Nonlinear Optimization and Applications, 1996, pp. 101–123.
Edited by G. Di Pillo, University of Rome "La Sapienza", Rome, Italy
and
F. Giannessi, University of Pisa, Pisa, Italy
Plenum Press, New York, 1996

SPACE-TRANSFORMATION TECHNIQUE: THE STATE OF THE ART

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Revised version 12 February 2004

Abstract. In this paper we give an overview of some current approaches to LP and NLP based on space transformation technique. A surjective space transformation is used to reduce the original problem with equality and inequality constraints to a problem involving only equality constraints. Continuous and discrete versions of the stable gradient projection method and the Newton method are used for treating the reduced problem. Upon the inverse transformation is applied to the original space, a class of numerical methods for solving optimization problems with equality and inequality constraints is obtained. The following algorithms are presented: primal barrier-projection methods, dual barrier-projection methods, primal barrier-Newton methods, dual barrier-Newton methods and primal-dual barrier-Newton methods. Using special space transformation, we obtained asymptotically stable interior-infeasible point algorithms. The main results about convergence rate analysis are given.

Key words. Linear programming, nonlinear programming, space transformation, surjective mapping, stable gradient projection method, Newton's method, interior point technique.

1. INTRODUCTION

In the past twenty five years quite general and effective space transformation technique has been developed for solving linear programming problems (LP) and nonlinear programming problems (NLP). The idea of this approach commonly occurs in the optimization literature, it came from nonlinear programming and projective geometry. Using a space transformation, the original problem with equality and inequality constraints is reduced to a problem with equality constraints only. Continuous and discrete versions of the stable gradient projection method and the Newton method are applied to the reduced problem. After an inverse transformation to the original space, a class of numerical methods for solving optimization problems with equality and inequality constraints was obtained. The proposed algorithms are based on the numerical integration of systems of ordinary differential equations. Vector fields described by these equations define flows leading to the optimal solution. As a result of the space transformation, the vector fields are changed and additional terms are introduced which serve as a barrier preventing the trajectories from leaving the feasible set. Therefore, we call these methods "barrier-projection" and "barrier-Newton" methods. In our algorithms we use the multiplicative barrier functions which are continuous and equal to zero on a boundary. We do not introduce conventional singular barriers and this feature provides a high rate of convergence. In this paper we give a survey of principal results which were published in the last two decades [6]–[18].

In Sect. 2 we describe a unified methodology for finding necessary and sufficient optimality conditions in extremal problems with functional equality constraints and nonfunctional inequality constraints. We show how numerous families of algorithms can be developed using various space transformations.

In Sect. 3, choosing an exponential space transformation, we obtain the Dikin algorithm [5] from the family of primal barrier-projection methods. This algorithm, however, does not posses the local convergence properties and, as a result, it converges only if starting points belong to relative interior of the feasible set. Furthermore, the convergence rate of a discrete version of the algorithms proves to be weaker than a linear one.

In 1984 N. Karmarkar [25] proposed a special sophisticated step-length rule in the method similar to discrete version of the Dikin affine scaling algorithm. Based on this rule the polynomial complexity was theoretically attained. After this publication an impressive number of papers have been published devoted to further modifications and improvements of the Dikin and Karmarkar algorithms. Many authors were trying to modify and explain these algorithms as classical methods. Various methods were obtained along this direction and the first of our algorithms published in seventies [8, 6, 9] were reinvented. Later on in eighties-nineties, we developed more efficient versions of these methods which are discussed here. These asymptotically stable methods are such that a feasible set is an attractor of the vector fields. They preserve feasibility, but a starting point can be infeasible. They belong to a class of interior-infeasible point algorithms. In Sect. 3 we show that among the barrier-projection algorithms there is a method which converges locally and exponentially fast to the optimal solution (in discrete case it converges locally with a linear rate).

In subsequent sections we apply our approach to primal and dual linear programming problems. For the sake of simplicity, we assume that these problems have unique non-degenerate solutions. In Sect. 4 we use a nonconventional representation of the dual linear programming problem and we propose a set of algorithms. Upon simplifying the problem and choosing a particular exponential space transformation we arrive at the dual affine scaling method proposed by I. Adler, N. Karmarkar, M. Resende and G. Veiga [1].

In Sect. 5 we describe the primal the and dual barrier-Newton methods. The primal-dual interior-infeasible Newton method is set forth in the final Sect. 6. For the steepest descent approach an upper bound for the number of iterations is indicated.

2. BASIC APPROACH AND OUTLINE OF A SPACE TRANSFORMATION TECHNIQUE

Define the following NLP problem:

$$\text{minimize } f(x) \text{ subject to } x \in X = \{x \in \mathbb{R}^n : g(x) = 0_m,\ x \in P\}, \tag{2.1}$$

where the functions f and g are twice continuously differentiable, $f(x)$ maps $\mathbb{R}^n$ onto $\mathbb{R}$ and $g(x)$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, P is a convex set with nonempty interior, 0_s is the s-dimensional null vector, 0_{sk} is the $s \times k$ rectangular null matrix.

Important particular cases of (2.1) are a linear programming problem given in standard form

$$\text{minimize } c^\top x \text{ subject to } x \in X := \{x \in \mathbb{R}^n : b - Ax = 0_m,\ x \geq 0_n\} \tag{2.2}$$

and its dual problem

$$\text{maximize } b^\top u \text{ subject to } u \in U := \{u \in \mathbb{R}^m : v = c - A^\top u \geq 0_n\}, \tag{2.3}$$

where v is the n-vector of slack variables; $A \in \mathbb{R}^{m\times n}$ $(m < n)$; $c, x \in \mathbb{R}^n$; $b, u \in \mathbb{R}^m$ and $\mathrm{rank}(A) = m$.

We define the relative interior set of X and the interior set of U as:

$$X_0 := \{x \in \mathbb{R}^n : Ax = b,\ x > 0_n\}, \qquad U_0 := \{u \in \mathbb{R}^m : v = c - A^\top u > 0_n\},$$

and assume that these sets are nonempty. We also introduce the following sets:

$$\begin{array}{rcl} \mathbb{R}^n_+ & := & \{x \in \mathbb{R}^n : x \geq 0_n\}, \qquad \mathbb{R}^n_{++} := \mathrm{int}\, \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x > 0_n\}, \\ V & := & \{v \in \mathbb{R}^n : \text{ there exists } u \in \mathbb{R}^m \text{ such that } v = c - A^\top u\}, \\ V_U & := & \{v \in \mathbb{R}^n : \text{ there exists } u \in U \text{ such that } v = c - A^\top u\}, \end{array}$$

where int denotes the interior. The set V_U is the image of U under the mapping $v(u) = c - A^\top u$. Therefore, $V_U = V \cap \mathbb{R}^n_+$.

For convenience, assume that the primal feasible set is bounded and both primal and dual problems are non-degenerate, which together imply that optimal solutions x_*, u_* exist and are unique. We split the vectors x_* and $v_* = v(u_*)$ in basic and nonbasic components. Without any loss of generality we assume that

$$x_* = \begin{bmatrix} x_*^B \\ x_*^N \end{bmatrix}, \quad v_* = \begin{bmatrix} v_*^B \\ v_*^N \end{bmatrix}, \qquad x_*^B > 0_m, \quad x_*^N = 0_d, \quad v_*^B = 0_m, \quad v_*^N > 0_d,$$

where $d = n - m$.

We denote the components of a vector by using superscripts and the iterate numbers by using subscripts; I_n denotes the identity matrix of the order n; $D(z)$ denotes the diagonal matrix whose entries are the components of the vector z. The dimensionality of $D(z)$ is determined by the dimensionality of z.

In order to construct a family of computational methods for solving the Problems (2.1) – (2.3) we use an approach based on space transformation. We introduce a new n-dimensional space with the coordinates $[y^1, \ldots, y^n]$ and make a differentiable transformation from this space to the original one: $x = \xi(y)$. This surjective transformation maps $\mathbb{R}^n$ onto P or $\mathrm{int}\, P$, i.e. $P = \overline{\xi(\mathbb{R}^n)}$, where $\overline{B}$ is the topological closure of B. Let $\tilde{J}(y) = dx/dy$ be the Jacobian matrix of the transformation $x = \xi(y)$ with respect to y.

Consider the transformed minimization problem

$$\text{minimize } \tilde{f}(y) \text{ subject to } y \in Y, \tag{2.4}$$

where $\tilde{f}(y) = f(\xi(y))$, $Y = \{y \in \mathbb{R}^n : \tilde{g}(y) = g(\xi(y)) = 0_m\}$.

Define the Lagrangian functions $L(x, u)$, $\tilde{L}(y, u)$ associated with the Problems (2.1) and (2.4), respectively:

$$L(x, u) = f(x) + u^\top g(x), \qquad \tilde{L}(y, u) = \tilde{f}(y) + u^\top \tilde{g}(y).$$

Then the first-order necessary conditions for a local minimum for the Problem (2.4) in the transformed space are

$$\tilde{L}_y(y, u) = \tilde{f}_y(y) + \tilde{g}_y^\top(y) u = 0_n, \qquad \tilde{g}(y) = 0_m, \tag{2.5}$$

where $\tilde{f}_y = \tilde{J}^\top f_x$, $\tilde{g}_y = g_x \tilde{J}$.

If $\tilde{J}$ is a nonsingular, then there exists an inverse transformation $y = \delta(x)$, so it is possible to return from the y-space to the x-space and we obtain in this way a matrix $J(x) = \tilde{J}(\delta(x))$

which is now a function of x. Using this substitution, we rewrite expressions (2.5) in terms of the variable x. They take the form

$$J^\top(x)L_x(x,u) = 0_n, \qquad g(x) = 0_m, \qquad x \in P. \tag{2.6}$$

Some properties of the nonlinear systems, which are obtained after space transformations, were investigated in [15].

Let $K(x \mid P)$ and $K^*(x \mid P)$, respectively, denote the cone of feasible directions at the point x relative to the set P and its dual:

$$K(x \mid P) = \{z \in \mathbb{R}^n : \; \exists \lambda(z) > 0 \text{ such that } x + \lambda z \in P, \; 0 < \lambda \leq \lambda(z)\},$$

$$K^*(x \mid P) = \{z \in \mathbb{R}^n : z^\top y \geq 0 \; \forall y \in K(x \mid P)\}.$$

Let $S(x \mid P)$ be a linear hull of the cone $K^*(x \mid P)$. The set of all vectors orthogonal to $S(x \mid P)$ is called orthogonal complement of $S(x \mid P)$ and is denoted by $S^\perp(x \mid P)$.

We will impose the following condition on the space transformation $\xi(y)$.

Condition 2.1. *At each point $x \in P$ the matrix $J(x)$ is defined and the null-space of $J^\top(x)$ coincides with the set $S(x \mid P)$.*

In particular, it follows from this condition that at all interior points $x \in \operatorname{int} P$ the matrix $J(x)$ is non-degenerate, becoming singular only on the boundary of the set P.

Definition 2.1. *Any pair $[x, u]$ is a weak KKT pair for the Problem (2.1) if it satisfies the conditions (2.6).*

It follows from this definition and Condition 2.1 that $L_x(x_*, u_*) \in S(x_* \mid P)$ at any weak KKT pair. Define the Gram matrix $G(x) = J(x)J^\top(x)$. Since the null-spaces of $G(x)$ and $J^\top(x)$ coincide, conditions (2.6) for the pair $[x_*, u_*]$ can be rewritten in the form

$$G(x)L_x(x,u) = 0_n, \qquad g(x) = 0_m, \qquad x \in P. \tag{2.7}$$

Definition 2.2. *A weak KKT pair $[x, u]$ is a KKT pair for the Problem (2.1) if $L_x(x,u) \in$ $\in K^*(x \mid P)$.*

Let $\operatorname{ri} B$ denote a relative interior of the set B.

Definition 2.3. *A KKT pair $[x, u]$, is a strong KKT pair if $L_x(x_*, u_*) \in \operatorname{ri} K^*(x_* \mid P)$.*

Definition 2.4. *The constraint qualification (CQ) for the Problem (2.1) holds at a point $x \in P$ if all vectors $g^i_x(x)$, $1 \leq i \leq m$, and any nonzero vector $p \in S(x \mid P)$ are linearly independent. We say that x is a regular point for the Problem (2.1) if the CQ holds at x.*

Theorem 2.1. *Let a regular point x_* be a solution of the Problem (2.1). Then there exists a vector $u_* \in \mathbb{R}^m$ such that $[x_*, u_*]$ forms a weak KKT pair for the Problem (2.1).*

The space transformation described above can be used to derive the second-order sufficient conditions for a point x_* to be an isolated minimum in the Problem (2.1). Introduce a null-space $N(x) = \{z \in \mathbb{R}^n : g_x(x)J(x)z = 0_m\}$.

Theorem 2.2. *Assume that f and g are twice-differentiable functions and the space transformation $\xi(y)$ satisfies Condition 2.1. Sufficient conditions for a point $x_* \in P$ to be an isolated local minimum of the Problem (2.1) are that there exists a strong KKT pair $[x_*, u_*]$ such that $z^\top J^\top(x_*)L_{xx}(x_*, u_*)J(x_*)z > 0$ for every $z \in N(x_*)$ such that $\|J(x_*)z\| \neq 0$.*

If $P = \mathbb{R}^n$ (in other words, the condition $x \in P$ is missing), we can take the trivial space transformation $x = y$. In this case we have

$$J(x) = I_n, \qquad N(x) = \{z \in \mathbb{R}^n : g_x(x)z = 0_m\},$$

$$K(x \mid \mathbb{R}^n) = \mathbb{R}^n, \qquad S(x \mid \mathbb{R}^n) = K^*(x \mid \mathbb{R}^n) = \operatorname{ri} K^*(x \mid \mathbb{R}^n) = 0_n.$$

The Theorem 2.2 reduces to the well-known second-order sufficient conditions for an isolated local minimum (see, for example, [20]).

Suppose that the function $\xi(y)$ is such that the matrix $G(x)$ is continuously differentiable. Let $p \in \mathbb{R}^n$ and $G_x(x;p)$ denote a square matrix of order n whose (i,j)-element equals to

$$G_x^{ij}(x;p) = \sum_{k=1}^{n} \frac{\partial G^{ik}(x)}{\partial x^j}\, p^k.$$

We impose two additional conditions on the space transformation $\xi(y)$:

Condition 2.2. *At each point $x \in P$ for any vector $p \in \operatorname{ri} K^*(x \mid P)$ the matrix $G_x(x;p)$ is symmetric and its null-space coincides with $S^{\perp}(x \mid P)$.*

Condition 2.3. *If $x \in P$, then $z^\top G_x(x;p)z > 0$ for any non-zero vector $z \in S(x \mid P)$ and any vector $p \in \operatorname{ri} K^*(x \mid P)$.*

Let us consider an important particular case of the Problem (2.1), where $P = \mathbb{R}^n_+$:

$$\text{minimize } f(x) \text{ subject to } x \in X = \{x \in \mathbb{R}^n : g(x) = 0_m,\ x \geq 0_n\}. \tag{2.8}$$

It is convenient for this set to use a component-wise space transformation:

$$x^i = \xi^i(y^i), \qquad 1 \leq i \leq n. \tag{2.9}$$

For such a transformation the inverse transformation $y = \delta(x)$ is also component-wise type $y^i = \delta^i(x^i)$, $1 \leq i \leq n$, and the corresponding matrices $J(x)$ and $G(x)$ are diagonal:

$$J(x) = D(\gamma(x)), \qquad \gamma^\top(x) = [\gamma^1(x^1), \dots, \gamma^n(x^n)], \qquad \gamma^i(x^i) = \dot{\xi}(\delta^i(x^i)),$$

$$G(x) = D(\theta(x)), \qquad \theta^\top(x) = [\theta^1(x^1), \dots, \theta^n(x^n)], \qquad , \theta^i(x^i) = [\gamma^i(x^i)]^2, \qquad 1 \leq i \leq n.$$

Let $\sigma(x) = \{i : x^i = 0\}$ be a set of active indices at the point $x \in \mathbb{R}^n_+$. In this case

$$\begin{aligned} K^*(x \mid \mathbb{R}^n_+) &= \{z \in \mathbb{R}^n_+ :\ \text{if } i \notin \sigma(x),\ \text{then } z^i = 0,\ 1 \leq i \leq n\}, \\ S(x \mid \mathbb{R}^n_+) &= \{z \in \mathbb{R}^n :\ \text{if } i \notin \sigma(x),\ \text{then } z^i = 0,\ 1 \leq i \leq n\}, \end{aligned}$$

condition 2.1 reduces to the following

Condition 2.4. *The vector function $\gamma(x)$ is defined at each point $x \in \mathbb{R}^n_+$ and $\gamma^i(x^i) = 0$ if and only if $i \in \sigma(x)$.*

In order to insure Conditions 2.2 and 2.3 we impose the

Condition 2.5. *The vector function $\theta(x)$ is differentiable in some neighborhood of $\mathbb{R}^n_+$ and $\dot{\theta}^i(0) > 0$, $1 \leq i \leq n$.*

As a rule, we perform the following quadratic and exponential transformations:

$$x^i = \xi^i(y^i) = \frac{1}{4}(y^i)^2, \qquad J(x) = D^{1/2}(x), \qquad G(x) = D(x), \tag{2.10}$$

$$x^i = \xi^i(y^i) = e^{y^i}, \qquad J(x) = D(x), \qquad G(x) = D^2(x). \tag{2.11}$$

In these two cases the Jacobian matrix is singular on the boundary of the set $P = \mathbb{R}^n_+$. These transformations satisfy Condition 2.4. The Condition 2.5 holds only for the quadratic transformation (2.10).

Let e^i denote the n-th order unit vector whose i-th component is equal to one. The CQ for the Problem (2.8) holds at a point x, if all the vectors $g^i(x)$, $1 \le i \le m$, and all e^j, such that $j \in \sigma(x)$, are linearly independent. The cone $N(x)$ takes the form $N(x) = \{z \in \mathbb{R}^n : : g_x(x)D(\gamma(x))z = 0_m\}$.

The strict complementary condition (SCC) holds at a pair $[x, u]$, if $L_{x^i}(x, u) > 0$ for all $i \in \sigma(x)$. From the Theorem 2.2 the following second-order sufficient optimality conditions is obtained.

Theorem 2.3. *Sufficient conditions for a point $x_* \in X$ to be an isolated local minimum of the Problem (2.8) are that there exists a Lagrange multiplier vector u_* such that*

$$D(\gamma(x_*))L_x(x_*, u_*) = 0_n,$$

that the SCC holds at $[x_*, u_*]$ and that

$$z^\top D(\gamma(x_*))L_{xx}(x_*, u_*)D(\gamma(x_*))z > 0$$

for all $z \in N(x_*)$, satisfying $\|D(\gamma(x_*))z\| \neq 0$.

Now we will construct numerical methods for solving the Problem (2.1). We use the stable version of the gradient projection method for solving the Problem (2.4). The numerical method is stated as an initial-value problem involving the following system of ordinary differential equations:

$$\frac{dy}{dt} = -\tilde{L}_y(y, \tilde{u}(y)), \quad \tilde{L}_y(y, \tilde{u}) = \tilde{f}_y(y) + \tilde{g}_y^\top(y)\tilde{u}, \quad y(0, y_0) = y_0 \in \mathbb{R}^n. \tag{2.12}$$

The function $\tilde{u}(y)$ is chosen to satisfy the following condition:

$$\frac{d\tilde{g}}{dt} = \tilde{g}_y \frac{dy}{dt} = -\tau \tilde{g}(y), \qquad \tau > 0. \tag{2.13}$$

From this condition we obtain the system of linear algebraic equations

$$\tilde{g}_y(y)\tilde{g}_y^\top(y)\tilde{u}(y) + \tilde{g}_y(y)\tilde{f}_y(y) = \tau \tilde{g}(y),$$

where $\tilde{f}_y = \tilde{J}^\top f_x$, $\tilde{g}_y = g_x \tilde{J}$. By differentiating $\xi(y)$ with respect to y and taking into account (2.12) and (2.13), we have

$$\frac{dx}{dt} = \frac{d\xi}{dy}\frac{dy}{dt} = J(x)\frac{dy}{dt} = -G(x)L_x(x, u(x)), \qquad x(0, x_0) = x_0 \in P, \tag{2.14}$$

$$\Gamma(x)u(x) + g_x(x)G(x)f_x(x) = \tau g(x), \tag{2.15}$$

where $\Gamma(x) = g_x(x)G(x)g_x^\top(x)$.

Lemma 2.1. *Let the space transformation $\xi(y)$ satisfy Condition 2.1, and let the CQ for the Problem (2.1) hold at a point $x \in P$. Then $\Gamma(x)$ is invertible and positive definite matrix.*

Lemma 2.2. *Let the space transformation $\xi(y)$ satisfy Condition 2.1. Then the regular point x_* is an equilibrium state of system (2.14) if and only if the pair $[x_*, u_*]$, where $u_* = u(x_*)$, is a weak KKT pair for the Problem (2.1).*

Hence, corresponding to any regular point x_* we can define a corresponding Lagrange multiplier $u(x_*)$ by solving linear algebraic equations (2.15). If a local solution of the original problem (2.1) occurs at a regular point $x_* \in P$, then $[x_*, u(x_*)]$ forms a weak KKT pair for the Problem (2.1) and x_* is an equilibrium state of (2.14).

Let W be a $m \times n$ rectangular matrix whose rank is m. We introduce the pseudo-inverse matrix $W^+ = W^\top (WW^\top)^{-1}$ and the orthogonal projector $\pi(W) = I_n - W^+ W$. If at a regular point x we define $u(x)$ and substitute it into the right-hand side of (2.14), then (2.14) can be rewritten in the following projective form:

$$\frac{dx}{dt} = -J(x)\{\pi[g_x(x)J(x)]J^\top(x)f_x(x) + \tau[g_x(x)J(x)]^+ g(x)\}. \tag{2.16}$$

Let $x(t, x_0)$ denote the solution of the Cauchy Problem (2.14) with an initial condition $x_0 = x(0, x_0)$. In what follows, we assume that the initial-value problem under consideration is always uniquely solvable. A trajectory $x(t, x_0)$ can be continued as long as points on it are regular. By differentiating $f(x(t, x_0))$ with respect to t we arrive at

$$\frac{df}{dt} = -\|J^\top(x)L_x(x, u(x))\|^2 + \tau u^\top(x)g(x). \tag{2.17}$$

Hence the objective function $f(x(t, x_0))$ monotonically decreases either on the feasible set X or when the trajectory is close to X, i.e. $\|g(x(t, x_0))\|$ is sufficiently small.

The system of ordinary differential equations (2.14), where $u(x)$ is given by (2.15), has the first integral

$$g(x(t, x_0)) = g(x_0)e^{-\tau t}.$$

This means that if $\tau > 0$, (2.14) has a remarkable property: all its trajectories approach the manifold $g(x) = 0_m$, as t tends to infinity, and this manifold is an asymptotically stable attractor (see [7, 10, 29]). Therefore, we can call the method (2.16) "the stable version of the barrier-projection method". If $x_0 \in X$, then the trajectory $x(t, x_0)$ of (2.14) remains on this manifold because $g(x(t, x_0)) \equiv 0_m$ for all $t \geq 0$ and the trajectories of (2.14) coincide with the trajectories of the following system:

$$\frac{dx}{dt} = -J(x)\pi[g_x(x)J(x)]J^\top(x)f_x(x), \tag{2.18}$$

which can be obtained from (2.16) if we put $\tau = 0$. But in contrast to (2.16) this system is neutrally stable with respect to equality constraints. It means that if $g(x_0) = c$, then $g(x(t, x_0)) \equiv c$ for all $t \geq 0$ and we have to introduce a correction procedure to remove the violation of constraints. If the condition $x \in P$ is missing, then we can put $\xi(y) = y$, hence $J(x) = I_n$ and (2.18) coincides with well-known gradient projection method [26], which is also neutrally stable.

Note that, according to the Condition 2.1, the subspace $S^\perp(x \mid P)$ coincides with the space of the columns of the matrix $J(x)$, and, since the vector $G(x)L_x(x, u(x))$ belongs to this space, the velocity vector $\dot{x}$ always lies in $S^\perp(x \mid P)$. Thus, if x is a boundary point of P, then the vector $\dot{x}$ belongs to the eigen subspace of the space $\mathbb{R}^n$, which coincides with the space $M(x) - x$, where $M(x)$ is the intersection of the support planes of the set P at the point x. If the cone $K^*(x \mid P)$ has a non-empty interior, then this subspace degenerates to a single point (the origin of coordinates).

Let us show that trajectories $x(t, x_0)$ of (2.14) cannot cross the boundary of P. To the contrary, suppose this is not true and a trajectory $x(t, x_0)$ starting inside P leaves P at $t_1 > 0$. Then there exists a vector $p \in K^*(x(t_1, x_0) \mid P)$ such that $p^\top \dot{x}(t_1, x_0) < 0$. But the vector

$\dot{x}(t, x_0)$ belongs to the orthogonal complement of the subspace $S(x \mid P)$ and the vector p belongs to $S(x \mid P)$. Hence $p^\top \dot{x}(t_1, x_0) = 0$ and, consequently, $x(t, x_0) \in P$ for all $t \geq 0$. Thus the matrix $G(x)$ plays the role of a "barrier" preventing $x(t, x_0)$ from intersecting the boundary of P. The trajectory $x(t, x_0)$ can approach the boundary points only in the limit as $t \to +\infty$. If the initial point x_0 lies on the boundary, then the entire trajectory of system (2.14) belongs to the boundary of P.

By applying the Euler method for solving system (2.16), we obtain

$$x_{k+1} = x_k - \alpha_k J(x_k)\{\pi[g_x(x_k)J(x_k)]J^\top(x_k)f_x(x_k) + \tau[g_x(x_k)J(x_k)]^+ g(x_k)\}, \tag{2.19}$$

where a step-size $\alpha_k > 0$.

Each equilibrium point x_* of the system (2.16) is a fixed point of iterations (2.19), i.e. $x_k = x_*$ implies $x_{k+1} = x_*$, and if iterates (2.19) converge to a regular point x_*, then the pair $[x_*, u(x_*)]$ satisfies conditions (2.7).

Theorem 2.4. *Let $[x_*, u_*]$ be a weak KKT pair of the Problem (2.1), where the CQ and the second-order sufficiency conditions of the Theorem 2.2 hold. Let the space transformation $\xi(y)$ satisfy the Conditions 2.1 – 2.3 and $\tau > 0$. Then x_* is an asymptotically stable equilibrium state of the system (2.16); there exists a positive number α_* such that for any fixed $0 < \alpha_k < \alpha_*$ the sequence $\{x_k\}$, generated by (2.19), converges locally with a linear rate to x_* while the corresponding sequence $\{u_k\}$, where $u_k = u(x_k)$, converges to u_*.*

The proof of this theorem is given in [15]. It is based on the Lyapunov linearization principle. The conditions for asymptotic stability to be valid are expressed in terms of eigenvalues determined by a matrix arisen from the right-hand side of (2.16) linearized about the equilibrium point x_*. The asymptotic stability of the point x_* implies the local, exponentially fast convergence of a trajectory $x(t, x_0)$ to the optimal solution x_*. The corresponding statement about the linear convergence of discrete versions follows from [7, Theorem 2.3.7].

The Theorem 2.4, being applied to the Problem (2.8), gives the the following statement:

Theorem 2.5. *Let $[x_*, u_*]$ be a weak KKT pair of the Problem (2.8), when the CQ and the second-order sufficiency conditions of the Theorem 2.3 hold. Let the component-wise space transformation (2.9) satisfy the Conditions 2.4, 2.5 and $\tau > 0$. Then x_* is an asymptotically stable equilibrium state of system (2.16); there exists a positive number α_* such that for any fixed $0 < \alpha_k < \alpha_*$ the sequence $\{x_k\}$, generated, by (2.19), converges locally with a linear rate to x_* while the corresponding sequence $\{u_k\}$ converges to u_*.*

This theorem was proved in [10, 12].

The preceding results and algorithms admit straightforward extensions for problems involving general functional inequality constraints by using space dilation. Consider a problem

$$\text{minimize } f(x) \text{ subject to } x \in X = \{x \in \mathbb{R}^n : g(x) = 0_m, \;\; h(x) \leq 0_c\}, \tag{2.20}$$

where $h(x)$ maps $\mathbb{R}^n$ into $\mathbb{R}^c$.

Space transformation approach can be used in this case by extension of the space and by converting the inequality constraints to equalities. We introduce an additional variable $p \in \mathbb{R}^c$, define $q = m + c$, combine primal, dual variables and all constraints:

$$z = \begin{bmatrix} x \\ p \end{bmatrix} \in \mathbb{R}^{n+c}, \qquad w = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{R}^q, \qquad \Phi(z) = \begin{bmatrix} g(x) \\ h(x) + p \end{bmatrix}.$$

Then the original Problem (2.20) is transformed into the equivalent problem

$$\text{minimize } f(x) \text{ subject to } z \in Z = \{z \in \mathbb{R}^{n+c} : \Phi(z) = 0_q, \;\; p \in \mathbb{R}^c_+\}. \tag{2.21}$$

Here P is a positive orthant in $\mathbb{R}^c$. This problem is similar to (2.8). In order to take into account the constraint $p \geq 0_c$ we introduce a surjective component-wise differentiable mapping $\xi : \mathbb{R}^c \to \mathbb{R}^c_+$ and make the space transformation $p = \xi(y)$, where $y \in \mathbb{R}^c$, $\overline{\xi(\mathbb{R}^c)} = \mathbb{R}^c_+$. Let $\xi_y(y)$ be the square $c \times c$ Jacobian matrix of the mapping $\xi(y)$ with respect to y. We assume that it is possible to define the inverse transformation $y = \psi(p)$ and hence we obtain the $c \times c$ Jacobian and Gram diagonal matrices:

$$J(p) = \xi_y(y)|_{y=\psi(p)} = D(\gamma(p)), \qquad G(p) = J(p)J^\top(p) = D(\theta(p)).$$

Combining variables and constraints for the reduced problem, let us define

$$\hat{z} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{n+c}, \qquad \hat{\Phi}(\hat{z}) = \begin{bmatrix} g(x) \\ h(x) + \xi(y) \end{bmatrix}, \qquad \hat{\Phi}_{\hat{z}} = \begin{bmatrix} g_x & 0_{mc} \\ h_x & \xi_y \end{bmatrix}.$$

The Problem (2.21) can be formulated as follows:

$$\text{minimize } f(x) \text{ subject to } \hat{z} \in \hat{Z} = \{\hat{z} \in \mathbb{R}^{n+c} : \hat{\Phi}(\hat{z}) = 0_q\}. \tag{2.22}$$

In the last Problem we have only equality constraints, therefore we can use the numerical method described above. After an inverse transformation to the space of x and p we obtain from (2.14), (2.15)

$$\frac{dz}{dt} = -\tilde{G}(p)L_z(z, w(z)). \tag{2.23}$$

Here

$$\begin{gathered} L(z,w) = f(x) + w^\top \Phi(z), \qquad L_z(z,w) = f_z(z) + \Phi_z^\top(z)w, \\ \Phi_z(z)\tilde{G}(p)L_z(z,w(z)) = \tau\Phi(z), \\ \tilde{G}(p) = \begin{bmatrix} I_n & 0_{nc} \\ 0_{cn} & G(p) \end{bmatrix}, \quad \Phi_z = \left[\begin{array}{c|c} \Phi_x & \begin{array}{c} 0_{mc} \\ I_c \end{array} \end{array}\right], \quad \Phi_x = \begin{bmatrix} g_x \\ h_x \end{bmatrix}, \quad f_x = \begin{bmatrix} f_x \\ 0_c \end{bmatrix}. \end{gathered} \tag{2.24}$$

System (2.23) can be rewritten in the more detailed form

$$\frac{dx}{dt} = -L_x(x, w(z)), \qquad \frac{dp}{dt} = -G(p)v(z), \tag{2.25}$$

where the function $w^\top(z) = [u^\top(z), v^\top(z)]$ is found from the following linear system of q equations:

$$\Gamma(z)w(z) + \Phi_x(x)f_x(x) = \tau\Phi(z), \qquad \Gamma(z) = \Phi_x(x)\Phi_x^\top(x) + \begin{bmatrix} 0_{mm} & 0_{mc} \\ 0_{cm} & G(p) \end{bmatrix}.$$

Condition (2.24) can be written as

$$\frac{dg(x)}{dt} = -\tau g(x), \qquad \frac{d(h(x)+p)}{dt} = -\tau(h(x)+p).$$

Therefore, system (2.25) has two first integrals:

$$g(x(t,z_0)) = g(x_0)e^{-\tau t}, \quad h(x(t,z_0)) + p(t,z_0) = (h(x_0)+p_0)e^{\tau t}, \quad z_0^\top = [x_0^\top, p_0^\top]. \tag{2.26}$$

Similarly to (2.17) we obtain

$$\frac{df}{dt} = -\|L_x\|^2 - \|J^\top(p)v\|^2 + \tau[u^\top g + v^\top(h+p)].$$

Consider the simplified version of method (2.25). Suppose that along the trajectories of system (2.25) the following condition holds:

$$h(x(t,z_0)) + p(t,z_0) \equiv 0_c.$$

From this equality we can define p as a function of h. We exclude from system (2.25) the additional vector p and integrate the system which does not employ this vector:

$$\frac{dx}{dt} = -L_x(x, w(x)), \tag{2.27}$$

where

$$\Gamma(x)w(x) + \Phi_x(x)f_x(x) = \tau \begin{bmatrix} g(x) \\ 0_c \end{bmatrix}, \quad \Gamma(x) = \Phi_x(x)\Phi_x^\top(x) + \begin{bmatrix} 0_{mm} & 0_{mc} \\ 0_{cm} & G(-h(x)) \end{bmatrix}. \tag{2.28}$$

Along the trajectories of (2.27) we have

$$\begin{gathered} \frac{dg}{dt} = -\tau g(x), \qquad \frac{dh}{dt} = -G(-h(x))v(x), \\ \frac{df}{dt} = -\|L_x(x,w(x))\|^2 - \|J^\top(-h(x))v(x)\|^2 + \tau u^\top(x)g(x). \end{gathered} \tag{2.29}$$

Let us show that the solution $x(t,x_0)$ does not leave the set X for any $t>0$, if $x_0 \in X$. Suppose to the contrary that this is not true and $h^i(x(t,x_0)) > 0$ for some $t>0$. Then there is an earlier instant $0 < t_1 < t$ such that $h^i(x(t_1,x_0)) = 0$ and $\dot{h}^i(x(t_1,x_0)) > 0$. This contradicts (2.29) since $\theta^i(0) = 0$. Hence $x(t,x_0) \in X$ for all $t \geq 0$. Thus the matrix $G(-h(x))$ plays the role of a "barrier" preventing $x(t,x_0)$ from intersecting the hypersurface $h^i(x) = 0$.

The method (2.27) is closely related to (2.14). Let us consider the Problem (2.8). We have the two alternatives ensuing from (2.14) or (2.27). The main body of computational work required when using any numerical integration method is to evaluate the right-hand sides of equations for various values of x. This could be done by solving the linear system (2.15) of m equations or the system (2.28) of $m+n$ equations, respectively. One might expect that introducing a slack variable p increases the computational work considerably. However, with allowance made for the simple structure of (2.28), we can reduce the computational time by using the Frobenius formula for an inverse matrix. Upon some transformations we find that (2.27), (2.28) can be written in the form of (2.14), (2.15), respectively, if we put

$$G(x) = D(\mu(x)), \quad \mu^i(x^i) = \frac{\theta^i(x^i)}{1+\theta^i(x^i)}, \qquad 1 \leq i \leq n.$$

Therefore, the performances of both methods which seem at a glance to be unrelated are in fact very similar.

3. PRIMAL BARRIER-PROJECTION METHODS FOR SOLVING LP PROBLEMS

Applying the stable barrier-projection method (2.14) for solving the Problem (2.2), we obtain the following continuous and discrete versions:

$$\frac{dx}{dt} = -G(x)[c - A^\top u(x)], \qquad x(0,x_0) = x_0, \tag{3.1}$$

$$x_{k+1} = x_k - \alpha_k G(x_k)[c - A^\top u_k], \qquad u_k = u(x_k), \tag{3.2}$$

where the function $u(x)$ is found from the linear equation (2.15) which can be rewritten as

$$AG(x)A^\top u(x) - AG(x)c = \tau(b - Ax). \tag{3.3}$$

By differentiating the objective function with respect to t we obtain

$$c^\top \frac{dx}{dt} = -\|J(x)(c - A^\top u(x))\|^2 + \tau u^\top(x)(b - Ax).$$

The system of ordinary differential equations (3.1) has the first integral

$$Ax(t, x_0) = b + (Ax_0 - b)e^{-\tau t}. \tag{3.4}$$

Under the non-degeneracy assumption all feasible points are regular, and each weak KKT pair $[x, u(x)]$ is such that x is an equilibrium state of system (3.1). The pair $[x, u(x)]$ is a strong KKT pair if and only if $x = x_*$.

Denote

$$\alpha_* = \frac{2}{\mu^*}, \quad \mu^* = \max\left[\tau, \max_{m+1\le i\le n} \dot\theta^i(x_*^i)v_*^i\right], \quad \mu_* = \min\left[\tau, \min_{m+1\le i\le n} \dot\theta^i(x_*^i)v_*^i\right].$$

Here μ^* and μ_* are, respectively, the largest and smallest eigenvalues of the first approximation matrix of the right-hand side of (3.1) at the optimal solution x_*.

Theorem 3.1. *Let x_*, u_* be unique non-degenerate solutions of the Problems (2.2) and (2.3), respectively. Assume that the component-wise space transformation $\xi(y)$ satisfy the Conditions 2.4, 2.5 and $\tau > 0$. Then the following statements are true:*

1. *The point x_* is an asymptotically stable equilibrium state of system (3.1).*
2. *The solutions $x(t, x_0)$ of system (3.1) converge locally exponentially fast to the optimal point x_*.*
3. *For any fixed $0 < \alpha_k < \alpha_*$ the sequence $\{x_k\}$, generated by (3.2), converges locally with a linear rate to x_* while the corresponding sequence $\{u_k\}$ converges to u_*.*
4. *All extreme feasible points of X are unstable equilibrium points of (3.1) and (3.2), except the optimal solution x_*.*

The exponential rate of convergence of the solution $x(t, x_0)$ of (3.1) to the equilibrium point x_* implies that there exist a neighborhood $\Delta(x_*)$ of x_* and a constant $C > 0$ such that

$$\|x(t, x_0) - x_*\| \le C\|x_0 - x_*\|e^{-\mu_* t}$$

for all $x_0 \in \Delta(x_*)$, $t > 0$.

In (3.1) we can take a starting vector x_0 such that $\|Ax_0 - b\| \neq 0$. Moreover, if the Condition 2.5 holds, then the components of x_0 corresponding to nonbasic components of the vector x_* may be negative. In this case the entire trajectory $x(t, x_0)$ is infeasible. Nevertheless, owing to the local convergence property, the trajectory $x(t, x_0)$ converges to x_*, if $\|x_0 - x_*\|$ is sufficiently small. It follows from (3.4) that, if initially $Ax_0 = b$, then $Ax(t, x_0) \equiv b$ for all $t \ge 0$. Still further if $x_0 \ge 0_n$, then the trajectory $x(t, x_0)$ of (3.1) remains in the feasible set X and the objective function monotonically decreases along $x(t, x_0)$. (3.1) preserves feasibility, hence it is an interior point method. If Condition 2.5 holds, then this method allows to start the computation from an infeasible point. Therefore, we can call it an interior-infeasible method.

If we use the quadratic and exponential space transformations, then (3.1) and (3.2) can be cast in the form

$$\frac{dx}{dt} = D^{\beta}(x)(A^{\top}u(x) - c), \qquad AD^{\beta}(x)A^{\top}u(x) = AD^{\beta}(x)c + \tau(b - Ax), \tag{3.5}$$

$$x_{k+1} = x_k + \alpha_k D^{\beta}(x_k)[A^{\top}u_k - c], \tag{3.6}$$

where $\beta = 1$ for (2.10), and $\beta = 2$ for (2.11).

The continuous version of (3.5) had been studied by Smirnov [27]. He proved that the estimate

$$\|x^N(t, x_0)\| = \begin{cases} O\left(e^{-\mu_* t}\right), & \text{if } \beta = 1, \\ O\left(t^{1/(1-\beta)}\right), & \text{if } \beta > 1 \end{cases}$$

holds for the vector of nonbasic components. Hence the trajectories of (3.5) derived with the help of the quadratic space transformation locally converge faster than the trajectories of (3.5), derived with the help of the exponential transformation. Therefore, in our codes we have mainly used the quadratic space transformation.

It was shown that if β is odd, then x_* is an asymptotically stable point of (3.5) and the local convergence takes place. If β is even, then the optimal point x_* is unstable and trajectories converge to x_* provided that $x_0 \in \Delta(x_*) \cap X_0$.

If we set $\beta = 2$, $\tau = 0$, then (3.5) leads to the Dikin affine scaling method

$$\frac{dx}{dt} = D^2(x)(A^{\top}u(x) - c), \qquad AD^2(x)A^{\top}u(x) = AD^2(x)c. \tag{3.7}$$

Discrete and continuous versions of (3.7) were investigated in numerous papers (see, for example, [3, 4, 5, 23, 24, 30, 31]). It is worthy of note that the optimal solution x_* proves to be unstable equilibrium point and, therefore, (3.7) does not possess the local convergence property. In the latter case a starting point x_0 should meet a constraint $x_0 \in \Delta(x_*) \cap X_0$ for the convergence to take place.

If we set $\beta = 1$ $\tau = 0$, then (3.5) yields

$$\frac{dx}{dt} = D(x)(A^{\top}u(x) - c), \qquad AD(x)A^{\top}u(x) = AD(x)c. \tag{3.8}$$

Since the quadratic space transformation is used here, the Condition 2.5 holds and we have the exponential rate of convergence (in discrete case a linear rate). However, owing to the simplification $\tau = 0$ a starting point x_0 must be such that $x_0 \in \Delta(x_*)$, $Ax_0 = b$. The convergence property of the method (3.8) is more attractive, than (3.7). Therefore, the method (3.8) is considered in the West as essential improvement of Dikin–Karmarkar method. The book [22] claims that (3.8) "is identical with the gradient flow for linear programming proposed and studied extensively by Faybusovich (1991). Independently this flow was also studied by Herzel, Recchioni and Zirilli (1991)... Starting from the seminal work of Khachian and Karmarkar, there has been a lot of progress in developing interior point algorithms for linear programming and nonlinear programming, due to Bayer, Lagarias and Faybusovich".

In point of fact, (3.8) has been proposed and analyzed as far back as 1977 in [9], and later on a comprehensive exposition of this method was described in numerous our papers and in the book [7] appeared in English in 1985 and three years before in Russian.

Still further, the nonlocal convergence analysis of (3.8) was carried out in [14]. We assumed that a condition $\sum_{i=1}^{n} x^i = 1$ was introduced among the other equality constraints. In this case the Lyapunov function

$$V(x) = \sum_{i=1}^{m} x_*^i[\ln x_*^i - \ln x^i] \tag{3.9}$$

decreases along the trajectories of the system (3.8) insofar as

$$\frac{dV}{dt} = c^\top (x_* - x) \leq 0 \tag{3.10}$$

everywhere on the feasible set X. Using (3.10) we evaluated a number of iterations required for finding ε-solution of (2.2), using step-sizes α_k determined by the steepest descent approach.

Numerous papers have been published in the West on the interior point techniques [3, 4, 19, 21, 23, 24, 30]. Some remarks deserve to be made concerning the main differences between the methodology adopted in these papers and our original approach:

1. Along with the LP problems we considered the NLP problems as well.
2. We developed asymptotically stable interior-infeasible-point algorithms. Our analysis was not confined to the interior point technique. For this reason the current points are in general allowed to be infeasible, however if the initial or current points are feasible, then the corresponding trajectory remains in the feasible set.
3. In all proposed methods multiplicative barriers were used and we did not resort to singular penalties.
4. The steepest descent approach was employed in computations. The trajectory could move along the boundary of the feasible set.

All these items can be considered as advantages of presented approach.

4. DUAL BARRIER-PROJECTION METHODS FOR SOLVING LP PROBLEMS

By extension of the space and by converting the inequality constraints to equalities, we transform the original dual problem (2.3) into the following equivalent problem:

$$\max_{u,y}(b^\top u) \text{ subject to } \xi(y) + A^\top u - c = 0_n, \tag{4.1}$$

where $v = \xi(y) \in \mathbb{R}^n_+$, $\overline{\xi(\mathbb{R}^n_+)} = \mathbb{R}^n_+$.

The Problem (4.1) is similar to (2.22). System (2.25) being applied to solving (4.1) can be rewritten in terms of u and v as follows:

$$\frac{du}{dt} = b - Ax(u,v), \qquad \frac{dv}{dt} = -G(v)x(u,v), \tag{4.2}$$

where $\Phi(v)x(u,v) = A^\top b + \tau(v + A^\top u - c)$ and $\Phi(v) = G(v) + A^\top A$.

If $u_0 \in U$, then we can get rid of the equation for v and this way simplify systems (4.2). In this case, (4.2) can be expressed as

$$\frac{du}{dt} = b - Ax(u), \qquad (G(v(u)) + A^\top A)x(u) = A^\top b, \tag{4.3}$$

where $u(0,u_0) = u_0 \in U$.

For this system we obtain the following inequality:

$$b^\top \frac{du}{dt} = \|b - Ax(u)\|^2 + x^\top(u)G(v(u))x(u) \geq 0.$$

Hence the objective function of the dual problem monotonically increases on a feasible set.

By applying the Euler numerical integration method we obtain the following iterative algorithm:

$$\begin{gathered} u_{k+1} = u_k + \alpha_k(b - Ax_k), \qquad v_{k+1} = v_k - \alpha_k G(v_k)x_k, \\ \left(G(v_k) + A^\top A\right) x_k = A^\top b + \tau(v_k + A^\top u_k - c). \end{gathered} \tag{4.4}$$

Similarly for the system (4.3) we have

$$u_{k+1} = u_k + \alpha_k(b - Ax_k), \qquad (G(v_k) + A^\top A)x_k = A^\top b, \tag{4.5}$$

where $v_k = v(u_k)$. Both variants solve the primal and dual problems simultaneously.

Denote

$$\alpha_* = \frac{2}{\lambda^*}, \qquad \lambda^* = \max\left[\tau, \max_{1\le i\le m} \dot{\theta}(0)x_*^i\right], \qquad \lambda_* = \min\left[\tau, \min_{1\le i\le m} \dot{\theta}(0)x_*^i\right],$$

where λ^* and λ_* are, respectively, maximum and minimum eigenvalues of the matrix of the equation of the first approximation about the optimal solution u_*.

Theorem 4.1. *Let x_* and u_* be unique non-degenerate solutions of the Problems (2.2) and (2.3), respectively, and let $v_* = c - A^\top u_*$. Assume that the component-wise space transformation $\xi(y)$ satisfies Conditions 2.4, 2.5 and $\tau > 0$. Then the following statements are true:*

1. *The pair $[u_*, v_*]$ is an asymptotically stable equilibrium state of system (4.2).*

2. *The solutions $u(t, z_0)$, $v(t, z_0)$ of system (4.2) converge locally exponentially fast to the pair $[u_*, v_*]$. The corresponding function $x(u(t, z_0), v(t, z_0))$ converges to the optimal solution x_* of the primal Problem (2.2).*

3. *The point u_* is an asymptotically stable equilibrium state of system (4.3).*

4. *The solutions $u(t, z_0)$ of system (4.3) converge locally exponentially fast to the optimal solution u_* of the dual Problem (2.3). The corresponding function $x(u(t, z_0))$ converges to the optimal solution x_* of the primal Problem (2.2).*

5. *For any fixed $0 < \alpha_k < \alpha_*$ the sequence $\{u_k, v_k\}$, generated by (4.4), converges locally with a linear rate to $[u_*, v_*]$ while the corresponding sequence $\{x_k\}$ converges to x_*.*

6. *There exists an $\alpha_* > 0$ such that for any fixed $0 < \alpha_k < \alpha_*$ the sequence $\{u_k\}$, generated by (4.5), converges locally with a linear rate to u_* while the corresponding sequence $\{x_k\}$ converges to x_*.*

The method (4.3) was proposed and studied in 1977 (see [9]). Method (4.2) was given in [11]. In [16] we describe nonlocal convergence analysis of dual method. We supposed the Problem (2.2) is such that $Ae = 0_m$, where e is a vector of ones in $\mathbb{R}^n$. We used the Lyapunov function similar to (3.9):

$$V(u) = \sum_{i=m+1}^{n} v_*^i[\ln v_*^i - \ln v^i(u)].$$

Let P be a full rank $d \times n$ matrix such that $AP^\top = 0_{md}$. Therefore, the columns of $P^\top$ are linearly independent and form a basis for the null-space of A. We partition A as $A = [B, N]$, where the square matrix B is nonsingular. We can now write the matrix P as .

$$P = [-N(B^\top)^{-1} \mid I_d].$$

The definitions of the sets V and V_U can be rewritten as follows:

$$V = \{v \in \mathbb{R}^n : P(v-c) = 0_d\}, \qquad V_U = \{v \in \mathbb{R}^n_+ : P(v-c) = 0_d\}.$$

Let $\bar{x} \in \mathbb{R}^n$ be an arbitrary vector which satisfies the constraint $A\bar{x} = b$. Then

$$\max_{u \in U} b^\top u = \max_{u \in U} \bar{x}^\top A^\top u = \max_{v \in V_U} \bar{x}^\top (c - v) = \bar{x}^\top c - \min_{v \in V_U} \bar{x}^\top v.$$

Hence the solution of the dual Problem (2.3) can be substituted by the following equivalent minimization problem:

$$\min_{v \in V_U} \bar{x}^\top v.$$

Applying the stable barrier-projection method (3.1) to this problem, we obtain

$$\frac{dv}{dt} = G(v)\left(P^\top x(v) - \bar{x}\right), \tag{4.6}$$

$$PG(v)P^\top x(v) = PG(v)\bar{x} + \tau P(c - v). \tag{4.7}$$

If a point v is such that the matrix $PG(v)P^\top$ is invertible, then we can solve the linear equation (4.7) and obtain

$$x(v) = (PG(v)P^\top)^{-1}(PG(v)\bar{x} + \tau P(c-v)).$$

Let $H(v) = G^{1/2}(v)$ and introduce the pseudo-inverse and projection matrices

$$(PH)^+ = (PH)^\top (PGP^\top)^{-1}, \qquad (PH)^\# = (PH)^+ PH.$$

The system (4.6), (4.7) can be rewritten in the following projective form:

$$\frac{dv}{dt} = H\left[\tau (PH)^+ P(c-v) - (I_n - (PH)^\#)H\bar{x}\right]. \tag{4.8}$$

The first vector in the square brackets belongs to the null-space of AH^{-1} and the second vector belongs to the row space of this matrix. Furthermore,

$$P\frac{dv}{dt} = \tau P(c - v), \qquad P(c - v(t, z_0)) = P(c - v_0)e^{-\tau t}.$$

Hence, the trajectories $v(t, z_0)$ approach the set V as $t \to \infty$.

If $v_0 \in V_U$ and $v_0 > 0$, then the entire trajectory does not leave the feasible set V_U, the objective function $\bar{x}^\top v(t, z_0)$ is a monotonically decreasing function of t and (4.8) can be rewritten as follows:

$$\frac{dv}{dt} = -G(v)\left(I_n - P^\top (PG(v)P^\top)^{-1} PG(v)\right)\bar{x}, \qquad v_0 \in \operatorname{ri} V_U. \tag{4.9}$$

Theorem 4.2. *Suppose that the conditions of the Theorem 4.1 hold. Then:*

1) *the point v_* is an asymptotically stable equilibrium point of system (4.6);*

2) *the solutions $v(t, v_0)$ of (4.8) converge locally to v_* with an exponential rate of convergence;*

3) *there exists an $\alpha_* > 0$ such that for any fixed $0 < \alpha_k < \alpha_*$ the discrete version*

$$v_{k+1} = v_k - \alpha_k G(v_k)(\bar{x} - P^\top x_k), \qquad x_k = x(v_k), \tag{4.10}$$

converges locally with a linear rate to v_* while the corresponding sequence $\{x_k\}$ converges to x_*.

Since for system (4.9) $P\dot{v} = 0_d$, it follows that the vector $\dot{v}$ belongs to null-space of P which coincides with the row space of A. Therefore, there exists a vector $\lambda \in \mathbb{R}^m$ such that

$$\dot{v} = A^\top \lambda. \tag{4.11}$$

If $v > 0_n$, then after left multiplying both sides of (4.11) with $AG^{-1}(v)$ and in view of (4.9) we obtain

$$\lambda = -(AG^{-1}(v)A^\top)^{-1}A\bar{x} = -(AG^{-1}(v)A^\top)^{-1}b.$$

Hence, on the set $\mathrm{ri}\, V_U$ the method (4.9) takes the form

$$\frac{dv}{dt} = -A^\top(AG^{-1}(v)A^\top)^{-1}b, \qquad v_0 \in \mathrm{ri}\, V_U.$$

In u-space this method can be written as

$$\frac{du}{dt} = (AG^{-1}(v(u))A^\top)^{-1}b, \qquad u_0 \in U_0.$$

If we use the quadratic and exponential space transformations (2.10), (2.11) we obtain

$$\frac{du}{dt} = (AD^{-1}(v(u))A^\top)^{-1}b, \qquad u_0 \in U_0, \tag{4.12}$$

and

$$\frac{du}{dt} = (AD^{-2}(v(u))A^\top)^{-1}b, \qquad u_0 \in U_0, \tag{4.13}$$

respectively. The system (4.13) coincides with the continuous version of the dual affine scaling method proposed by I. Adler, N. Karmarkar, M. Resende and G. Veiga in 1989 (see [1]).

According to the Theorem 4.2, the solution of (4.6) converges locally with an exponential rate to the equilibrium point $v_* = v(u_*)$. Therefore, the solutions of (4.12) also converge to the point u_* on the set U_0.

The discrete version of (4.12) consists of the iteration

$$u_{k+1} = u_k + \alpha_k(AD^{-1}(v_k)A^\top)^{-1}b, \qquad u_0 \in U_0, \tag{4.14}$$

where $v_k = v(u_k)$. Taking into account the Theorem 2.3.7 from [7] we conclude that the exponential rate of convergence of (4.12) insures local linear convergence of the discrete variant (4.14) if the step-length α_k is sufficient small.

5. PRIMAL AND DUAL BARRIER-NEWTON METHODS

Let us consider the Problem (2.1) supposing that all functions $f(x)$, $g^i(x)$, $1 \le i \le m$, are at least twice continuously differentiable. Equation (2.15) can be written as

$$g_x G(x) L_x(x, u(x)) = \tau g(x). \tag{5.1}$$

If the regular point x is such that

$$G(x) L_x(x, u(x)) = 0, \tag{5.2}$$

then $[x, u(x)]$ is a weak KKT pair for the Problem (2.1). In Section 2 we actually used a method of successive approximation for finding a solution of this nonlinear equation. Now we will apply the Newton method for this purpose. The continuous version of the Newton method leads to the initial value problem for the following system of ordinary differential equations:

$$\Lambda(x)\frac{dx}{dt} = -G(x)L_x(x,u(x)), \qquad x(0,x_0) = x_0, \tag{5.3}$$

where $\Lambda(x)$ is a Jacobian matrix of the mapping $G(x)L_x(x,u(x))$ with respect to x. By following the trajectories satisfying (5.3), we can theoretically obtain a solution of the system of nonlinear equations (5.2). In practice, we build the iterative procedures using a discretization of dynamical systems.

Along the trajectories of (5.3) we have

$$G(x(t,x_0))L_x(x(t,x_0),u(x(t,x_0))) = e^{-t}G(x_0)L_x(x_0,u(x_0)),$$
$$g(x(t,x_0)) = \frac{e^{-t}}{\tau}g_x(x(t,x_0))G(x_0)L_x(x_0,u(x_0)).$$

If the trajectories of (5.3) can be continued as $t \to +\infty$ and the norm of the vector

$$g_x(x(t,x_0))G(x_0)L_x(x_0,u(x_0))$$

is bounded, then the pair $[x(t,x_0),u(x(t,x_0))]$ converges to a weak KKT pair.

The brief analysis of method (5.3) for the case where $P = \mathbb{R}^n_+$ is given in [12]. Here we apply the primal barrier-Newton method (5.3) for solving the linear programming Problem (2.2). In this case we have

$$\Lambda(x) = [I_n - H(x)]D(\dot\theta(x))D(c - A^\top u(x)) + \tau H(x),$$
$$H(x) = D(\theta(x))A^\top (AD(\theta(x))A^\top)^{-1}A.$$

Using relation (5.1) we obtain that the system (5.3) has first integrals

$$D(\theta(x(t,x_0)))(c - A^\top u(x(t,x_0))) = D(\theta(x_0))(c - A^\top u(x_0))e^{-t},$$
$$b - Ax(t,x_0) = (b - Ax_0)e^{-t}.$$

Introduce a Lebesgue level set in $\mathbb{R}^n$

$$\Omega = \{x \in \mathbb{R}^n_+ : \|Ax - b\| \le \|Ax_0 - b\|,\ 0_n \le D(\theta(x))(c - A^\top u(x)) \le D(\theta(x_0))v_0\},$$

where x_0 is an initial point in (5.3), $v_0 = c - A^\top u_0$, $u_0 = u(x_0)$.

Theorem 5.1. *Suppose that the set Ω is compact and contains a unique stationary point x_*. Assume that the space transformation (2.9) satisfies Conditions 2.4, 2.5 and is such that the matrix $\Lambda(x)$ is nonsingular everywhere on Ω. If the starting point x_0 is such that $x_0 > 0_n$, $v_0 > 0_n$, then*

$$\lim_{t\to\infty} x(t,x_0) = x_*, \qquad \lim_{t\to\infty} u(x(t,x_0)) = u_*, \tag{5.4}$$

where x_*, u_* are the solutions of the Problems (2.2) and (2.3), respectively.

Integrating (5.3) using the Euler method, we obtain the following iterative process:

$$x_{k+1} = x_k - \alpha\Lambda^{-1}(x_k)D(\theta(x_k))(c - A^\top u(x_k)), \tag{5.5}$$

where $\alpha > 0$ is a step-size.

If the conditions of the Theorem 3.1 hold and the space transformation function satisfies Conditions 2.4, 2.5, then the matrix $\Lambda(x_*)$ is nonsingular. Therefore, if the step-size α is fixed and $0 < \alpha < 2$, then the discrete version (5.5) locally converges to the point x_* with at least linear rate. If matrix $\Lambda(x)$ satisfies the Lipschitz condition in a neighborhood of x_* and $\alpha = 1$, then the sequence $\{x_k\}$ converges quadratically to x_*.

Let a function $x(u)$ be defined from (4.3). Substituting this function in feasibility condition we obtain the following nonlinear system:

$$Ax(u) - b = 0_m.$$

Applying the Newton method for solving this system, we obtain the following continuous and discrete versions:

$$Q(u)\frac{du}{dt} = Ax(u) - b, \tag{5.6}$$

$$u_{k+1} = u_k + \alpha Q^{-1}(u_k)(Ax_k - b), \tag{5.7}$$

where

$$Q(u) = -A\left(D(\theta(v)) + A^\top A\right)^{-1} D(\dot{\theta}(v))D(x(u))A^\top.$$

Theorem 5.2. *Let the mapping $\xi(y)$, defined in Section 4, be such that Conditions 2.4 and 2.5 hold. Then the matrix $Q(u_*)$ is nonsingular, the solution u_* of the Problem (2.3) is an asymptotically stable equilibrium state of the system (5.6), the discrete version (5.7) converges locally to the point u_* with at least linear rate if* $0 < \alpha < 2$.

The more detailed information about this method can be found in [17].

6. PRIMAL-DUAL BARRIER-NEWTON METHODS

Here we construct a primal-dual barrier-Newton method for solving the Problem (2.8). Introduce an additional mapping

$$\phi(z) = [\phi^1(z^1), \dots, \phi^n(z^n)]$$

and assume that $\phi(z)$ satisfies the Conditions 2.4 and 2.5.

Then the necessary optimality conditions (2.7) for the Problem (2.8) can be rewritten in the form

$$D(\theta(x))\phi(L_x(x,u)) = 0_n, \qquad g(x) = 0_m, \qquad x \in \mathbb{R}^n_+. \tag{6.1}$$

For solving this system we use the continuous version of Newton's method. The computation process is described by the system of ordinary differential equations

$$W(x,u)\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = -\begin{pmatrix} \alpha D(\theta(x))\phi(L_x(x,u)) \\ \tau g(x) \end{pmatrix}, \tag{6.2}$$

where $\alpha > 0$, $\tau > 0$, W is a square $(n+m)^2$ matrix,

$$W(x,u) = \begin{pmatrix} M & D(\theta(x))D(\dot{\phi})g_x^\top \\ g_x & 0_{mm} \end{pmatrix}, \qquad M = D(\dot{\theta})D(\phi) + D(\theta)D(\dot{\phi})L_{xx}. \tag{6.3}$$

Lemma 6.1. *Let $[x_*, u_*]$ be a weak KKT pair, where the conditions of the Theorem 2.3 are satisfied. Assume that x_* is a regular point for the Problem (2.8) and the functions $\theta(x)$, $\phi(z)$ satisfy Conditions 2.4, 2.5. Then the matrix $W(x_*, u_*)$ is nonsingular.*

Let $x(t,z_0)$, $u(t,z_0)$ denote the solutions of the Cauchy problem (6.2) with initial conditions $x_0 = x(0,z_0)$, $u_0 = u(0,z_0)$, $z_0^\top = [x_0^\top, u_0^\top]$. Using this notation, we rewrite the system of equations (6.2) as

$$W(z)\frac{dz}{dt} = -D(\gamma)\mathbb{R}(z), \qquad z(0,z_0) = z_0, \tag{6.4}$$

where γ has the first n components equal to α and all other components equal to τ. We denote $\gamma_* = \min[\alpha, \tau]$.

Theorem 6.1. *Suppose that the conditions of the Lemma 6.1 hold. Then for any $\alpha > 0$, $\tau > 0$ the pair $z_*^\top = [x_*^\top, u_*^\top]$ is an asymptotically stable equilibrium point of system (6.4). If step-size h_k is fixed and $0 < h_k < 2/\gamma_*$, then the discrete version*

$$z_{k+1} = z_k - h_k W^{-1}(z_k) D(\gamma)\mathbb{R}(z_k) \tag{6.5}$$

locally converges to the point z_* with at least linear rate. If $W(z)$ satisfies a Lipschitz condition in a neighborhood of z_* and $h_k = \alpha = \tau = 1$, then the sequence $\{z_k\}$ converges quadratically to z_*.

Let us use a homogeneous function $\theta(x)$ of order λ, i.e. $\theta^i(x) = (x^i)^\lambda$, $1 \le i \le n$. In this case we modify (6.2) and use the system

$$\tilde{W}(x,u)\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = -\begin{pmatrix} \alpha D(x)\phi(L_x(x,u)) \\ \tau g(x) \end{pmatrix}, \tag{6.6}$$

where $\alpha > 0$, $\tau > 0$, $\tilde{W}$ is a square matrix

$$\tilde{W}(x,u) = \begin{pmatrix} M & D(x)D(\dot{\phi})g_x^\top \\ g_x & 0_{mm} \end{pmatrix}, \qquad M = \lambda D(\phi) + D(x)D(\dot{\phi})L_{xx}.$$

Lemma 6.2. *Let x be a regular point, and let the pair $[x,u]$ be such that $x^i \neq 0$, $L_{x^i}(x,u) \neq$ $\neq 0$ for all $1 \le i \le n$, and $M(x,u)$ is nonsingular. Then $W(x,u)$ is nonsingular.*

Define the nonnegative Lyapunov function

$$F(x,u) = \|D(\theta(x))\phi(L_x(x,u))\| + \|g(x)\|$$

and introduce two sets:

$$\Omega_0 = \{[x,u] : F(x,u) \le F(x_0,u_0),\ x \ge 0_n,\ L_x(x,u) \ge 0_n\},$$
$$\tilde{\Omega}_0 = \{[x,u] \in \Omega_0 : x > 0_n,\ L_x(x,u) > 0_n\}.$$

Theorem 6.2. *Suppose that the set Ω_0 is bounded and contains the unique KKT pair $[x_*, u_*]$. Suppose also that for any pair $[x,u] \in \tilde{\Omega}_0$ the conditions of the Lemma 6.2 are satisfied. Then all trajectories of (6.4) starting from a pair $[x_0,u_0] \in \tilde{\Omega}_0$, converge to $[x_*, u_*]$.*

The system of ordinary differential equations (6.4) has the first integrals

$$D(\theta(x(t,z_0)))\phi(L_x(x(t,z_0),u(t,z_0))) = D(\theta(x_0))\phi(L_x(x_0,u_0))e^{-\alpha t}, \tag{6.7}$$

$$g(x(t,z_0)) = g(x_0)e^{-\tau t}. \tag{6.8}$$

The solutions of (6.4) belong to Ω_0 and are, therefore, bounded. The right-hand sides of (6.7), (6.8) are strictly positive and tend to zero only as $t \to \infty$. By moving along the trajectories of (6.4) we do not violate nonnegativity of x and L_x. Therefore, the trajectories do not cross the boundary of the set Ω_0. The transformation functions $\theta(x)$ and $\phi(v)$ thus play the role of the multiplicative barriers preserving nonnegativity. All trajectories that emanate from $\tilde{\Omega}_0$ remain in the interior of Ω_0. According to La Salle's Invariance Principle [2] the solutions $x(t, z_0)$, $u(t, z_0)$ can be prolonged as $t \to \infty$, the positive limit set of the solution is a compact connected set contained in Ω_0 and coincides with the equilibrium pair $[x_*, u_*]$, which is unique on Ω_0.

Now we apply the Newton method (6.6) to LP problem. For the sake of simplicity we consider the case, where $\phi(v) = v$ and $\lambda = \alpha/\tau$. Introduce new vectors $\mu \in \mathbb{R}^m$, $\eta \in \mathbb{R}^n$:

$$\mu = \left(AD(x)D^{-1}(v)A^\top\right)^{-1} b, \qquad \eta = D^{-1}(v)A^\top \mu.$$

Let e be the vector of ones in $\mathbb{R}^n$. Now the system (6.6) can be written as

$$\frac{dx}{dt} = \tau D(x)[\eta - e], \qquad \frac{du}{dt} = \alpha\mu. \tag{6.9}$$

Theorem 6.3. *Let $[x_*, u_*]$ be a non-degenerate optimal pair for the Problems (2.2) and (2.3). Assume that Ω is bounded. Let the starting point $x_0 \in \mathbb{R}^n_{++}$ and $u_0 \in U_0$ be interior, then the trajectories of (6.9) are such that:*

1) *the matrix $AD(x(t, z_0))D^{-1}(v(t, z_0))A^\top$ is non-degenerate $\forall t \geq 0$;*

2) *$z(t, z_0) \in \Omega$ and $v(t, z_0) \in V$ for all $t > 0$;*

3) *the objective function $b^\top u(t, z_0)$ of the dual problem increases monotonically;*

4) *the pair $[x(t, z_0), u(t, z_0)]$ is bounded and converges to $[x_*, u_*]$ as $t \to \infty$;*

5) *all components of vectors $D^\lambda(x(t, z_0))v(t, z_0)$, $Ax(t, z_0)$ change monotonically and*

$$D^\lambda(x(t, z_0))v(t, z_0) = e^{-\alpha t} D^\lambda(x_0)v_0, \qquad Ax(t, z_0) - b = e^{-\tau t}(Ax_0 - b).$$

By applying the Euler numerical integration method to system (6.9) we obtain the simplest discrete version of the method:

$$x_{k+1} = D(x_k)(e + \tau_k(\eta_k - e)), \qquad u_{k+1} = u_k + \alpha_k \mu_k. \tag{6.10}$$

Note that if we set in (6.9), (6.10) $\alpha = 1$, $\tau = 0$ and $x = e$, then we obtain (4.12) and (4.14), respectively.

We specify three classes of procedures for determining the step lengths:

1) step lengths are fixed and small enough, hence the discrete process (6.10) is close to a continuous one (6.9);

2) step-sizes are close to one and, therefore, the discrete process has properties of Newton's method;

3) step-sizes are chosen from steepest descent conditions or from another auxiliary optimization problem.

The investigation of all these cases can be found in [18]. Here we consider the third approach to step-size choice, which proved to be computationally the most efficient.

The iterates produced by algorithm (6.10) are well-defined if vectors x_k, v_k are strictly positive for all k. In order to ensure the positiveness of x_{k+1} and v_{k+1} we have to choose the step lengths α_k and τ_k such that

$$e \geq \alpha_k \eta_k, \qquad e \geq \tau_k(e - \eta_k).$$

It is now straightforward to verify that non-negativity conditions hold if α_k and τ_k satisfy

$$0 < \alpha_k \leq \alpha_k^* = \frac{1}{[\eta_k^*]_+}, \qquad 0 < \tau_k \leq \tau_k^* = \frac{1}{[1 - \eta_*^k]_+},$$

where $[\alpha]_+ = \max[0, \alpha]$, η_k^* and η_*^k are maximal and minimal components of the vector η_k, respectively. Define the steps $\alpha_k = \omega \alpha_k^*$, $\tau_k = \omega \tau_k^*$, where $0 < \omega < 1$ is a safety factor.

Introduce two functions

$$\Phi(x, u) = x^\top v(u) + \|Ax - b\|, \qquad \phi(\alpha_k, \tau_k) = \Phi(x_{k+1}, u_{k+1}).$$

The steepest descent step-sizes $\bar{\alpha}_k$, $\bar{\tau}_k$ are found from the solution of the following auxiliary problem:

$$\phi(\bar{\alpha}_k, \bar{\tau}_k) = \min_{0 \leq \alpha_k \leq \omega \alpha_k^*} \min_{0 \leq \tau_k \leq \omega \tau_k^*} \phi(\alpha_k, \tau_k).$$

Here ϕ is a bilinear function of α and τ. In [18] this problem was solved analytically. Denote $\|\eta\|_\infty = \max_{1 \leq i \leq n} |\eta^i|$. Introduce the function

$$\mathcal{K}(x, v) = \left\lceil \frac{1 + C}{\omega} \ln \frac{\Phi(x, v)}{\varepsilon} \right\rceil,$$

where $\lceil a \rceil$ is the least integer larger than or equal to a and $\varepsilon > 0$.

Theorem 6.4. *Let $x_0 \in \mathbb{R}^n_{++}$, $u_0 \in U_0$ and suppose that the sequence $\{x_k, u_k\}$, generated by algorithm (6.10) with steepest descent rule, is such that $\|\eta_k\|_\infty \leq C$ for all k. Then the sequence $\{x_k, u_k\}$ converges to $[x_*, u_*]$ at finite number of steps or at least superlinearly and the function $\Phi(x_k, v_k)$ becomes less then ε in at most $\mathcal{K}(x_0, v_0)$ iterations.*

Method (6.10) possesses local convergence property, therefore, it is possible take starting points outside the positive orthant, but in this case we must take into account that the matrix $AD(x)D^{-1}(v)A^\top$ is singular on some manifold and we must complete the step-size rule. A phase portrait analysis of (6.9), proof of the Theorem 6.4 and illustrative computational example can be found in [13, 18].

Algorithm (6.10) has one important disadvantage connected with the necessity to know a starting point $u_0 \in U_0$. It is possible to get rid of this restriction if we use the barrier-Newton method in the extended space of variables x, u, v. The simplest version of the method is described by following system of $2n + m$ differential equations:

$$\frac{dx}{dt} = \tau G(A^\top \zeta - c), \qquad \frac{dv}{dt} = \alpha(c - v - A^\top \zeta), \qquad \frac{du}{dt} = \alpha(\zeta - u), \tag{6.11}$$

where $\zeta = (AGA^\top)^{-1}[b - Ax + AGc]$, $G = D(x)D^{-1}(v)$. The essential difference in the requirements on the initial conditions between systems (6.9) and (6.11) is that in the latter case we impose only the simplest restrictions: $x_0 > 0_n$, $v_0 > 0_n$.

Let $Ax_0 = b$. If we set in (6.11) $\alpha = 0$, $\tau = 1$ and $v = e$, then we obtain (3.8). The generalization of systems (6.9) and (6.11) for the LP problem with box constraints is given in [28].

ACKNOWLEDGEMENT

The research has been supported by the Russian Scientific Fund (grant N° 94-01-01379), which is here gratefully acknowledged.

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