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RATIONAL SOLUTIONS OF LINEAR DIFFERENTIAL AND DIFFERENCE EQUATIONS WITH POLYNOMIAL COEFFICIENTS*

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Linear differential and difference equations whose coefficients and right-hand sides are polynomials are considered. The problem of constructing all rational solutions of an equation is solved.

Introduction.

The quest for algorithms for solving differential equations is one of the major problems of computer algebra. No less important is the solution of difference equations, but the algorithmic theory of such equations is less well developed. In this paper we consider linear differential and difference of equations of arbitrary order with polynomial coefficients. The equations are allowed to be inhomogeneous, but then the right-hand side is also assumed to be a polynomial. We shall present a procedure for constructing all the rational solutions of an equation.

As far as differential equations of this type are concerned, the study of this problem goes back to the nineteenth century. The algorithm proposed here does not involve factorization of the polynomials into irreducible factors; in particular, there is no need to determine all the roots of algebraic equations. In addition, quite wide assumptions will be made concerning the field to which the coefficients of the polynomials belong. As to difference equations, it seems that problems of this kind have been considered hitherto only for linear equations with polynomial coefficients of special forms.

In a previous paper /1/, we considered difference equations with constant coefficients and rational right-hand sides. In this paper we extend the approach proposed in /1/. As before, the fields of the coefficients will be what we call "adequate fields".

Definition. An adequate field is a field K of characteristic 0 with an algorithm for determining the integer roots of equations $p(x)=0$, where $p(x)\in K[x]$. By an integer root we mean a root $n1$, where $n\in Z$, and 1 is the unit of the field K.

The field Q of rational numbers is clearly an adequate field. It is readily seen that a simple algebraic extension $K(\theta)$ (algebraic or transcendental) of an adequate field K is itself adequate. Hence it follows, in particular, that the following fields are adequate: the field $Q(\sqrt{-1})$ of rational complex numbers, the field $Q(t_1,\ldots,t_m)$ of rational functions over Q in arbitrarily many variables, the field of algebraic functions over Q in arbitrarily many variables $t_1,\ldots,t_m$, and so on.

Throughout the sequel, K will denote an arbitrary adequate field.

We will adopt the following convention. If $p(x)\in K[x]$ is a coefficient in an equation, then the statement that $p(x)\neq 0$, will mean that $p(x)$ is not zero as an element of the ring of polynomials in x, i.e., at least one of its coefficients does not vanish. The statement that a polynomial is equal to zero should be understood in a similar spirit

We mention that algorithms to determine polynomial solutions of equations of the above type were considered in /2/, and in fact the algorithms presented below will be based on these purely polynomial algorithms. The cases of differential and difference equations will be discussed separately.

1. *Rational solutions of differential equations.*

Consider the equation

$$\sum_{\nu=0}^{n} a_\nu(x)F^{(\nu)}(x)=b(x), \qquad (1)$$

where $a_0(x),\ldots,a_n(x)$, $b(x)\in K[x]$. Temporarily, replace the original coefficient field K by its closure $\bar K$. Let $F(x)\in\bar K(x)$ be a solution of (1). Expanding the function $F(x)$ in the field $\bar K(x)$ as a sum of partial fractions, one can show, first, that every root $\xi\in\bar K$ of the denominator of $F(x)$ (more precisely, of the denominator of its irreducible form) is also a root of the polynomial $a_n(x)$; and, second, the exponent of the power of the factor $x-\xi$ in the denominator of the irreducible form of $F(x)$ is the absolute value of a certain negative integer root of an algebraic equation, known in the theory of differential equations as the defining equation /3/. The defining equation is found as follows. Express the coefficients $a_0(x),\ldots,a_n(x)$ in (1) as

$$a_\nu(x)=(x-\xi)^{\alpha_\nu}h_\nu(x), \qquad \nu=0,1,\ldots,n, \qquad (2)$$

where $h_\nu(x)$ is a polynomial indivisible by $x-\xi$ if $a_\nu(x)\neq 0$; if $a_\nu(x)=0$ then $\alpha_\nu=\infty$, $h_\nu(x)=0$. Let the polynomial $d(r)$ be the coefficient of the lowest power of $x-\xi$ in the expression

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$$\sum_{\nu=0}^{n} r(r-1)\dots(r-\nu+1)h_\nu(\xi)(x-\xi)^{\alpha_\nu-\nu}; \tag{3}$$

then $d(r)=0$ is the defining equation.

We note at once that by Taylor's formula

$$h_\nu(\xi)=a_\nu^{(\alpha_\nu)}(\xi)/\alpha_\nu!\,,$$

and so (3) can be rewritten as

$$\sum_{\nu=0}^{n} r(r-1)\dots(r-\nu+1)\frac{a_\nu^{(\alpha_\nu)}(\xi)}{\alpha_\nu!}(x-\xi)^{\alpha_\nu-\nu}. \tag{4}$$

The defining equation can be used to derive an upper bound for the power to which $x-\xi$ occurs in the denominator of a rational solution of the differential Eq.(1): the absolute value of the least non-positive integer root of the defining equation is such a bound. The essential point here is that it is not necessary to go from K to $\bar{K}$; all we need is to consider successive simple algebraic extensions of the type $K(\xi)$, $u_\mu(\xi)=0$, where $u_1(x)$, $u_2(x),\dots \in K[x]$ are the irreducible factors in $K[x]$ of $a_n(x)$. Nevertheless, there is a simpler algorithm for deriving upper bounds for these multiplicities, not requiring factorization of the polynomial. Before describing this algorithm, we note that we shall sometimes speak of roots of equations $u(x)=0$, $u(x)\in K[x]$. These roots may be elements of either K itself or of some extension (such as $\bar{K}$). The consideration of these roots is necessary only to prove that the algorithm is correct; the algorithm itself does not involve construction of these extensions.

First, simple operations take us from $a_n(x)$ to a polynomial $a_n^*(x)$ which has the same irreducible factors but is square-free: if $u_1^{\beta_1}(x)u_2^{\beta_2}(x)\dots$ is the factorization of $a_n(x)$, then $u_1(x)u_2(x)\dots$ is that of $a_n^*(x)$.

Let Eq.(1) have a solution $F(x)\in K(x)$. Let $u(x)$ be an irreducible factor in $K[x]$ of $a_n(x)$. Suppose that $u(x)$ factors into linear factors $x-\xi_1, x-\xi_2,\dots$ in $\bar{K}[x]$. Then $u(x)$ occurs in the denominator of the irreducible form of $F(x)$ to the same power as each of $x-\xi_1$, $x-\xi_2,\dots$, provided that $F(x)$ is considered as a solution of Eq.(1) over $\bar{K}(x)$. Now the tuple of numbers $\alpha_0,\dots,\alpha_n$ defined in (2) is the same for all $\xi_1,\xi_2,\dots$. We have

$$a_\nu(x)=u(x)^{\alpha_\nu}\hat{h}_\nu(x),\quad \nu=0,1,\dots,n, \tag{5}$$

where $\hat{h}_\nu(x)\in K[x]$, and $\hat{h}(x)$ is divisible by $u(x)$ only if this polynomial is zero.

Thus, with any irreducible polynomial $u(x)\in K[x]$ we can associate a tuple $\alpha_0,\dots,\alpha_n$, which we shall call the multi-exponent of $u(x)$ in Eq.(1). A polynomial which is the product of several irreducible polynomials having the same multi-exponents will be called a balanced polynomial relative to Eq.(1).

Using the operation of finding the greatest common divisor (LCM) of polynomials, with no need to appeal to factorization, we can express $a_n^*(x)$ as a product of balanced polynomials relative to (1). Together with these polynomials we obtain their multi-exponents. The procedure is easily obtained from the simpler procedure which, given a square-free polynomial $f(x)\in K[x]$ and an arbitrary non-zero $p(x)\in K[x]$, produces a representation of $f(x)$ in $K[x]$ as a product $v_1(x)\dots v_\sigma(x)$ such that

$$p(x)=\bar{p}(x)v_1(x)^{\beta_1}\dots v_\sigma(x)^{\beta_\sigma},$$

where $\bar{p}(x)$ is prime to $f(x)$ and the non-negative integers $\beta_1,\dots,\beta_\sigma$ are pairwise distinct. We describe this last-mentioned procedure, which returns an output in the form of a sequence of pairs $(v_1(x),\beta_1),\dots(v_\sigma(x),\beta_\sigma)$ with $0\leqslant\beta_1<\dots<\beta_\sigma$.

We first find $g(x)=\mathrm{LCM}(f(x),p(x))$. If $\deg g(x)=0$, the application of the procedure ends and the output is a sequence consisting of a single pair $(f(x),0)$. Suppose, then, that $\deg g(x)>0$. Let $f(x)=\check{f}(x)g(x)$, $p(x)=\check{p}(x)g(x)^\beta$, where $\beta>0$, and $\check{p}(x)$ is not divisible by $g(x)$; we may assume that either $\check{f}(x)=1$ or $\deg\check{f}(x)>0$. We now apply our procedure recursively to the polynomials $g(x)$ and $\check{p}(x)$. Suppose the outcome is $(w_1(x),\gamma_1),\dots,(w_\rho(x),\gamma_\rho)$. If $\check{f}(x)=1$, then the output for input data $f(x)$, $p(x)$ will be $(w_1(x),\gamma_1+\beta),\dots,(w_\rho(x),\gamma_\rho+\beta)$; otherwise it will be $(f(x),0)$, $(w_1(x),\gamma_1+\beta),\dots(w_\rho(x),\gamma_\rho+\beta)$.

To obtain a representation of $a_n^*(x)$ as a product of balanced factors relative to (1) and to evaluate its multi-exponents, we first apply the above procedure to $a_n^*(x)$ and $a_{i_1}(x)$, where $a_{i_1}(x)$ is the first non-zero polynomial among $a_0(x),\dots,a_n(x)$. Having obtained the corresponding sequence of pairs $(v_1(x),\beta_1),\dots,(v_\tau(x),\beta_\tau)$, we can then apply the procedure to $v_1(x)$ and $a_{i_2}(x),\dots,v_\tau(x)$ and $a_{i_2}(x)$, where $a_{i_2}(x)$ is the second non-zero polynomial among $a_0(x),\dots,a_n(x)$. Continuing in this way up to the penultimate non-zero polynomial among $a_0(x),\dots,a_n(x)$ (the last is $a_n(x)$, which remains unused), we obtain the required representation. The multi-exponent of each factor will have i_1-1 first components equal to ∞, then comes one of the numbers $\beta_1,\dots,\beta_\tau$, and then, up to the component with index i_2-1, the symbol ∞, and so on.

Suppose, now, that we have constructed a representation of $a_n^*(x)$ as a product

$$a_n^*(x)=c_1(x)\dots c_m(x), \qquad m\geqslant 1, \tag{6}$$

of balanced polynomilals with respect to (1) and evaluated their multi-exponents. We consider $c_1(x),\dots,c_m(x)$ one by one. Let $c(x)$ be one of these polynomials and $\alpha_0,\dots,\alpha_n$ its multi-exponent. Consider the differences $\alpha_\nu-\nu$, $\nu=0, 1,\dots,n$, and choose the values of ν for which $\alpha_\nu-\nu$ is a minimum. Let these values be $\nu_1,\dots,\nu_k$, $0\leqslant\nu_1<\dots<\nu_k\leqslant n$. We now use (4) to obtain the defining equation. We do not know the value of ξ, and instead use a variable, e.g., x:

$$r(r-1)\dots(r-\nu_1+1)\frac{a_{\nu_1}^{(\alpha_{\nu_1})}(x)}{\alpha_{\nu_1}!}+\dots+r(r-1)\dots(r-\nu_k+1)\frac{a_{\nu_k}^{(\alpha_{\nu_k})}(x)}{\alpha_{\nu_k}!}=0. \tag{7}$$

This gives the defining equation as an equality $d(r, x)=0$, $d(r, x)\in K[r, x]$. Write $d(r, x)$ as $\tilde{d}(x)$, including r in the coefficients: $\tilde{d}(x)\in K[r][x]$. If there is an integer r_0 such that we can choose a root ξ of the equation $c(x)=0$ (we recall that $c(x)$ is one of the factors on the right of (6)) such that $d(r_0, \xi)=0$, then r_0 itself must be a root of the equation $u(r)=0$, where $u(r)$ is the resultant of the polynomials $\tilde{d}(x)$ and $c(x)$. Therefore, determining the least negative root of the equation $u(r)=0$, $u(r)\in K[r]$, and taking its absolute value, we obtain an upper bound for the power to which $c(x)$ occurs in the denominator of the rational solution $F(x)$ of Eq.(1).

To prove that this derivation of upper bounds for the exponents is fully rigorous, we must still show that the resultant $u(r)$ does not vanish identically. Indeed, if this were the case, $d(r, x)$ and $c(x)$ would have a non-trivial common multiple - a polynomial in x of degree greater than zero. But any irreducible factor in $K[x]$ of $c(x)$ occurs in $a_{\nu_i}(x)$ to the power α_{ν_i}, $i=1, 2,\dots, k$. Therefore, the polynomial $a_{\nu_i}^{(\alpha_{\nu_i})}(x)$ is prime to $c(x)$. At the same time, any element of the ring $K[r, x]$ admits of a unique representation in terms of the polynomials 1, r, $r(r-1)$, $r(r-1)(r-2),\dots$ with coefficients in $K[x]$. Thus a polynomial $d(r, x)$ of the form (7) and the polynomial $c(x)$ cannot possibly have a non-trivial common multiple.

If the equation $u(r)=0$ has no negative integer roots, then the denominator of the rational solution $F(x)$ of Eq.(1) need not have a factor $c(x)$. But if such negative integer roots exist then, finding the absolute value of the least such root, we obtain an upper bound for the exponent of the power to which $c(x)$ occurs in the denominator of $F(x)$. Let us suppose that such non-negative integers $\tau_1,\dots,\tau_m$ have been determined for the polynomials $c_1(x),\dots,c_m(x)$ on the right of (6). Then, if Eq.(1) has a rational solution $F(x)$ the denominator of the irreducible form of $F(x)$ divides $c_1(x)^{\tau_1}\dots c_m(x)^{\tau_m}$. But this means that the rational solution may be found in the form

$$\frac{y(x)}{c_1(x)^{\tau_1}\dots c_m(x)^{\tau_m}}, \qquad y(x)\in K[x]. \tag{8}$$

Substituting (8) for $F(x)$ in Eq.(1), we obtain an equation for $y(x)$ with polynomial coefficients. This equation can be dealt with using the algorithm of /2/.

The implementation of our algorithm may be summarized as follows. 1) Free the polynomial $a_n(x)$ of squares (i.e., construct $a_n^*(x)$). 2) Express $a_n^*(x)$ as a product $c_1(x)\dots c_m(x)$ of balanced polynomials relative to (1) and evaluate the corresponding multi-exponents. 3) For each $c_i(x)$, $i=1, 2,\dots, m$, find a polynomial $\tilde{d}_i(x)$ with coefficients in $K[r]$, equal to the left-hand side of Eq.(7) (the tuple $\nu_1,\dots,\nu_k$ is determined from the multi-exponent of $c_i(x)$). 4) For each $i=1, 2,\dots,m$, evaluate the resultant $u_i(x)$ of $\tilde{d}_i(x)$ and $c_i(x)$, find τ_i ($\tau_i=0$ if $u_i(r)$ has no negative integer, roots, otherwise τ_i is the absolute value of the least integer root). 5) Substitute (8), with unknown polynomial $y(x)$ for $F(x)$ in Eq. (1). 6) Use the algorithm of /2/ to investigate and determine polynomial solutions $y(x)$ of the equation.

This completes our description of the algorithm for determining rational solutions of the differential Eq.(1). It remains to observe that, in order to find the least negative r such that $d(r, x)$ and $c(x)$ have a common divisor - a polynomial in x of degree greater than one - one can use Sylvester's form of the resultant. Collins /4/ has proposed an economical modular algorithm for constructing resultants. However, one can avoid explicit construction of the resultant. If $d(r, x)$ and $c(x)$ are considered as polynomials in x whose coefficients are rational functions in r and Euclid's algorithm is used, the required negative integer must be a root of one of the equations obtained by equating the leading coefficients of the polynomials occurring in the sequence of remainders to zero.

A situation requiring the determination of integers r such that given polynomials $f(r, x)$, $g(r, x)$ have a non-trivial common multiple $h(x)$ occurs in the following sections as well. Any of the above methods may be used.

2. *Rational solutions of difference equations.*

In this section it will be convenient to use a recurrent notation for difference equations:

$$\sum_{\nu=0}^{n} a_\nu(x)F(x+\nu)=b(x), \tag{9}$$

where $a_0(x),\dots,a_n(x)$, $b(x)\in K[x]$. The case of constant coefficients was discussed in /1/; the algorithm of this section essentially relies on use of the quantity $\operatorname{Dis}F(x)$ introduced in /1/. We will present all the necessary definitions here, generalizing some of the definitions of /1/.

Let $s(x)$, $t(x)\in K[x]$. In some cases one can choose a non-negative integer r such that $s(x+r)$ and $t(x)$ have a non-trivial common multiple (i.e., a multiple which is a polynomial of degree greater than zero). It follows from the uniqueness of factorization into irreducible polynomials in $K[x]$ that the set of such r is at most finite. Define

$$\operatorname{dis}(s(x),\ t(x))=\max\{r|r\in Z,\ r\geqslant 0,\ \deg \operatorname{LCM}\ (s(x+r),\ t(x))\geqslant 1\}.$$

It should be stressed from the start that " dis " is not defined for all $s(x)$, $t(x)$; for example, it remains undefined for $s(x)=x$, $t(x)=x^2+1$. Nevertheless, $\operatorname{dis}(s(x), s(x))$ is always defined when $\deg s(x)>0$; thus $\operatorname{dis}(x, x)=0$, $\operatorname{dis}(x(x+1), x(x+1))=1$ and so on.. Given $s(x)$ and $t(x)$, the value of $\operatorname{dis}(s(x), t(x))$ can be evaluated, e.g., as the largest non-negative integer root of the equation $d(r)=0$, where $d(r)$ is the resultant of the polynomials $s(x+r)$ and $t(x)$, considered as polynomials in x over $K[r]$.

Now let $F(x)\in K(x)$, and suppose that the irreducible form of the rational function $F(x)$ is $t(x)/s(x)$ and $\deg s(x)>0$. Define

$$\operatorname{Dis}F(x)=\operatorname{dis}(s(x),\ s(x)).$$

Thus, $\operatorname{Dis}F(x)$ is defined for all rational functions which are not polynomials.

The role of $\operatorname{Dis}F(x)$ for difference equations is analogous to that of the maximum order of poles of functions for differential equations.

Assuming that Eq.(9) has a solution $F(x)\in K(x)$ which is not a polynomial, we can find an upper bound for $\operatorname{Dis}F(x)$. It turns out that $\operatorname{Dis}F(x)\leqslant\operatorname{dis}(a_0(x),\ a_n(x))-n$. In fact, let $\operatorname{Dis}F(x)=m\geqslant 0$ and let the irreducible form of $F(x)$ be $t(x)/s(x)$. Let $p_1(x)$, $p_2(x)$ be irreducible factors in K of $s(x)$ such that $p_1(x+m)=p_2(x)$. Then the decomposition of $F(x)$ as a sum of partial fractions contains fractions with denominators $p_1(x)^i$ and $p_2(x)^j$, where i and j are natural numbers. At the same time, the decomposition of $F(x+n)$ contains a fraction with denominator $p_2(x+n)^j=p_1(x+m+n)^j$. Thus $F(x)$ gives a partial fraction with denominator $p_1(x)^i$ and $F(x+n)$ a fraction with denominator $p_1(x+m+n)^j$. The decompositions of $F(x+1),\dots,F(x+n-1)$ do not contain partial fractions with denominators $p_1(x)^k$ or $p_1(x+m+n)$, since $\operatorname{Dis}F(x)=m$: for example, if the decomposition of $F(x+1)$ contained a partial fraction with denominator $p_1(x)^k$, then together with a fraction with denominator $p_1(x+m)^j$ the function $F(x)$ would contain a fraction with denominator $p_1(x-1)^k$, and this is possible only if $\operatorname{Dis}F(x)\geqslant m+1$. Thus, for the sum on the left of (9) to be a polynomial, it is necessary that $a_0(x)$ contain a factor $p_1(x)$ (with exponent greater than or equal to i) and $a_n(x)$ a factor $p_1(x+n+m)$ (with exponent greater than or equal to j). Hence the value of $\operatorname{dis}(a_0(x), a_n(x))$ must be defined, and moreover $\operatorname{dis}(a_0(x), a_n(x))\geqslant n+m$, as claimed.

It follows that if $\operatorname{dis}(a_0(x), a_n(x))$ is undefined, then Eq.(9) cannot have rational solutions which are not polynomials. In that case the problem is completely solved by the algorithm of /2/. The situation is similar when $\operatorname{dis}(a_0(x), a_n(x))<n$. Throughout the sequel we shall assume that $\operatorname{dis}(a_0(x), a_n(x))\geqslant n$.

We claim that for any natural h we can construct an equations

$$\sum_{\nu=0}^{m} f_\nu(x)F(x+\nu h)=g(x) \tag{10}$$

(where $f_0(x),\dots,f_m(x)$, $g(x)\in K[x]$, $m\leqslant n$), which has exactly the same solutions as (9).

We first note that, using (9), we can express $F(x+k)$, where k is an arbitrary natural number, as

$$\sum_{\nu=0}^{n-1} v_{\nu k}(x)F(x+\nu)+w_k(x), \tag{11}$$

where $v_{0k}(x),\dots,v_{n-1,k}(x)$, $w_k(x)\in K(x)$. Indeed, if $k\leqslant n-1$, then $F(x+k)$ itself is such an expression, and then $v_{\nu k}(x)=\delta_{\nu k}$ (the Kronecker delta), $w_k(x)=0$. If $k=n$, then by (9)

$$F(x+n)=-\sum_{\nu=0}^{n-1}\frac{a_\nu(x)}{a_n(x)}F(x+\nu)+\frac{b(x)}{a_n(x)}. \tag{12}$$

We proceed by induction: suppose that $F(x+k-1)$, where $k>n$, can be expressed as

$$F(x+k-1)=\sum_{\nu=0}^{n-1} v_{\nu,k-1}(x)F(x+\nu)+w_{k-1}(x),$$

then we must find an expression

$$F(x+k)=\sum_{\nu=1}^{n} v_{\nu-1,k-1}(x)F(x+\nu)+w_{k-1}(x+1)$$

and this is done by substituting the right-hand side of (12) for $F(x+n)$ on the right of this inequality; this gives an expression for $F(x+k)$ in terms of $F(x+n-1), F(x+n-2),\dots, F(x)$. Throughout, the coefficients $v_{\nu k}(x)$ and the free term $w_k(x)$ will be represented by rational functions, each with denominator $a_n(x)^{k-n}$. It is not hard to write out recurrent relations expressing the numerators of $v_{\nu k}(x)$ and $w_k(x)$ in terms of the numerators of $v_{\nu,k-1}(x)$ and $w_{k-1}(x)$, $\nu=0, 1,\dots, n$.

Now, considering expressions of type (11) for $F(x+\nu h)$, $\nu=0, 1,\dots,$ and introducing the notation

$$F(x+\nu h)=u_{\nu 0}(x)F(x)+\dots+u_{\nu,n-1}(x)F(x+(n-1)h)+u_{\nu n}(x),$$
$$\nu=0, 1,\dots,$$

we readily see that of the rows $\mathrm{u}_\nu=(u_{\nu 0}(x),\dots,u_{\nu n}(x))$, $\nu=0, 1,\dots,$ at most n are linearly independent over $K(x)$. Therefore there exist a non-negative integer $m\leqslant n$ and $A_0(x),\dots, A_m(x)\in K(x)$, $A_m(x)\neq 0$, such that $A_0(x)\mathrm{u}_0+\dots+A_m(x)\mathrm{u}_m=0$. But this means that

$$A_m(x)F(x+mh)+\dots+A_0(x)F(x)=A_m(x)u_{mn}(x)+\dots$$
$$+A_0(x)u_{0n}(x).$$

Multiplying the last equality by a common denominator of the right-hand side and the coefficients, we obtain the required Eq.(10) with polynomial coefficients.

If we let h be a number which is certainly larger than $\operatorname{Dis}F(x)$, say $\operatorname{dis}(a_0(x), a_n(x))-n+1$, then for any non-negative integer m the rational functions $F(x), F(x+h),\dots,F(x+mh)$, in their irreducible forms, will have relatively prime denominators. Therefore, the sum on the left of Eq.(10) can be a polynomial provided only that each of the products $f_\nu(x)F(x+\nu h)$, $\nu=0, 1, \dots, m$, is a polynomial. Let the irreducible form of $F(x)$ be $t(x)/s(x)$. Then $s(x)$ divides $f_0(x)$, $s(x+h)$ divides $f_1(x),\dots, s(x+mh)$ divides $f_m(x)$, i.e., $s(x)$ divides any of the polynomials $f_\nu(x-\nu h)$, $\nu=0, 1,\dots, m$, and therefore it divides the LCM $(f_0(x), f_1(x-h),\dots,f(x-mh))$. Hence, it Eq.(9) has a rational solution other than a polynomial, this solution can be expressed as (an irreducible fraction)

$$\frac{y(x)}{\mathrm{LCM}\,(f_0(x),f_1(x-h),\dots,f_m(x-mh))},\quad y(x)\in K[x]. \tag{13}$$

Substituting (13) for $F(x)$ into (9) we obtain an equation with polynomial coefficients for the polynomial $y(x)$. This equation can be dealt with using the algorithm of (2).

Thus, the implementation of our algorithm may be described as follows. 1) Evaluate dis $(a_0(x), a_n(x))$. 2) if $\operatorname{dis}(a_0(x), a_n(x))$ is undefined or less than h, then (9) cannot have rational solutions that are not polynomials and therefore can be solved by the algorithm of /2/. 3) Evaluate $h=\operatorname{dis}(a_0(x), a_n(x))-n+1$, to obtain a strict upper bound for $\operatorname{Dis}F(x)$ (here and below we assume that $h>0$). 4) Construct Eq.(10), which is satisfied by any solution of Eq.(9). 5) Starting from the polynomials $f_0(x),\dots,f_m(x)$ - the coefficients of Eq.(10) - construct the polynomial $l(x)=\mathrm{LCM}\ f_0(x), f_1(x-h),\dots,f_m(x-mh))$, which is divisible by the denominator of any rational solution of Eq.(9). 6) Substitute $y(x)/l(x)$, where $y(x)$ is an unknown polynomial, for $F(x)$ into (9) (to obtain an equation with polynomial coefficients and polynomial right-hand side for $y(x)$)). 7) Apply the algorithm of /2/ to this equation.

Conclusion

The novelty of the first algorithm, compared with the usual technique for constructing solutions (including what are known as normal solutions /3/) of linear differential equations, is contained in steps 2-4 of our description, i.e., it consists in the rapid combination of several factors of the polynomial $a(x)$ into a product and examination of the latter together with a single defining equation. In principle, in the case of a homogeneous equation the algorithm may be made suitable for determining solutions of the form $q_1(x)^{r_1}\dots q_m^{r_m}(x)R(x)$, where $R(x)$ is a rational function, $q_1(x)\dots q_m(x)$ are certain factors of $a_n(x)$ and each $r_1,\dots,r_m$ is a non-integral root of some defining equation. The computations that this requires are much more complicated: if any irreducible factors $p_1(x)$ and $p_2(x)$ of $a_n(x)$ have the same multi-exponent, they can nevertheless be combined into a single factor only if the difference between any two roots of the defining equation is an integer. But if no appeal is made to such cmbinations, it becomes necessary to check a large number of different cases. In addition, $R(x)$ may be a rational function over some extension of the field K (e.g., over $K(r_1,\dots,r_m)$), and the construction of $R(x)$ will involve computations in that field. Therefore, although the construction of solutions of this type is possible, the main idea that makes our algorithm for determining rational solutions at all economical turns out to be almost useless.

If the initial equation is inhomogeneous, a knowledge of any particular solution makes it possible to deal instead with the corresponding homogeneous equation. In the case of a homogeneous equation, any known rational solution $R(x)$ can be used to lower the order of the equation, again obtaining a homogeneous linear equation with polynomial coefficients. Therefore, apart from the fact that every rational solution is interesting *per se* (simply as a known solution of the equation), it is particularly valuable in the case of linear differential and difference equations because it enables one to continue the search for other solutions with simpler equations of the same sort, i.e., again considering linear equations with polynomial coefficients.

We recall that the reduction in the order of the equation is achieved by the substitution $y(x)=R(x)\cdot z(x)$, where $R(x)$ is a known solution of the n-th order equation and $z(x)$ is a new unknown function. This substitution yields an equation not involving $z(x)$, and therefore the equation for $t(x)=z'(x)$ correspondingly for $t(x)=\Delta z(x))$ is of order $n-1$. In the case of a difference equation it is essential here that $\Delta^k(R(x)z(x))$ can be expressed as a linear combination of $\Delta^k z(x),\ldots,\Delta z(x)$, $z(x)$ with known functions as coefficients, such that the coefficient of $z(x)$ is $\Delta^k R(x)$, for example, $\Delta(R(x)z(x))=R(x+1)\Delta z(x)+(\Delta R(x))z(x)$.

Order-reduction may lead to an equation for which solutions can be identified from tables (for example, the handbook /3/ contains information about the solutions of a considerable number of differential equations of low orders with polynomial coefficients). On the other hand, if the operation results in a differential equation of second order, it may be possible to use Kovacic's algorithm /5/, for which ready-made programs are available /6/. This algorithm produces all Liouville solutions of the equation (though ina form rather inconvenient for practical use). We nevertheless reiterate that Kovacic's algorithm is applicable only to homogeneous linear differential equations of second order with polynomial coefficients.

Of course, if the function $z'(x)$ (correspondingly, $\Delta z(x)$) is rational, then $z(x)$ may not be rational, and therefore the order-reduction procedure may sometimes enable one to obtain, using the algorithms of this paper, non-rational solutions as well. A simple example: the equation

$$x^2y''-xy'+y=0 \tag{14}$$

has the polynomial solution $y=x$. Transforming to a new unknown function z such that $y=zx$, we obtain the equation

$$x^3z''+x^2z'=0;$$

after this we cancel out x^2 and introduce the function $t=z'$. This gives the equation

$$xt'+t=0.$$

Applying our algorithm for determining rational solutions, we obtain $t=1/x$, which gives $z'=1/x$, $y=zx$. Ordinary integration now gives the new solution $y=x\ln x$. Thus, using our algorithm for rational solutions and integrating, we have obtained two linearly independent solutions of Eq.(14): x and $x\ln x$.

Thus, the algorithms proposed above may become a useful component of any computer-algebra system for solving linear differential or difference equations with polynomial coefficients.

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