On valuations of meromorphic solutions of arbitrary-order linear difference systems with polynomial coefficients*

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ABSTRACT

Algorithms for computing lower bounds on valuations (e.g., orders of the poles) of the components of meromorphic solutions of arbitrary-order linear difference systems with polynomial coefficients are considered. In addition to algorithms based on ideas which have been already utilized in computer algebra for treating normal first-order systems, a new algorithm using tropical calculations is proposed. It is shown that the latter algorithm is rather fast, and produces the bounds with good accuracy.

Categories and Subject Descriptors

I.1.2 [**Symbolic And Algebraic Manipulation**]: Algorithms—*Algebraic algorithms*

General Terms

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1. INTRODUCTION

Finding and studying the singularities of solutions of linear difference equations and systems is a part of various algorithms.

Let k be a field of characteristic 0. We consider systems of the form

$$A_r(x)y(x+r)+\cdots+A_1(x)y(x+1)+A_0(x)y(x) = b(x), \quad (1)$$

where

- $A_0(x), A_1(x), \ldots, A_r(x)$ are square matrices of order m with entries from $k[x]$ (which is denoted as $A_0(x), A_1(x), \ldots, A_r(x) \in \mathrm{Mat}_m(k[x])$) with the assumption that the leading and trailing matrices $A_r(x), A_0(x)$ are nonzero,
- $b(x) = (b_1(x), b_2(x), \ldots, b_m(x))^T \in k[x]^m$ is the right-hand side of the system,
- $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T$ is a column of unknown functions.

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The number r is called the order of the system.

Let the homogeneous system S' be obtained by dropping the right-hand side of the original system. We assume that the equations of S' are independent over $k[x, \phi]$, where ϕ is the shift operator:

$$\phi(y(x)) = y(x+1).$$

This means that if a linear combination of equations of S' which has coefficients in $k[x, \phi]$ is equal to zero then all these coefficients are equal to zero.

If k is a numeric field, i.e., $k \subseteq \mathbb{C}$, then one can consider analytical and, in particular, meromorphic solutions of the system (1). For a meromorphic function $f(x)$ and $\alpha \in \mathbb{C}$, the valuation $\mathrm{val}_{x-\alpha} f(x)$ is defined as the lowest degree of $x-\alpha$ for which the Laurent series expansion of $f(x)$ about the point α has a nonzero coefficient (by convention, $\mathrm{val}_{x-\alpha} 0 = \infty$). For two meromorphic functions the following relations hold:

$$\begin{gathered}\mathrm{val}_{x-\alpha}(f(x)g(x)) = \mathrm{val}_{x-\alpha} f(x) + \mathrm{val}_{x-\alpha} g(x), \\ \mathrm{val}_{x-\alpha}(f(x)+g(x)) \geqslant \min\{\mathrm{val}_{x-\alpha} f(x),\ \mathrm{val}_{x-\alpha} g(x)\}.\end{gathered} \quad (2)$$

For a vector $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T$ consisting of meromorphic functions, $\mathrm{val}_{x-\alpha} y(x)$ is defined to be $\min_{i=1}^m \mathrm{val}_{x-\alpha} y_i(x)$.

There is a significant difference between the solution spaces of linear ordinary differential and linear ordinary difference systems: the solutions of the latter may be multiplied not only by constants, but also by functions with the period equal to 1. Along with a meromorphic solution $y(x)$ the system has also, for example, solutions $(\sin 2\pi(x+\beta))y(x)$ and $(\sin 2\pi(x+\beta))^{-1}y(x)$ for any $\beta \in \mathbb{C}$. The singularities of solutions of a differential system similar to (1) with equations which are independent over $k[x, \frac{d}{dx}]$ constitute a finite set (in [6] an algorithm is proposed which constructs a finite superset of this set). The situation is different for difference systems and even for scalar difference equations with polynomial coefficients — it is enough to mention the gamma function, which satisfies the scalar equation $y(x+1)-xy(x)=0$. This equation is satisfied not only by $y(x)=\Gamma(x)$,

$$\mathrm{val}_{x-\alpha}y(x) = \begin{cases} -1, & \text{if } \alpha \text{ is a non-positive integer,} \\ 0, & \text{otherwise,} \end{cases}$$

but also by, e.g., $y(x)=\sin^2(2\pi x)\Gamma(x)$:

$$\mathrm{val}_{x-\alpha}y(x) = \begin{cases} 2, & \text{if } \alpha \in \frac{1}{2}+\mathbb{Z} \text{ or } \alpha \text{ is a positive integer,} \\ 1, & \text{if } \alpha \text{ is a non}-\text{positive integer,} \\ 0, & \text{otherwise.} \end{cases}$$

For this reason, before discussing the valuations of solutions of a given scalar equation or a system, it is necessary to formulate some initial conditions, e.g., in the form of lower bounds on the valuations $\mathrm{val}_{x-\alpha}y(x+n)$ for some consecutive integer values of n (see below). Naturally, our algorithms presuppose that explicit bounds on the valuations of the desired solution are known at certain points.

If one assumes that the matrices $A_r(x)$, $A_0(x)$ in (1) are invertible in $\mathrm{Mat}_m(k(x))$, then for an arbitrary meromorphic solution $y(x)$ of (1) and $\alpha\in\mathbb{C}$ the value $\mathrm{val}_{x-\alpha-n}y(x)$ has a lower bound for n running through $\mathbb{Z}$ (Proposition 1 in Section 3.1). Consideration of the so-called embracing systems, described in Section 2.2, allows us to avoid the assumption of invertibility of the matrices $A_r(x)$, $A_0(x)$.

It is obvious that for any n the equation $\mathrm{val}_{x-\alpha-n}y(x) = \mathrm{val}_{x-\alpha}y(x+n)$ holds. In the rest of the paper we consider the valuations of the form $\mathrm{val}_{x-\alpha}y(x+n)$.

We are interested in two problems. The first problem is the computation of a lower bound for $\mathrm{val}_{x-\alpha}y(x)$, i.e., a global lower bound for all components of the solution $y(x)$. The second problem, a refinement of the first one, is the computation of individual lower bounds on each component: $\mathrm{val}_{x-\alpha}y_i(x)$, $i=1,2,\ldots,m$. In both problems, $y(x)=(y_1(x),y_2(x),\ldots,y_m(x))^T$ is a meromorphic solution of the system (1), and α is a fixed point in the complex plane.

In the first problem, it is assumed that a common lower bound for the valuations $\mathrm{val}_{x-\alpha}y(x+n)$ for some r consecutive integer values of n is given. Proposition 1 in Section 3.1 that we mentioned above, shows that in some cases we do not need to mention explicitly which values of n are used (when we have conditions "at infinity"). This is detailed in Remark 1.

In the second problem, it is assumed that individual lower bounds on the valuations $\mathrm{val}_{x-\alpha}y_i(x+n)$, $i=1,2,\ldots,m$, for some r consecutive integer values of n are given.

In some interesting cases — for example, in constructing denominator bounds for rational solutions — one can take the a priori known valuation bounds to be equal to zero.

Note that for the scalar case, a study of the space of those meromorphic solutions for which the valuations are non-negative for all large-enough values of $\mathrm{Re}\,x$, is contained in the unpublished work [15], in the thesis [9], and in the paper [13].

The first problem is considered separately, since a simpler algorithm can be given for solving it. The idea of this algorithm has been, to some extent, already utilized in algorithms for finding rational solutions of first-order normal difference systems of the form

$$y(x+1)=A(x)y(x) \tag{3}$$

where $A(x)\in\mathrm{Mat}_m(k(x))$ is an invertible matrix (see [11, 4, 5]). Concerning (3) the first problem is rather simple in comparison with the systems of the general form. The algorithm proposed in Section 3.2 is applicable to arbitrary-order systems of the form (1).

As for the second problem, in Section 3.3.1 we propose an algorithm which is applicable to arbitrary-order systems of the form (1), and which uses the so-called tropical operations on matrices with entries from $\mathbb{Z}\cup\{\infty\}$.

Our complexity analysis (see Section 3.3.2) and experiments (see Section 4) show that the proposed algorithm is rather fast, and produces bounds of good accuracy. The idea of using the tropical calculations to estimate the valuations of solutions of difference equations and systems seems to be quite natural. However we have been unable to find in the literature any explicit mention of this.

2. PRELIMINARIES

2.1 Valuation of rational functions at irreducible polynomials

The set of monic irreducible polynomials from $k[x]$ is denoted as $\mathrm{Irr}(k[x])$. If $p(x)\in\mathrm{Irr}(k[x])$ and $f(x)\in k[x]$, the valuation $\mathrm{val}_{p(x)}f(x)$ is defined to be the greatest $n\in\mathbb{N}$ such that $p^n(x)\mid f(x)$ ($\mathrm{val}_{p(x)}0=\infty$), and $\mathrm{val}_{p(x)}F(x) = \mathrm{val}_{p(x)}f(x)-\mathrm{val}_{p(x)}g(x)$ for $F(x)=\frac{f(x)}{g(x)}$, $f(x),g(x)\in k[x]$. For two arbitrary nonzero rational functions $r(x),s(x)$ and for $p(x)\in\mathrm{Irr}(k[x])$, the following relations hold:

$$\begin{aligned} &\mathrm{val}_{p(x)}(r(x)s(x)) = \mathrm{val}_{p(x)}r(x)+\mathrm{val}_{p(x)}s(x), \\ &\mathrm{val}_{p(x)}(r(x)+s(x)) \geqslant \min\{\mathrm{val}_{p(x)}r(x),\ \mathrm{val}_{p(x)}s(x)\}. \end{aligned} \tag{4}$$

If $F(x)\in k(x)$ then we denote by $\mathrm{den}\,F(x)$ the denominator of $F(x)$, i.e., a monic polynomial such that $F(x)=\frac{f(x)}{\mathrm{den}\,F(x)}$ for a polynomial $f(x)\in k[x]$ which is co-prime with $\mathrm{den}\,F(x)$. If $F(x)=(F_1(x),F_2(x),\ldots,F_m(x))^T\in k(x)^m$ then $\mathrm{den}\,F(x)=\mathrm{lcm}_{i=1}^m\,\mathrm{den}\,F_i(x)$ and $\mathrm{val}_{p(x)}F(x)=\min_{i=1}^m \mathrm{val}_{p(x)}F_i(x)$, where lcm denotes the least common multiple.

For an arbitrary matrix $A(x)=(a_{ij}(x))\in\mathrm{Mat}_m(k(x))$ we define $\mathrm{den}\,A(x)=\mathrm{lcm}_{i=1}^m\mathrm{lcm}_{j=1}^m\,\mathrm{den}\,a_{ij}(x)$.

2.2 Embracing systems

For any system S of the form (1) one can construct an l-*embracing* system $\bar{S}$

$$\bar{A}_r(x)y(x+r)+\cdots+\bar{A}_1(x)y(x+1)+\bar{A}_0(x)y(x)=\bar{b}(x), \tag{5}$$

with the leading matrix $\bar{A}_r(x)$ being invertible in $\mathrm{Mat}_m(k(x))$, and with the set of solutions containing all the solutions of the system S. Similarly, one can construct a *t-embracing* system $\bar{\bar{S}}$

$$\bar{\bar{A}}_r(x)y(x+r)+\cdots+\bar{\bar{A}}_1(x)y(x+1)+\bar{\bar{A}}_0(x)y(x) = \bar{\bar{b}}(x), \quad (6)$$

with the trailing matrix $\bar{\bar{A}}_0(x)$ being invertible in $\mathrm{Mat}_m(k(x))$, and with the set of solutions containing all the solutions of the system S. All the entries of the matrices and of the right-hand sides of (5), (6) are in $k[x]$. It is possible that the matrices $\bar{A}_0(x)$, $\bar{\bar{A}}_r(x)$ are zero, either both or one of them.

The construction of the embracing systems can be performed with the algorithms EG ([1]) or EG$'$ ([2]); the algorithm EG$'$ is an improved version of the algorithm EG. The resulting l- and t-embracing systems (5), (6) might be rewritten in the form

$$y(x) = B_1(x)y(x-1) + \cdots + B_r(x)y(x-r) + \varphi(x), \quad (7)$$

where $\varphi(x) = \bar{A}_r^{-1}(x-r)\bar{b}(x-r)$, $B_i(x) = -\bar{A}_r^{-1}(x-r)\bar{A}_{r-i}(x-r)$, as well as in the form

$$y(x) = C_1(x)y(x+1) + \cdots + C_r(x)y(x+r) + \psi(x), \quad (8)$$

where $\psi(x) = \bar{\bar{A}}_0^{-1}(x)\bar{\bar{b}}(x)$, $C_i(x) = -\bar{\bar{A}}_0^{-1}(x)\bar{\bar{A}}_i(x)$, $i = 1, 2, \ldots, r$.

3. MEROMORPHIC SOLUTIONS: LOWER BOUNDS ON VALUATIONS

In this section we suppose that $k \subseteq \mathbb{C}$. If $r(x) \in k(x)$, $p(x) \in \mathrm{Irr}(k[x])$, $\alpha \in \bar{k}$ and $p(\alpha) = 0$, then evidently $\mathrm{val}_{x-\alpha} r(x) = \mathrm{val}_{p(x)} r(x)$.

3.1 On a property of valuations of meromorphic solutions of difference systems

Let $y(x)$ be a meromorphic solution of (1) and $\alpha \in \mathbb{C}$. Then in view of the existence of the l- and t-embracing systems, the value $\mathrm{val}_{x-\alpha} y(x+n)$ is bounded from below when n runs through $\mathbb{Z}$. This can be formulated as the following property of a meromorphic solution $y(x)$:

PROPOSITION 1. *Let $p(\alpha) = 0$ for $p(x) \in \mathrm{Irr}(k[x])$. Let $\bar{A}_r(x)$ be the leading matrix of an l-embracing system and $\bar{\bar{A}}_0(x)$ be the trailing matrix of a t-embracing system for (1). Let*

$$V(x) = \mathrm{den}\, \bar{A}_r^{-1}(x-r), \quad W(x) = \mathrm{den}\, \bar{\bar{A}}_0^{-1}(x).$$

In this case

(i) If N_0 is such that $p(x) \nmid V(x+n_0)W(x+n_0)$ for all integers $n_0 \geqslant N_0$ then the values $\min_{n=n_0}^{n_0+r-1} \mathrm{val}_{x-\alpha} y(x+n)$ are equal for all integers $n_0 \geqslant N_0$, i.e., there exists $\lambda \in \mathbb{Z}$ such that

$$\forall_{n_0 \in \mathbb{Z},\, n_0 \geqslant N_0} \min_{n=n_0}^{n_0+r-1} \mathrm{val}_{x-\alpha} y(x+n) = \lambda.$$

(ii) If N_1 is such that $p(x) \nmid V(x+n_1)W(x+n_1)$ for all integers $n_1 \leqslant N_1$ then the values $\min_{n=n_1-r+1}^{n_1} \mathrm{val}_{x-\alpha} y(x+n)$ are equal for all integer $n_1 \leqslant N_1$, i.e., there exists $\mu \in \mathbb{Z}$ such that

$$\forall_{n_1 \in \mathbb{Z},\, n_1 \leqslant N_1} \min_{n=n_1}^{n_1+r-1} \mathrm{val}_{x-\alpha} y(x+n) = \mu.$$

Proof. Going back to the systems (7), (8), it is easy to see that the denominators of the matrices $B_i(x)$ and of the vector $\varphi(x)$ divide $V(x)$, while the denominators of the matrices $C_i(x)$ and of the vector $\psi(x)$ divide $W(x)$, for $i = 1, 2, \ldots, r$. As a consequence, if the inequality $\min_{n=l}^{l+r-1} \mathrm{val}_{x-\alpha} y(x+n) < \min_{n=l+1}^{l+r} \mathrm{val}_{x-\alpha} y(x+n)$ is valid for an integer l then $p(x) \mid V(x+l)$. Similarly, if the inequality $\min_{n=l}^{l+r-1} \mathrm{val}_{x-\alpha} y(x+n) > \min_{n=l+1}^{l+r} \mathrm{val}_{x-\alpha} y(x+n)$ is valid for an integer l then $p(x) \mid W(x)$. This implies the validity of assertions (i) and (ii). $\square$

It follows from Proposition 1 that if the polynomial $V(x)W(x)$ has no root belonging to the set $\alpha+\mathbb{Z}$ then values $\min_{n=n_0}^{n_0+r-1} \mathrm{val}_{x-\alpha} y(x+n)$ are equal for all $n_0 \in \mathbb{Z}$.

From here on we will assume that

$$\alpha,\ p(x),\ V(x),\ W(x),\ N_0,\ N_1,\ \lambda,\ \mu$$

are as described above.

Below we consider problems of computing lower bounds on the valuations of meromorphic solutions of a system of the form (1) at a point α.

3.2 The first problem of computing lower bounds

Here we consider the first of the problems formulated in the Introduction: the problem of computing a lower bound on $\mathrm{val}_{x-\alpha} y(x)$ assuming that

$$\min_{n=n_0}^{n_0+r-1} \mathrm{val}_{x-\alpha} y(x-n) \geqslant v \quad (9)$$

for some non-negative integer n_0 and integer v, or, similarly, assuming that

$$\min_{n=n_1}^{n_1+r-1} \mathrm{val}_{x-\alpha} y(x+n) \geqslant w \quad (10)$$

for some non-negative integer n_1 and integer w.

THEOREM 1. *Let $y(x)$ be a meromorphic solution of a system of the form (1). We distinguish two cases: the system is homogeneous (i.e., $b(x)$ is equal to zero identically), and the general case.*

(i) The homogeneous case. If a non-negative integer n_0 and an integer v are such that inequality (9) holds, then

$$\mathrm{val}_{x-\alpha} y(x) \geqslant v - \sum_{n=0}^{n_0-1} \mathrm{val}_{p(x)} V(x-n). \quad (11)$$

Similarly, if a non-negative integer n_1 and an integer w are such that inequality (10) holds, then

$$\mathrm{val}_{x-\alpha} y(x) \geqslant w - \sum_{n=0}^{n_1-1} \mathrm{val}_{p(x)} W(x+n). \quad (12)$$

(ii) The general case. The statement (i) is correct under the additional precondition that v, w are non-positive.

Proof. Considering the equivalent form (7) of the system we see that $\operatorname{den} B_i(x) \mid V(x)$ for $i = 1, 2, \ldots, r$, and $\operatorname{den} \varphi(x) \mid V(x)$. By means of the valuation properties (2), (4) it is straightforward to prove (11) by induction from $n_0 - 1$ to 0.

In the inhomogeneous case the valuation of the last term of the right-hand side of (7) does not depend on valuations of $y(x-1), y(x-2), \ldots, y(x-r)$ which can be positive. In general, formula (11) is not correct if $v > 0$. The estimate (12) can be proved similarly for both cases. $\square$

Remark 1. *If $v \leqslant \lambda$ and $w \leqslant \mu$ then in the homogeneous case the following inequality holds for any mutual disposition of the point α and the roots of the polynomials $W(x), V(x)$:*

$$\operatorname{val}_{x-\alpha} y(x) \geqslant \max \left\{ v - \sum_{n \in \mathbb{N}} \operatorname{val}_{p(x)} V(x-n), \right.$$
$$\left. w - \sum_{n \in \mathbb{N}} \operatorname{val}_{p(x)} W(x+n) \right\} \quad (13)$$

(the sums on the right-hand side of the inequality are finite). In the general case, inequality (13) is correct under the additional precondition that v, w are non-positive.

In a straightforward manner, Theorem 1 and Remark 1 yield an algorithm for solving the first problem of computing lower bounds. We will refer to it as $\mathbf{LB^1}$ further on.

3.3 The second problem of computing lower bounds

Here we consider the second of the problems formulated in the Introduction: the problem of computing lower bounds on $\operatorname{val}_{x-\alpha} y_i(x)$, $i = 1, 2, \ldots, m$, assuming that for some non-negative integer n_0 lower bounds on the valuations

$$\operatorname{val}_{x-\alpha} y_i(x-n)$$

are given (separately for $n = n_0, n_0+1, \ldots, n_0+r-1$ and $i = 1, 2, \ldots, m$), or that for some non-negative integer n_1 lower bounds on the valuations

$$\operatorname{val}_{x-\alpha} y_i(x+n)$$

are given (separately for $n = n_1, n_1+1, \ldots, n_1+r-1$ and $i = 1, 2, \ldots, m$).

3.3.1 Tropical calculations

We will consider the set $\mathbb{Z}^\circ = \mathbb{Z} \cup \{\infty\}$ with operations

$$a \odot b = a + b, \quad a \oplus b = \min\{a, b\}, \quad (14)$$

which replace the usual operations of multiplication $\cdot$ and addition $+$. The neutral element for $\odot$ is 0, and ∞ plays the analogous role for $\oplus$. Both operations are associative, and $\odot$ is distributive over $\oplus$.

In recent years, calculations in semi-rings of this type were named "tropical". In 1962 such calculations were used (without any tropical terminology, and with $\mathbb{R}$ instead of $\mathbb{Z}$) by R. W. Floyd ([10]) for finding shortest paths between all pairs of vertices of a graph (see also [7, Sect. 5.6, 5.8]). In our paper we use such calculations for estimating valuations, lowering the level of operations in the sense noted by V. I. Arnold in [8, §2]: '... The modern term "tropical", taken by me to mean "exotic", is used when one lowers the level of the algebraic operations, transforming multiplication to addition, and replacing addition by the lower-level "tropical addition" operation, with respect to which the normal addition is distributive, as is normal multiplication with respect to normal addition ...'

The operations (14) can be extended to matrices and vectors with entries from $\mathbb{Z}^\circ$, e.g., if

$$A = (a_{ij})_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m}, \ B = (b_{ij})_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m}$$

are matrices and whose entries belong to $\mathbb{Z}^\circ$ then we define $C = A \oplus B$ and $D = A \odot B$ by

$$c_{ij} = a_{ij} \oplus b_{ij}, \ \ d_{ij} = \bigoplus_{l=1}^{m} a_{il} \odot b_{lj},$$

$i, j = 1, 2, \ldots, m$. If $g = (g_l)_{1 \leqslant l \leqslant m}$ is a vector whose entries belong to $\mathbb{Z}^\circ$ and $h = Ag$ then $h_i = \bigoplus_{l=1}^{m} a_{il} \odot g_l$, $i = 1, 2, \ldots, m$, and so on.

Let $p(x) \in \operatorname{Irr}(k[x])$ be fixed. For an arbitrary function $f(x) \in k(x)$ we consider the double-sided sequence

$$f^\circ(n) = \operatorname{val}_{p(x)} f(x+n), \ \ n = 0, \pm 1, \pm 2, \ldots$$

of elements of $\mathbb{Z}^\circ$. Similarly, for an arbitrary matrix $A(x) \in \operatorname{Mat}_m(k(x))$ we consider the matrix $A^\circ(n)$ whose entries are sequences of the mentioned form. The same for rational function vectors.

If we know the values of the components of $y(x+\alpha+n)$ for some r consecutive values of n then we can try to use, e.g., the system (7) for computing the values of the components for some other values of n. However, in this way we can be faced with some obstacles since $B_i(x), \varphi(x)$ can have poles in $\alpha + \mathbb{Z}$. The following theorem shows that using in a similar way the tropical calculations, we encounter no problems when computing lower bounds on the valuations of the components of $y(x+\alpha+n)$.

Theorem 2. *Let the components of vectors*

$$v(n) = (v_1(n), v_2(n), \ldots, v_m(n))^T,$$

$$w(n) = (w_1(n), w_2(n), \ldots, w_m(n))^T$$

be sequences of elements of $\mathbb{Z}^\circ$, and let $y(x)$ be a meromorphic solution of (7). In this case:

(i) If an integer n_0 is such that $\operatorname{val}_{x-\alpha} y_i(x-n) \geqslant v_i(n)$ for $n = n_0, n_0+1, \ldots, n_0+r-1$, $i = 1, 2, \ldots, m$, and for all $n < n_0$ the sequence $v(n)$ is defined by

$$v(n) = B_1^\circ(n) \odot v(n+1) \oplus \cdots \oplus B_r^\circ(n) \odot v(n+r) \oplus \varphi^\circ(n) \quad (15)$$

then

$$\operatorname{val}_{x-\alpha} y_i(x-n) \geqslant v_i(n), \ \ i = 1, 2, \ldots, m,$$

for all $n = n_0 - l$, $l = 1, 2, \ldots$

(ii) If an integer n_1 is such that $\operatorname{val}_{x-\alpha} y_i(x+n) \geqslant w_i(n)$ for $n = n_1, n_1+1, \ldots, n_1+r-1$, $i = 1, 2, \ldots, m$, and for

all $n < n_1$ the sequence $v(n)$ is defined by

$$w(n) = C_1^\circ(n) \odot w(n+1) \oplus \cdots \oplus C_r^\circ(n) \odot w(n+r) \oplus \psi^\circ(n) \quad (16)$$

then

$$\mathrm{val}_{x-\alpha} y_i(x+n) \geqslant w_i(n), i = 1, 2, \ldots, m,$$

for all $n = n_1 - l$, $l = 1, 2, \ldots$

Proof. If we substitute $x - n$ for x into (7), it relates $y(x-(n+1)), \ldots, y(x-(n+r))$ and $y(x-n)$ and gives a possibility to compute $y(x-n)$ if $y(x-(n+1)), \ldots, y(x-(n+r))$ are already known, in the same way (15) relates the bounds on the valuations of $y(x-(n+1)), \ldots, y(x-(n+r))$ and the valuation of $y(x-n)$ and, hence, gives a possibility to compute $v(n)$ if $v(n+1), \ldots, v(n+r)$ are already known. Similarly, if we substitute $x+n$ for x into (8), it relates $y(x+(n+1)), \ldots, y(x+(n+r))$ and $y(x+n)$ and gives a possibility to compute $y(x+n)$ if $y(x+(n+1)), \ldots, y(x+(n+r))$ are already known, in the same way (16) relates the bounds on the valuations of $y(x+(n+1)), \ldots, y(x+(n+r))$ and the valuation of $y(x+n)$ and, hence, gives a possibility to compute $w(n)$ if $w(n+1), \ldots, w(n+r)$ are already known. By means of the valuation properties (2), (4) and the definition of the operations $\odot, \oplus$ both (i) and (ii) can be proven by induction on l. □

Thus, if we know lower bounds on

$$\mathrm{val}_{x-\alpha} y_i(x-n), \quad i = 1, 2, \ldots, m, \tag{17}$$

for $n = n_0, n_0+1, \ldots, n_0+r-1$ then using (15) we can step by step compute lower bounds on (17) for $n = n_0 - 1, n_0 - 2, \ldots$

Similarly, if we know lower bounds on

$$\mathrm{val}_{x-\alpha} y_i(x+n), \quad i = 1, 2, \ldots, m, \tag{18}$$

for $n = n_1, n_1+1, \ldots, n_1-r+1$ then using (16) we can step by step compute lower bounds on (18) for $n = n_1 - 1, n_1 - 2, \ldots$ This is an algorithm for solving the second problem of computing lower bounds. We will refer to it as $\mathbf{LB_T^2}$ further on. Note that if $-n_0 < 0 < n_1$ then we can compute two lower bounds on $\mathrm{val}_{x-\alpha} y_i(x)$, and take the minimal one, $i = 1, 2, \ldots, m$.

REMARK 2. *Let the matrices $B_1^\circ(n), B_2^\circ(n), \ldots, B_r^\circ(n)$ and $\varphi^\circ(n)$ be zero for some $n \in \mathbb{Z}$. Then all the components of the vector $B_1^\circ(n) \odot v(n-1) \oplus \cdots \oplus B_r^\circ(n) \odot v(n-r) \oplus \varphi^\circ(n)$ have the same value which is equal to the minimum of all the components of the vector $(v(n-1), v(n-2), \ldots, v(n-r))^T$. A similar assertion holds for $C_1^\circ(n) \odot w(n+1) \oplus \cdots \oplus C_r^\circ(n) \odot w(n+r) \oplus \psi^\circ(n)$.*

3.3.2 Complexity Analysis

We now give a complexity analysis of $\mathbf{LB^1}$ and $\mathbf{LB_T^2}$.

For the sake of simplicity we assume that the given system is homogeneous and its leading and trailing matrices are invertible (if it is not the case then both algorithms use the same embracing systems construction approach and this step makes no difference for the complexity of the algorithms). Additionally we assume that all matrix entries are polynomials (i.e., the denominators are cleared) and that the upper bound on their degrees is d. In our analysis, the complexity is the number of field operations in $k(x)$ in the worst case. Both in $\mathbf{LB^1}$ and $\mathbf{LB_T^2}$, the degrees of the polynomials involved are bounded by md.

Let

$$q = \deg p(x), \quad s = n_0 + n_1$$

and let the complexity of $m \times m$ matrix multiplication be $\Theta(m^\omega)$ with $2 < \omega \leqslant 3$ (we quote the definition of Θ from [14]: $f(n) = \Theta(g(n))$ if there exist positive constants C, C' and n_0 such that $Cf(n) \leqslant g(n) \leqslant C'f(n)$ for all $n \geqslant n_0$). The valuations at $p(x)$ are computed by iterative divisions.

The complexity of computing $V(x)$ and $W(x)$ is $\Theta(m^\omega)$. Since $\deg V(x)$ and $\deg W(x)$ are bounded by md, it follows that the computations using (11) require no more than $\frac{md}{q} + n_0$ divisions — it is obvious that the summands on the right-hand side of the inequality may be considered as the valuations of the same polynomial $V(x)$ at consecutive points and, hence, on the one hand the sum of the valuations is not greater than $\frac{md}{q}$, and on the other hand, each summand requires at least one division even if the valuation at this point is zero. We do not take into account the complexity of the summation itself since the complexity of integer addition is negligible compared to the complexity of operations in $k(x)$. Similarly, the computations using (12) require no more than $\frac{md}{q} + n_1$ divisions. The total complexity of $\mathbf{LB^1}$ for fixed $p(x)$ is therefore

$$\Theta(m^\omega + md + s). \tag{19}$$

The complexity of computing (7) and (8) is $\Theta(rm^\omega)$. The resulting degrees of the numerator and denominator of each of the matrix entries are bounded by md. It follows that computations using (15) require no more than $2m^2 r(\frac{md}{q} + n_0)$ divisions – the factor 2 stems from the fact that in order to compute the valuation of a rational function we compute the valuations of its numerator and denominator. Again we do not take into account the complexity of integer addition and multiplication. Similarly, the computations using (16) require no more than $2m^2 r(\frac{md}{q} + n_1)$ divisions. The total complexity of $\mathbf{LB_T^2}$ for fixed $p(x)$ is therefore

$$\Theta(rm^\omega + rm^3 d + rm^2 s). \tag{20}$$

It is natural to assume $d \geqslant 1$ which, together with $\omega \leqslant 3$, implies that the complexity of $\mathbf{LB_T^2}$ is

$$\Theta(rm^3 d + rm^2 s). \tag{21}$$

We know that the degrees of all involved polynomials are bounded by md. This fact may be easily used to rewrite the complexity estimates (19), (20), (21) as the worst-case estimates of the numbers of operations in k. Here linearity of both kinds of estimates in r and s is of prime importance.

3.3.3 Operating on matrices with entries from $k(x)$

If $M(x) \in \mathrm{Mat}_m(k(x))$ and $1 \leqslant i \leqslant m$ then the minimum of the valuations of the i-th row entries of a matrix $M(x)$ will be denoted by $\mathrm{val}_{x-\alpha}^{(i)} M(x)$.

The algorithm of M. van Hoeij ([12]) for finding denominator bounds for rational solutions of a system of the form (3) is based on the following observation. If a rational solution $y(x)$ is such that $\mathrm{val}_{x-\alpha} y(x-n_0) = 0$ for a non-negative integer n_0, then for any $1 \leqslant i \leqslant m$ we have

$$\mathrm{val}_{x-\alpha} y_i(x) \geqslant \mathrm{val}^{(i)}_{x-\alpha} \left(A(x-1)A(x-2)\dots A(x-n_0)\right). \quad (22)$$

Similarly, if $\mathrm{val}_{x-\alpha} y(x+n_1) = 0$ for a non-negative integer n_1, then for any $1 \leqslant i \leqslant m$ we have

$$\mathrm{val}_{x-\alpha} y_i(x) \geqslant \mathrm{val}^{(i)}_{x-\alpha} \left(A^{-1}(x)A^{-1}(x+1)\dots A^{-1}(x+n_1-1)\right). \quad (23)$$

Denominator bounds computed using the formulas (22), (23) are in some sense optimal ([12, Thm 1]).

This algorithm can be considered as an algorithm for solving the second problem of computing lower bounds for the specific case of normal first-order systems and zero a-priori known bounds. We will refer to it as $\mathbf{LB^2_M}$ further on. (The computational complexity of this approach is quite high since the entries of the matrix product "swell" quickly when the number of factors grows.)

In our experiments with systems of the form (3) the algorithms $\mathbf{LB^2_M}$ and $\mathbf{LB^2_T}$ always produce the same bounds. Taking into account some optimality of the algorithm $\mathbf{LB^2_M}$ mentioned above, it is not excluded that examples could exist where $\mathbf{LB^2_M}$ gives more accurate bounds than $\mathbf{LB^2_T}$. However anyway, this would be achieved at the cost of a significant increase in the computation time.

REMARK 3. *Tropical conversion of the algorithm of van Hoeij (formulas (22), (23)) leads to tropical products*

$$(A^{-1})^\circ(n) \odot (A^{-1})^\circ(n+1) \odot \cdots \odot (A^{-1})^\circ(n+n_0-1),$$

and

$$A^\circ(n-1) \odot A^\circ(n-2) \odot \cdots \odot A^\circ(n-n_1)$$

instead of products of matrices with entries from $k(x)$. Considerations similar to those of Section 3.3.2 show that the complexity of this version is linear on s:

$$\Theta(m^\omega + m^3 d + m^2 s).$$

It coincides with the complexity of $\mathbf{LB^2_T}$ for first-order systems ($r = 1$).

4. IMPLEMENTATION & EXPERIMENTS

We implemented in Maple ([16]) the algorithms $\mathbf{LB^1}$ and $\mathbf{LB^2_T}$. To construct the embracing systems we used the implementation of algorithm EG′ described in [3] [1]. The implementation of algorithm $\mathbf{LB^1}$ is similar to the implementation of algorithm $\mathbf{A}'_U$ from [4] for constructing universal denominators. For the purpose of comparison, the algorithm $\mathbf{LB^2_M}$ is also implemented. The latter implementation is partially based on our implementation of the version of van Hoeij's algorithm from [5].

EXPERIMENT 1.

We use the first-order system given on the help page of the procedure `RationalSolutions` from the package `LinearFunctionalSystems` in Maple:

$$\begin{array}{rcl}
\{(x+3)(x+6)(x+1)(x+5)xy_1(x+1)- & & \\
-(x-1)(x+2)(x+3)(x+6)(x+1)y_1(x)- & & \\
-x(x^6+11x^5+41x^4+65x^3+50x^2-36)y_2(x)+ & & \\
+6(x+2)(x+3)(x+6)(x+1)xy_4(x) & = & 0, \\
(x+6)(x+2)y_2(x+1)-x^2y_2(x) & = & 0, \\
(x+6)(x+1)(x+5)xy_3(x+1)+ & & \\
+(x+6)(x+1)(x-1)y_1(x)- & & \\
-x(x^5+7x^4+11x^3+4x^2-5x+6)y_2(x)- & & \\
-y_3(x)(x+6)(x+1)(x+5)x+ & & \\
+3(x+6)(x+1)x(x+3)y_4(x) & = & 0, \\
(x+6)y_4(x+1)+x^2y_2(x)- & & \\
-(x+6)y_4(x) & = & 0\}
\end{array} \quad (24)$$

For this system, $W(x) = (x-1)(x+2)(x+3)(x+6)(x+1)(x+5)x^2$, $V(x) = (x+1)(x+2)(x+5)x(x+4)(x-1)$.

Let $\lambda = 0, \mu = 0$ for the solutions to be found. $\mathbf{LB^1}$ allows us to compute with the help of inequality (13) that, for example:

$$\begin{array}{rcl}
\mathrm{val}_{x-4} y(x) & \geqslant & 0, \\
\mathrm{val}_{x-1} y(x) & \geqslant & -1, \\
\mathrm{val}_{x+4} y(x) & \geqslant & -2, \\
\mathrm{val}_{x+8} y(x) & \geqslant & 0.
\end{array} \quad (25)$$

If $\lambda = 0, \mu = -1$ instead then

$$\begin{array}{rcl}
\mathrm{val}_{x-4} y(x) & \geqslant & 0, \\
\mathrm{val}_{x-1} y(x) & \geqslant & -1, \\
\mathrm{val}_{x+4} y(x) & \geqslant & -3, \\
\mathrm{val}_{x+8} y(x) & \geqslant & -1.
\end{array} \quad (26)$$

If it is known that $\mathrm{val}_{x+4} y(x+10) \geqslant 0$ and $\mathrm{val}_{x+4} y(x-10) \geqslant 0$ then by applying $\mathbf{LB^2_T}$ one can find

$$\begin{array}{rcl}
\mathrm{val}_{x+4} y_1(x) & \geqslant & -2, \\
\mathrm{val}_{x+4} y_2(x) & \geqslant & -1, \\
\mathrm{val}_{x+4} y_3(x) & \geqslant & -2, \\
\mathrm{val}_{x+4} y_4(x) & \geqslant & -1.
\end{array} \quad (27)$$

The same result is found by applying $\mathbf{LB^2_M}$ as well. As it has been expected, these bounds are more accurate than the bound produced by $\mathbf{LB^1}$. However the computation of the bound with $\mathbf{LB^1}$ took only 0.093 seconds, with $\mathbf{LB^2_T}$ it took 0.405 seconds, and with $\mathbf{LB^2_M}$ it took 0.967 seconds[2].

[1] It is available at http://www.ccas.ru/ca/doku.php/eg. The extended version of the implementation is available as the procedure `MatrixTriangularization` from the package `LinearFunctionPackage` in Maple — the extension includes so called vertical shifts which make the use of this version less straightforward.

[2] For all the experiments: Maple 13, Windows XP, Pentium 4 1.7 GHz, 1.5 GB RAM.

Let us mention the interesting fact that in the case when the problem of bounding the valuation is an auxiliary task for solving another problem, it can happen that the bounds obtained with $\mathbf{LB^2_T}$ (or $\mathbf{LB^2_M}$), although more accurate than the bounds obtained with $\mathbf{LB^1}$, save no computation time, or even lead to additional costs when used on the further steps in solving the main problem. For example, our experiments which use the valuation bounding as an auxiliary task for computing rational solutions of systems of the form (1), show that this phenomenon occurs for the system (24). This is related to the fact that the more accurate bounds for denominators of the desired rational solutions, obtained by means of $\mathbf{LB^2_T}$ (or $\mathbf{LB^2_M}$), in this case yield a system whose polynomial solutions take longer to find on the next step than those of the system resulting by using the less accurate bounds obtained with $\mathbf{LB^1}$. Nevertheless, in most cases the more accurate bounds lead to shorter overall running times, which is why efficient computation of more accurate bounds is of practical value for this problem as well.

Experiment 2.

Let us consider an example of a system of higher order. For that we modify system (24) from experiment 1 by shifting some of the equations ($x \to x+1$ in the first and second equations, $x \to x+4$ in the fourth equation). For this system, $W(x) = (x-1)(x+3)(x+1)(x+5)(x+6)(x+2)x^2$, $V(x) = (x+2)x(x+4)(x-1)(x+5)(x+1)$.

Let $\lambda = 0, \mu = 0$ for the solutions to be found. By means of $\mathbf{LB^1}$ and inequality (13) we find that, for example, the bounds (25) are valid. If instead $\lambda = 0, \mu = -1$ for the solutions to be found, we find that the bounds (26) are valid.

If we know that

$$\mathrm{val}_{x+4} y_i(x+10+k) \geqslant 0, \quad \mathrm{val}_{x+4} y_i(x-10-k) \geqslant 0 \quad (28)$$

for $i = 1, \ldots, 4$, $k = 0, \ldots, 4$, then we can compute with $\mathbf{LB^2_T}$ that the bounds (27) are valid. As it has been expected, the bounds are more accurate than the bound produced by $\mathbf{LB^1}$. However the computation of the bound with $\mathbf{LB^1}$ took only 0.125 seconds while computation with $\mathbf{LB^2_T}$ took 0.515 seconds ($\mathbf{LB^2_M}$ is not applicable in this case since it works with first-order systems only).

All the results coincide with the analogous results for the original first-order system. Taking into account the way the considered system was constructed this is quite expected, since the solutions of the system obtained after the shifts coincide with those of the original system.

Let us find bounds on the valuations of the solutions satisfying conditions which are different from (28). If we know that

$$\mathrm{val}_{x+4} y_i(x+10+k) \geqslant 1, \quad \mathrm{val}_{x+4} y_i(x-10-k) \geqslant 1 \quad (29)$$

for $i = 1, \ldots, 4$, $k = 0, \ldots, 4$, then we can find with $\mathbf{LB^2_T}$ the following bounds:

$$\begin{array}{rcl} \mathrm{val}_{x+4} y_1(x) & \geqslant & -1, \\ \mathrm{val}_{x+4} y_2(x) & \geqslant & 0, \\ \mathrm{val}_{x+4} y_3(x) & \geqslant & -1, \\ \mathrm{val}_{x+4} y_4(x) & \geqslant & 0. \end{array} \quad (30)$$

For conditions that differ from (29) only in a single component in a single point: $\mathrm{val}_{x+4} y_2(x-10) \geqslant 0$, we find with $\mathbf{LB^2_T}$ the following bounds:

$$\begin{array}{rcl} \mathrm{val}_{x+4} y_1(x) & \geqslant & -2, \\ \mathrm{val}_{x+4} y_2(x) & \geqslant & 0, \\ \mathrm{val}_{x+4} y_3(x) & \geqslant & -2, \\ \mathrm{val}_{x+4} y_4(x) & \geqslant & -1. \end{array}$$

If, in addition, $\mathrm{val}_{x+4} y_2(x+10) \geqslant 0$ (thus, the difference from (29) is in one component in two points), then we find out with $\mathbf{LB^2_T}$ that the bounds (27) are valid again.

Experiment 3.

We generate systems of order $r > 1$ containing five equations, whose coefficients are polynomials with random integer roots from $[-9, 9]$. The zero a-priori known bounds are given in the points $20, 21, \ldots, 20+r-1$ and $-20-r+1, -20-r+2, \ldots, -20$.

As in Experiment 2, the bounds are found with $\mathbf{LB^1}$ and $\mathbf{LB^2_T}$. When the order of the systems generated in this way grows, the chance to obtain non-zero valuation bounds decreases. When the bounds are non-zero, algorithm $\mathbf{LB^2_T}$ turns out to produce more accurate results than algorithm $\mathbf{LB^1}$, in accord with the previous experiments.

In the table below we list the total time taken by each of the algorithms to compute bounds at the points -2 and -4 for 10 generated systems of each of the orders $r = 2, 6, 10$.

	r=2	r=6	r=10
$\mathbf{LB^1}$	8.766	12.155	13.954
$\mathbf{LB^2_T}$	56.294	181.676	288.379

As expected, computation time of algorithm $\mathbf{LB^2_T}$ on systems of order r grows linearly with r, provided that all the other size-related parameters are fixed.

It seems rather natural that in all the examples considered in experiments 1 – 3, computation times of algorithm $\mathbf{LB^2_T}$ (which produces more accurate bounds) are greater than the corresponding computation times of algorithm $\mathbf{LB^1}$ (which is faster). To take advantage of the strengths of both algorithms, we can apply the following strategy: In the first step, we compute bounds with $\mathbf{LB^1}$. If the obtained bounds are sufficiently accurate for our purpose, we stop. Otherwise, these bounds are refined with $\mathbf{LB^2_T}$ in the second step. For example, if $\mathbf{LB^1}$ gives the bound 0 for the valuation at $x = 0$, then the solution to be found has no pole there.

If this information is sufficient for our purpose, the costly application of $\mathbf{LB_T^2}$ in the second step can be avoided.

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