(\frac{d}{dx} - \frac{\psi'}{\psi} \right). \tag{24}$$

Indeed, $A \circ \psi = B \circ \frac{d}{dx}$ for some operator B (the left-hand side of this operator is to be understood as the composition of the operators A and ψ, and not as the result of applying A to ψ). We have

$$A = B \circ \frac{d}{dx} \circ \frac{1}{\psi} = B \circ \left(\frac{1}{\psi} \frac{d}{dx} + \left(\frac{1}{\psi} \right)' \right) =$$

$$= B \circ \left(\frac{1}{\psi} \frac{d}{dx} - \frac{\psi'}{\psi^2} \right) = B \circ \left(\frac{1}{\psi} \left(\frac{d}{dx} - \frac{\psi'}{\psi} \right) \right).$$

It is easy to see that P on the right-hand side of (23) is equal to $B \circ \frac{1}{\psi}$.

Example. $x + a$ is a solution of the equation $y'' = 0$ for any constant a. The operator d^2/dx^2 has the right-hand factor $d/dx - 1/(x+a)$:

$$\frac{d^2}{dx^2} = \left(\frac{d}{dx} + \frac{1}{x+a} \right) \circ \left(\frac{d}{dx} - \frac{1}{x+a} \right)$$

(since a is arbitrary, the last equality gives a good example of a non-unique factorization of operators). D'Alembert substitution connected with $x + a$ gives $(x+a)y' + 2y = 0$. It is easy to see that in accordance with the above

$$(x+a)\frac{d}{dx} + 2 = \left(\frac{d}{dx} + \frac{1}{x+a} \right) \circ (x+a).$$

6 D'Alembert substitutions and nonhomogeneous equation

Let a nonhomogeneous linear equation be given, then we can consider the corresponding homogeneous equation of the form (1). If we can find some linearly independent solutions $\varphi_1, \ldots, \varphi_k$, then it is possible by d'Alembert substitutions to reduce the order of the original nonhomogeneous equation by k (the right-hand side of the equation is not changed). Repeating this operation as long as possible we will obtain either an equation of order zero $by = d$ where b and d are known functions, or an equation of order $m > 0$

$$b_m y^{(m)} + \cdots + b_0 y = d \tag{25}$$

such that we are not able to find any solution of the corresponding homogeneous equation. In the first case we obtain from d/b the general solution of the original equation, in the second case we build up a formula allowing to express the general solution of the original equation via the general solution of (25). Neither case requires solving systems of linear algebraic equations (as does the method of variation of constants).

Example. $(x^2 - 1)y'' + 4xy' + 2y = \sin x$. The corresponding homogeneous equation has solutions $x/(x^2 - 1)$ and $1/(x^2 - 1)$. D'Alembert substitution connected with the first solution gives the equation $xy' + 2y = \sin x$. Its homogeneous equation has to have the solution $((1/(x^2-1))/(x/(x^2-1)))' = -1/x^2$. D'Alembert substitution gives $(-1/x)y = \sin x$, i.e. $y = -x \sin x$. The general solution of the original equation is

$$y = \frac{x}{x^2-1} \int \frac{1}{x^2} \int x \sin x \, dx \; dx \; = \\ -\frac{\sin x}{x^2-1} + \frac{C_1 x + C_2}{x^2-1}.$$

Example. $x^3 y^{(\mathrm{iv})} - (x^4 - 6x^2)y''' - (2x^3 + 3x^2)y'' + (6x^2 - 12)y' + 18y = \sin x$. The corresponding homogeneous equation has the solution $1/x^2$. D'Alembert substitution and multiplication by $-x$ give $x^2 y''' - (x^3 + 2x)y'' + (4x^2 - 3x)y' - (4x-12)y = -x \sin x$. Its homogeneous equation has the solution x^4. D'Alembert substitution and multiplication by $1/x^4$ give

$$x^2 y'' - (x^3 - 10x)y' - (4x^2 + 3x - 20)y = -\frac{\sin x}{x^3}. \tag{26}$$

Let us suppose that we can find no solution of Eq.(26) and of its homogeneous equation. Then we write the general solution of the original equation in the form

$$y = \frac{1}{x^2} \int x^4 \int \xi \, dx \; dx \,,$$

where ξ is a general solution of Eq. (26).

7 Difference equations

It has been remarked in the paper that all our propositions are valid for difference equations as well. Little change is needed in the theory. The factor in the right-hand side of (12) is equal to $(-u_2(x+1)/u_1(x+1))$. The analogue of the right-hand side factor in (24) is

$$\frac{1}{\psi(x+1)}\left(\Delta - \frac{\Delta\psi(x)}{\psi(x)}\right).$$

There is complete analogy between functions with rational logarithmic derivative and hypergeometric functions, i.e. functions $f(x)$ such that $f(x+1)/f(x)$ (or $\Delta f(x)/f(x)$) is a rational function. The Section 4 is valid for difference equations and hypergeometric functions as well.

When some linear difference equations with rational function coefficients is given we can use d'Alembert substitutions based on the algorithm to search for rational ([Abr89a, Abr89b]) and hypergeometric ([Pet90, Pet92a]) solutions (both algorithms were implemented by B.Salvy in Maple).

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