Hypergeometric Solutions of First-Order Linear Difference Systems with Rational-Function Coefficients

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Abstract. Algorithms for finding hypergeometric solutions of scalar linear difference equations with rational-function coefficients are known in computer algebra. We propose an algorithm for the case of a first-order system of such equations. The algorithm is based on the resolving procedure which is proposed as a suitable auxiliary tool, and on the search for hypergeometric solutions of scalar equations as well as on the search for rational solutions of systems with rational-function coefficients. We report some experiments with our implementation of the algorithm.

1 Introduction

As a rule, both in scientific literature and in practice, algorithms for finding solutions of a certain kind for scalar differential or difference equations appear earlier than for systems of such equations. It may also be that a direct algorithm for systems is known in theory but does not yet have an available computer implementation (e.g., there is no such implementation in commonly used software packages). In this case, one makes an effort to find solutions of a system through some auxiliary scalar equations which are constructed for the system.

In [20, 18, 15], algorithms for finding hypergeometric solutions of scalar linear difference equations with rational-function coefficients were described. Using those algorithms, we propose below an algorithm to find hypergeometric solutions of linear normal first-order systems of the form $y(x+1) = A(x)y(x)$, where $A(x)$ is a square matrix whose entries are rational functions. Our algorithm differs somewhat from the algorithms based on the cyclic vector approach and is faster, as our experiments show. It is also worthy to note that even if $A(x)$ is singular, this is not an obstacle for our algorithm.

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Generally, direct algorithms work faster than algorithms that first uncouple the system. Thus, very likely, also the search for hypergeometric solutions of systems will become faster with the advent of full direct algorithms (it is known, e.g., that work is under way on such an algorithm for normal first-order systems with rational-function coefficients [11]). Until then, our algorithm can be useful for solving systems – all the more so since our experiments show that it works in reasonable time.

As an example of a computational problem which requires finding hypergeometric solutions of a first-order linear difference systems with rational-function coefficients we mention the important OPERATOR FACTORIZATION PROBLEM: Given a linear difference operator L of order n with rational-function coefficients and a positive integer $r < n$, find a linear difference operator R of order r with rational-function coefficients which divides L from the right, or prove that no such R exists. This problem can be solved by noticing that the $N = \binom{n}{r}$ maximal minors of the nth generalized Casoratian of a fundamental set of solutions of R must satisfy a system of first-order linear difference equations with rational-function coefficients easily obtainable from the coefficients of L, and that the coefficients of R must be proportional to certain $r + 1$ of these minors (cf. [13]).

2 The Problem

Let K be an algebraically closed field of characteristic 0. Denote by H_K the K-linear space of finite linear combinations of hypergeometric terms over K (i.e., $\frac{h(x+1)}{h(x)} \in K(x)$ for each hypergeometric term $h(x)$ under consideration) with coefficients in K.

Let E be the shift operator: $Ev(x) = v(x+1)$, and let $A(x)$ be an $m \times m$-matrix whose entries are in $K(x)$. We consider systems of the form

$$Ey = A(x)y, \quad y = (y_1(x), \ldots, y_m(x))^T, \tag{1}$$

and propose an algorithm which for a given system of the form (1) constructs a basis for the space of its solutions belonging to H_K^m. The basis consists of elements of the form

$$h(x)R(x), \tag{2}$$

where $h(x)$ is a hypergeometric term and $R(x) \in K(x)^m$.

We will say that an element of H_K^m is *related* to a hypergeometric term $h(x)$ if it can be represented in the form (2) (i.e., if each of its nonzero components is similar to $h(x)$).

For a system of the form (1), we will use the short notation $[A(x)]$.

3 The Reasoning Behind the Algorithm

3.1 The Resolving Equation and Matrix

Let y be any solution of (1). It follows by induction on j that

$$E^j y \;=\; \left(\prod_{i=1}^{j} E^{j-i} A(x)\right) y, \quad j = 0, 1, 2, \ldots$$

Let $c \in K(x)^m$ be an arbitrary row vector, and $t = cy$ a scalar function. Then $E^j t = (E^j c)\,(E^j y) = c^{[j]} y$ where

$$c^{[j]} = (E^j c) \left(\prod_{i=1}^{j} E^{j-i} A(x)\right) \in K(x)^m, \quad j = 0, 1, 2, \ldots,$$

are row vectors. We can construct the sequence $c^{[0]}, c^{[1]}, \ldots$ step by step, using the recurrence relation

$$c^{[0]} = c, \;\; c^{[i]} = (Ec^{[i-1]})A, \;\; i = 1, 2, \ldots \tag{3}$$

As $c^{[0]}, c^{[1]}, \ldots, c^{[m]}$ are $m+1$ vectors of length m, they are linearly dependent over $K(x)$. Let $k \in \{0, 1, \ldots, m\}$ be the least integer such that $c^{[0]}, c^{[1]}, \ldots, c^{[k]}$ are linearly dependent over $K(x)$. Then there are $u_0(x), u_1(x), \ldots, u_k(x) \in K(x)$, with $u_k(x) \neq 0$, such that $\sum_{j=0}^{k} u_j(x) c^{[j]} = 0$. So $\sum_{j=0}^{k} u_j(x) E^j t = \sum_{j=0}^{k} u_j(x) c^{[j]} y = 0$ as well. In particular, for

$$c = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{m-i}) \tag{4}$$

we have $t = cy = y_i$, hence

$$\sum_{j=0}^{k} u_j(x) E^j y_i = 0 \tag{5}$$

is a scalar equation of order k satisfied by y_i for any solution y of $[A(x)]$.

Definition 1. *Let row vectors $c^{[0]}, c^{[1]}, \ldots, c^{[k]}$ and $u_0(x), u_1(x), \ldots, u_k(x) \in K(x)$ be constructed as it is described above. Then we call (5) the* y_i-resolving equation, *and the full rank $k \times m$-matrix $B(x)$ whose jth row, for $j = 1, \ldots, k$, is $c^{[j-1]}$, the* y_i-resolving matrix *of $[A(x)]$.*

With the y_i-resolving matrix $B(x)$ we have

$$B(x)y = (y_i, Ey_i, \ldots, E^{k-1} y_i)^T \tag{6}$$

for any solution y of $[A(x)]$.

3.2 The Minimal Subspace Containing All Solutions with $y_i \neq 0$

Fix i and pick one solution from each set of similar hypergeometric terms satisfying (5). Let the selected hypergeometric terms be

$$h_1(x), \dots, h_l(x). \tag{7}$$

For each $h_j(x)$ substitute $y(x) = h_j(x)z(x)$ into $[A(x)]$, where $z(x) = (z_1(x), \dots, z_m(x))^T \in K(x)^m$ is a new unknown vector. If $\frac{h_j(x+1)}{h_j(x)} = r_j(x) \in K(x)$ then we get the system

$$Ez(x) = \frac{1}{r_j(x)} A(x)z(x). \tag{8}$$

If $R_{j,1}(x), \dots, R_{j,s_j}(x) \in K(x)^m$ is a basis for rational solutions of system (8) then we obtain K-linearly independent hypergeometric solutions

$$h_j(x)R_{j,1}(x), \dots, h_j(x)R_{j,s_j}(x) \tag{9}$$

of $[A(x)]$. Consider all such $h_j(x)$ for $j = 1, \dots, l$. The solutions

$$h_1(x)R_{1,1}(x), \dots, h_1(x)R_{1,s_1}(x), \dots, h_l(x)R_{l,1}(x), \dots, h_l(x)R_{l,s_l}(x) \tag{10}$$

of the system $[A(x)]$ generate over K all the solutions of $[A(x)]$ which have the form $h(x)R(x)$, where $R(x) \in K(x)^m$ and $h(x)$ is a hypergeometric term similar to one of (7). In particular, they generate all the solutions with $y_i(x) \neq 0$. In this sense, (10) is a basis of the minimal subspace containing all solutions with $y_i \neq 0$.

3.3 The Use of RNF

If $h_j(x) \in K(x)$ for some j then the system transformation leading to (8) is not needed since if a solution is related to a rational function then it is related to 1.

More generally, if $h_j(x)$ is a hypergeometric term and $\frac{h_j(x+1)}{h_j(x)} = r_j(x)$ then we can construct the rational normal form (RNF) of $r_j(x)$, i.e., represent $r_j(x)$ in the form $U_j(x)\frac{V_j(x+1)}{V_j(x)}$ with $U_j(x), V_j(x) \in K(x)$ where $U_j(x)$ has the numerator and the denominator of minimal possible degrees [7]. We can use $U_j(x)$ instead of $r_j(x)$ in (8). In this case we have to use in (9) the hypergeometric term $\frac{1}{V_j(x)}h_j(x)$ instead of $h_j(x)$.

3.4 The Space of Solutions with $y_i = 0$

Here we are interested in the solutions of $[A(x)]$ belonging to H_K^m with $y_i(x) = 0$.

Proposition 1. *Let equation (5) with $k < m$ be the y_i-resolving equation for $[A(x)]$, $1 \leq i \leq m$. Then there are $m - k$ indices $1 \leq i_1 < i_2 < \cdots < i_{m-k} \leq m$, and an $(m-k) \times (m-k)$-matrix $\tilde{A}(x)$ with entries in $K(x)$ such that if in some space Λ over $K(x)$, the system $[A(x)]$ has a solution $y(x)$ with $y_i(x) = 0$, then*

1. *the vector* $\tilde{y}(x) = (y_{i_1}(x), \dots, y_{i_{m-k}}(x))^T$ *satisfies* $E\tilde{y} = \tilde{A}(x)\tilde{y}$,
2. *each* $y_j(x)$ *with* $j \notin \{i_1, \dots, i_{m-k}\}$ *can be expressed as a linear form in* $y_{i_1}, \dots, y_{i_{m-k}}$ *having coefficients from* $K(x)$.

If $k = m$ *in (5) then* $y_i(x) = 0$ *implies* $y_j(x) = 0$ *for all* $j = 1, \dots, m$.

Proof. Note that if $y_i(x)$ is zero then $Ey_i(x), E^2y_i(x), \dots, E^{k-1}y_i(x)$ are zero as well. Since $c^{[j]}y = E^j(c\,y) = E^jy_i = 0$, this yields a system of k independent linear algebraic equations

$$B(x)y = 0 \tag{11}$$

for the unknown $y(x)$, where $B(x)$ is the y_i-resolving matrix of $[A(x)]$ (see Section 3.1). The matrix $B(x)$ has full rank, and hence there exist $m-k$ entries $y_{i_1}, \dots, y_{i_{m-k}}$ of y such that by means of this system, the other k entries of y can be expressed as linear forms in $y_{i_1}, \dots, y_{i_{m-k}}$ having coefficients from $K(x)$. Now we can transform $[A(x)]$ as follows:

For each $1 \le j \le m$ such that $j \notin \{i_1, \dots, i_{m-k}\}$ we

(a) remove the equation

$$Ey_j = a_{j1}y_1 + \cdots + a_{jm}y_m$$

from $[A(x)]$,

(b) in all other equations, replace y_j by the corresponding linear form in $y_{i_1}, \dots, y_{i_{m-k}}$ (in particular, y_i will be replaced by 0, since the first row of $B(x)$ is $c^{[0]} = c$ as given in (4), hence system (11) contains the equation $y_i = 0$, and $i \notin \{i_1, \dots, i_{m-k}\}$).

Denote the matrix of the resulting system by $\tilde{A}(x)$. If $[A(x)]$ has a solution $y(x) \in \Lambda$ such that $y_i(x) = 0$, then the vector $\tilde{y}(x) = (y_{i_1}(x), \dots, y_{i_{m-k}}(x))^T$ satisfies $E\tilde{y} = \tilde{A}(x)\tilde{y}$, and each $y_j(x)$ with $j \notin \{i_1, \dots, i_{m-k}\}$ can be expressed as a linear form in $y_{i_1}, \dots, y_{i_{m-k}}$, having coefficients from $K(x)$.

If $k = m$ then $B(x)$ is an invertible $m \times m$-matrix of the linear algebraic system (11). Thus, if in addition $y_i = 0$ then $y(x)$ satisfies (11), and $y(x) = 0$.

The proof of Proposition 1 contains an algorithm for constructing the matrix $\tilde{A}$. This matrix is independent of the space Λ. We will use it in Section 4 for the case $\Lambda = H_K^m$.

3.5 When $k = m$

Suppose that $k = m$ in (5). Then $B(x)$ is an invertible $m \times m$-matrix, and c in (4) is a *cyclic vector* (see, e.g., [16, 8, 12, 14] and Section 6.3 below) for the original system $[A(x)]$.

Let $h_1(x), \dots, h_l(x)$ be a basis for solutions of (5) that belong to H_K. Then by solving the inhomogeneous linear algebraic systems

$$B(x)y = (h_i(x), Eh_i(x), \dots, E^{m-1}h_i(x))^T, \;\; i = 1, \dots, l, \tag{12}$$

we obtain a basis for solutions of $[A(x)]$ that belong to H_K^m.

3.6 Selection of y_i

The simplest way to select y_i is just to pick the first unknown from those under consideration. However, it is probably more reasonable to find such a row of the matrix of the difference system which is the least "cumbersome" of all the rows which contain the largest number of zero entries (the "cumbersome" criterion should be clarified). Then we select the unknown y_i so that Ey_i corresponds to the selected row in the matrix of the difference system.

4 The Algorithm

Input: A system of the form (1) (or, equivalently, $[A(x)]$).
Output: A basis for the space of all solutions of $[A(x)]$ belonging to H_K^m, with basis elements in the form (2).

1. $\ell := \emptyset$; $M(x) := A(x)$.

2. Select y_i (Section 3.6). Construct the y_i-resolving equation and the y_i-resolving matrix $B(x)$ of the system $[M(x)]$ (Section 3.1). Let k be the order of the y_i-equation. Compute a basis b for solutions of the constructed y_i-equation that belong to H_K; the elements of b are hypergeometric terms.

3. Include into ℓ those elements of b that are not similar to any of the elements already in ℓ; in each moment the elements of ℓ are pairwise non-similar hypergeometric terms.

4. If $m = k$ then compute a basis $h_1(x), \dots, h_l(x)$ for the solutions of (5) that belong to H_K. Then by solving inhomogeneous linear algebraic systems (12) find a basis for solutions of $[A(x)]$ that belong to H_K^m. (All the systems (12) can be considered as one system with the left hand side $B(x)y$ and a finite collection of right hand sides.) STOP.

Comment: The equality $k = m$ may be satisfied only at the first execution of Step 4. In all the subsequent executions, k will be less than m.

5. If the order k of equation (5) is less than the order of the matrix $M(x)$ then construct $[\tilde{M}(x)]$ using $B(x)$ (Section 3.4), set $M(x) := \tilde{M}(x)$, and go to 2.

6. For each $h_j(x)$ belonging to ℓ, apply the RNF transformation (Section 3.3) to the rational function $\frac{h_j(x+1)}{h_j(x)}$. If the result is $U_j(x)\frac{V_j(x+1)}{V_j(x)}$ then set $r_j(x) := U_j(x)$ and use it in (8) to construct a basis for the space of those solutions of the system $[A(x)]$ which are related to $h_j(x)$ (Section 3.2). (If $r_j(x) = 1$ the original system does not change.) The union of all such bases gives a basis for the space of solutions of $[A(x)]$ that belong to H_K^m.

In Examples 1 and 2, the unknown y_i is always selected as the first unknown from all the unknowns of the current system.

Example 1. Let

$$A(x) = \begin{pmatrix} \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ 1 & 0 & \frac{2}{x+1} & -x \\ -1 & 1 & x-1 & 1 \\ -\frac{x+2}{x} & \frac{x+1}{x} & \frac{x^2-x-1}{x(x+1)} & \frac{x^2+x+1}{x} \end{pmatrix}$$

With this matrix as input the algorithm proceeds as follows:

1. $\ell := \emptyset$; $M(x) := A(x)$.

2. Set $y_i = y_1$. The y_i-resolving matrix and equation of $[M(x)]$ are

$$B(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{x-1}{x} & 0 & -\frac{x-1}{x+1} & 0 \\ \frac{2(x^2+x-1)}{(x+1)(x+2)} & -\frac{x}{x+2} & -\frac{x(x-1)(x^2+3x+3)}{(x+1)^2(x+2)} & -\frac{x}{x+2} \end{pmatrix},$$

and

$$\begin{aligned} -\,x(x-1)(x+2)(x+1)(x^2-x-1)y_1(x) +\\ 2x(x+2)(x^4+x^3-x^2-x-1)y_1(x+1) -\\ (x-1)(x+1)(x^4+6x^3+12x^2+8x+4)y_1(x+2) +\\ x^2(x-1)(x+3)(x+2)y_1(x+3) = 0. \end{aligned} \tag{13}$$

A basis b for the solutions of the resolving equation that belong to H_K consists of only one element which happens to be a rational function:

$$h_1(x) = \frac{1}{x-1}.$$

3. In accordance with Section 3.3 we set

$$\ell = \left\{ \frac{1}{x-1} \right\}.$$

4. Since $k = 3$ and $m = 4$, we go to Step 5.

5. Using $B(x)$, the matrix $M(x)$ is transformed into $\tilde{M}(x)$ which is a 1×1-matrix, and the system $[\tilde{M}(x)]$ is

$$y_4(x+1) = xy_4(x).$$

The set ℓ is extended by the hypergeometric term $\Gamma(x)$:

$$\ell = \left\{ \frac{1}{x-1}, \Gamma(x) \right\}.$$

6. For the first element of ℓ, we get $r_1(x) = 1$ since the RNF of $\frac{x-1}{x}$ is $1 \cdot \frac{1/x}{1/(x-1)}$. The system (8) with $r_1(x) = 1$ has no rational solutions, thus, there is no solution of the original system which is related to $\frac{1}{x-1}$.

Since $\frac{\Gamma(x+1)}{\Gamma(x)} = x$ and the RNF of x is $x \cdot \frac{1}{1}$, we use $r_2(x) = x$ in (8). This system has a one-dimensional space of rational solutions, generated by

$$R(x) = (0, -1, 0, 1)^T.$$

Finally, we obtain the basis of the (one-dimensional) space of all solutions of $[A(x)]$ belonging to H_K^4. It contains the single element

$$\Gamma(x)R(x) = (0, -\Gamma(x), 0, \Gamma(x))^T.$$

Remark 1. *Example 1 shows that the proposed resolving approach is not a modification of the block-diagonal form algorithm [8]: if the constructed y_1-resolving equation (13) corresponds to a diagonal block of the original system then the system would have a rational solution. However, this is not the case.*

Example 2. Let

$$A(x) = \begin{pmatrix} \frac{x^3+4x^2+4x-2}{(x+4)(x+2)(x+1)} & \frac{x^2+3x+1}{(x+2)(x+1)} & \frac{x+1}{x+4} & \frac{2(x+2)}{x+4} \\ -\frac{x^3+4x^2+4x-2}{(x+4)(x+2)(x+1)} & \frac{1}{(x+2)(x+1)} & -\frac{x+1}{x+4} & -\frac{x}{x+4} \\ -\frac{x(2x^2+8x+9)}{(x+4)(x+2)(x+1)} & -\frac{x^2+3x+1}{(x+2)(x+1)} & -\frac{2(x+1)}{x+4} & -\frac{2x}{x+4} \\ \frac{x+1}{x+4} & 0 & \frac{x+1}{x+4} & \frac{x}{x+4} \end{pmatrix}.$$

With this matrix as input the algorithm proceeds as follows:

1. $\ell := \emptyset$; $M(x) := A(x)$.

2. Set $y_i = y_1$. The y_i-resolving matrix is a 4×4-matrix and the y_i-resolving equation is

$$(x^2+9x+21)(x^5+17x^4+111x^3+339x^2+453x+167)(x+2)^2 y_1(x)+$$
$$(x^8+30x^7+383x^6+2727x^5+11919x^4+33035x^3+$$
$$57308x^2+57507x+25746)y_1(x+1)-$$

$$(x^2+5x+5)(x^9+28x^8+341x^7+2360x^6+10158x^5+27884x^4+$$
$$47833x^3+47264x^2+21093x+462)y_1(x+2)+$$
$$(x+4)(x^9+25x^8+267x^7+1564x^6+5268x^5+9116x^4+$$
$$1933x^3-21905x^2-36519x-19110)y_1(x+3)+$$
$$(x+5)(x+4)(x+7)(x+1)(x^2+7x+13)(x^5+12x^4+53x^3+98x^2+$$
$$45x-42)y_1(x+4)=0$$

A basis b for the solutions of the y_i-equation belonging to H_K consists of four elements:

$$h_1(x)=\frac{(-1)^x(2x+5)}{(x+2)(x+3)},\ h_2(x)=\frac{1}{(x+3)(x+2)},$$

$$h_3(x)=\frac{(-1)^x(x^3+7x^2+16x+12)}{\Gamma(x+4)},\ h_4(x)=\frac{x^3+5x^2+6x}{\Gamma(x+4)}.$$

3. ℓ is the same as b.

4. We decide to solve (6) and get the following basis for solutions of $[A(x)]$:

$$\left(\frac{(-1)^x(2x+5)}{(x+2)(x+3)},-\frac{(-1)^x(2x+5)}{(x+2)(x+3)},-\frac{(-1)^x(6x^2+23x+19)}{(x+3)(x+2)(x+1)},\frac{(-1)^x(2x+5)}{(x+2)(x+3)}\right)^T,$$

$$\left(\frac{1}{(x+3)(x+2)},-\frac{1}{(x+3)(x+2)},-\frac{x-1}{(x+3)(x+2)(x+1)},\frac{1}{(x+3)(x+2)}\right)^T,$$

$$\left(\frac{(-1)^x(x^3+7x^2+16x+12)}{\Gamma(x+4)},\frac{(-1)^x(x^2+5x+6)}{-\Gamma(x+4)},\frac{(-1)^x(x^3+7x^2+16x+12)}{-\Gamma(x+4)},0\right)^T,$$

$$\left(\frac{x(x^2+5x+6)}{\Gamma(x+4)},\frac{x^2+5x+6}{\Gamma(x+4)},-\frac{x(x^2+5x+6)}{\Gamma(x+4)},0\right)^T.$$

5 On the Resolving Procedure

Now we investigate the complexity of the *resolving procedure* that was presented in Sections 3.1, 3.4. The procedure transforms a linear difference system $[A(x)]$, where $A(x)$ is an $m\times m$-matrix ($m\geq 2$) whose entries belong to $K(x)$, into one or several scalar difference equations with coefficients in $K(x)$. The sum of the orders of those scalar equations does not exceed m. If $[A(x)]$ has a solution of the form (2) then at least one of the obtained equations has a solution of the form $h(x)r(x)$ with $r(x)\in K(x)$.

Proposition 2. *The complexity (the number of field operations in $K(x)$ in the worst case) of the resolving procedure is $O(m^3)$.*

Proof. Let the size of the input matrix resp. the resolving matrix be $n \times n$ resp. $l \times n$ where $l \leq n$. Then there is a constant C such that the number of field operations in $K(x)$ needed for constructing the resolving equation, the resolving matrix and the matrix $\tilde{A}(x)$ (see the proof of Proposition 1) does not exceed Cln^2. Using this, it is easy to prove by induction on the number of steps needed to construct the y_i-resolving equations and matrices that the number of field operations needed to perform the resolving procedure does not exceed Cm^3. Indeed, if the number of steps is 1 the claim is evident. Otherwise, let the size of the y_i-resolving matrix constructed on the first step be $k \times m$ where $k < m$. In this case, the general number of operations in $K(x)$ does not exceed $C(m-k)^3 + Ckm^2$. It remains to note that $m^3 - (m-k)^3 - km^2 = k^2(m-k) + 2k(m-k)^2 \geq 0$.

Algorithms for constructing a cyclic vector and related uncoupling algorithms for normal first-order systems are well known in computer algebra. The paper [12] contains a review of such algorithms and corresponding references. In that paper, a new algorithm for constructing a cyclic vector based on fast linear algebra algorithms ([21]) is also proposed. The existence of cyclic vector algorithms suggests that there is no need for resolving procedures – all the more so since cyclic vector procedures and the uncoupling are multi-purpose procedures which may be useful not only for finding hypergeometric or exponential-logarithmic solutions of systems. Besides, the resolving procedure only solves part of the problem. Solutions of the operators belonging to the resolving sequence are a "half-finished product", since we have in addition to find rational solutions of other difference systems. However, the resolving equations approach can have some advantage over the cyclic vector approach. First, a sequence of resolving equations is constructed in a single pass, while in practical cyclic vector algorithms numerous random candidates are considered (if a candidate is not appropriate then another one is generated, and all calculations are resumed from the beginning).

Second, asymptotically fast linear algebra algorithms have advantages over classical algorithms only for large inputs. The orientation on asymptotic complexity estimates does not seem to be very productive here. The use of classical linear algebra algorithms as auxiliary tools for searching for solutions of difference systems can be advantageous in many cases.

Third, the resolving system constructed by our algorithm is such that the sum of the orders of its operators does not exceed the order of the operator obtained by a cyclic vector algorithm. As a rule, it is easier to solve a few equations of small orders than a single equation of a large order (which provides motivation to develop factorization algorithms). Even when we obtain a unique resolving operator (in the case $k = m$, Section 3.5), we can profit since such a cyclic vector is of a very simple form, and the scalar equation to solve will not have "too cumbersome" coefficients. In general, we have an opportunity to use various heuristics to obtain operators with possibly less cumbersome coefficients.

As for the second stage when some solutions of additional difference systems have to be found, this can quite often be done in reasonable time. The search for rational solutions of difference systems having rational-function coefficients is not time consuming ([2, 4, 5, 6, 9, 10, 17]).

Our experimental comparison demonstrates a definite advantage of a resolving procedure over the cyclic vector approach.

6 Implementation and Experiments

6.1 Implementation

We have implemented the algorithm in Maple 18 ([22]). The implemented procedures are put together in the package `LRS` (Linear Recurrence Systems). The main procedure of the package is `HypergeometricSolution`.

To find a basis of hypergeometric solutions of the y_i-resolving equation (5), the procedure `hypergeomsols` from the package `LREtools` is used. It implements the algorithm from [18]. To find a basis of rational solutions of the system (8) we use the procedure `RationalSolution` from the package `LinearFunctionalSystems`. This procedure implements the algorithms from [2, 19, 1, 3]. To perform RNF transformation, the procedure `RationalCanonicalForm` from the package `RationalNormalForms` is used. It implements the algorithms from [7].

To select y_i, the procedure `SelectIndicator` from `LRS` finds all the rows of $A(x)$ with the greatest number of zero entries. In this set of rows, it finds the rows having the least sum of degrees in x of all the numerators and denominators of nonzero entries. If the resulting set has several rows, one of them is selected by the standard Maple procedure `rand`. The procedure returns the number i of the selected row. The arguments of the procedure `HypergeometricSolution` are a square matrix with rational-function entries and a name of an independent variable. The output is a list of vectors whose entries are hypergeometric terms.

6.2 Some Experiments

Example 3. Applying `HypergeometricSolution` to the matrix $A(x)$ from Example 1 we get the result in 0.303 CPU seconds[1]:

```
> A1 := Matrix([[(x-1)/x, 0, -(x-1)/(x+1), 0],
               [1, 0, 2/(x+1), -x],
               [ -1, 1, x-1, 1],
               [-(x+2)/x, (x+1)/x, (x^2-x-1)/((x+1)*x),
                                             (x^2+x+1)/x]]):

> st := time():
  LRS:-HypergeometricSolution(A1, x);
  time()-st;
```

[1] By Maple 18, Ubuntu 8.04.4 LTS, AMD Athlon(tm) 64 Processor 3700+, 3GB RAM.

$$\left[\begin{bmatrix} 0 \\ -\Gamma(x) \\ 0 \\ \Gamma(x) \end{bmatrix}\right]$$

0.303

Example 4. For $A(x)$ from Example 2 we get the result:

```
> A2 := Matrix([
   [(x^3+4*x^2+4*x-2)/((x+4)*(x+2)*(x+1)),
         (x^2+3*x+1)/((x+2)*(x+1)), (x+1)/(x+4), (2*x+4)/(x+4)],
   [ -(x^3+4*x^2+4*x-2)/((x+4)*(x+2)*(x+1)),
           1/((x+2)*(x+1)), -(x+1)/(x+4),  -x/(x+4)],
   [-x*(2*x^2+8*x+9)/((x+4)*(x+2)*(x+1)),
        -(x^2+3*x+1)/((x+2)*(x+1)), -(2*x+2)/(x+4), -2*x/(x+4)],
   [(x+1)/(x+4), 0, (x+1)/(x+4), x/(x+4)]]):


> st := time():
  LRS:-HypergeometricSolution(A2, x);
  time()-st;
```

$$\left[\begin{bmatrix} \frac{(-1)^x(2x+5)}{(x+2)(x+3)} \\ -\frac{(-1)^x(2x+5)}{(x+2)(x+3)} \\ -\frac{(-1)^x(6x^2+23x+19)}{(x+3)(x+2)(x+1)} \\ \frac{(-1)^x(2x+5)}{(x+2)(x+3)} \end{bmatrix}, \begin{bmatrix} \frac{1}{(x+3)(x+2)} \\ -\frac{1}{(x+3)(x+2)} \\ -\frac{x-1}{(x+3)(x+2)(x+1)} \\ \frac{1}{(x+3)(x+2)} \end{bmatrix}, \begin{bmatrix} -\frac{(-1)^x(x+2)}{\Gamma(x+2)} \\ \frac{(-1)^x}{\Gamma(x+2)} \\ \frac{(-1)^x(x+2)}{\Gamma(x+2)} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{x}{\Gamma(x+2)} \\ -\frac{1}{\Gamma(x+2)} \\ \frac{x}{\Gamma(x+2)} \\ 0 \end{bmatrix}\right]$$

0.410

Here, `SelectIndicator` selects $i = 4$, because the fourth matrix row has one zero. The corresponding resolving equation is of order $k = 2$ and has a two-dimensional hypergeometric solutions space. The reduced system $[\tilde{M}(x)]$ has a second-order resolving equation with a two-dimensional hypergeometric solutions space.

Example 5. We tested `HypergeometricSolution` for systems with a 16×16-matrix. We cannot present here this matrix and a basis of the hypergeometric solutions space since they are too large. The matrix is such that 80% of its entries are zeros. The maximum degree of the numerators of its entries is 13. The maximum degree of their denominators is 11. The procedure finds a two-dimensional hypergeometric solutions space in 346.046 CPU seconds.

The code and examples of applications of `HypergeometricSolution` are available from http://www.ccas.ru/ca/doku.php/lrs.

6.3 Comparison with the Cyclic Vector Approach

Besides the resolving procedure, we implemented also the search for hypergeometric solutions based on the cyclic vector approach (procedure `CyclicVector`). In our experiments, we used a traditional randomized version of a cyclic vector algorithm which allows to obtain

- a scalar difference equation of order m with rational-function coefficients, and
- a matrix $B(x)$ with rational-function entries.

These objects are such that if $h_1(x), \dots, h_l(x)$ form a basis for solutions of the scalar equation that belong to H_K then by solving the inhomogeneous linear algebraic systems $B(x)y = (h_i(x), Eh_i(x), \dots, E^{m-1}h_i(x))^T$, $i = 1, \dots, l$, one obtains a basis for solutions of $[A(x)]$ that belong to H_K^m:

1. Randomly choose a row vector $c^{[0]}$ containing polynomials of degree 0.

2. Create an $m \times m$-matrix $B(x)$:
    for i from 1 to m do
        the i-th row of $B(x)$ is $c^{[i-1]}$;
        $c^{[i]} := (Ec^{[i-1]})A(x)$.

3. If $B(x)$ is not invertible, go to step 2 with a new random row vector $c^{[0]}$ of polynomials of degree $m-1$. Otherwise, solve the linear system $u(x)B(x) = c^{[m]}$ to obtain a vector $u(x) = (u_0(x), \dots, u_{m-1}(x))$.

4. Return the scalar equation for a new unknown $t(x)$

$$E^m t = u_{m-1}(x)E^{m-1}t + \cdots + u_0(x)t \tag{14}$$

and the matrix $B(x)$.

Instead of the CPU time demonstrated in Examples 3, 4, 5, i.e., 0.303, 0.410, and 346.046 seconds, the computation using the cyclic vector procedure takes, resp., 0.410, 0.474, 1063.747 seconds.

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