Descriptive Image Algebras with One Ring[1]

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Abstract—The paper describes new results of investigations into developing mathematical tools for the analysis and estimation of information represented in the form of images. It is a continuation of the study of a new class of image algebras, descriptive image algebras (DIA). The use of DIA in image analysis applications requires examination of a great many operations, which may lead (or may not) to the construction of DIA with or without physical meaning. The questions about the kinds of operations which can be used for DIA construction and about the dependence of this process on the physical interpretability of these operations remain open. In general, the problem consists in formulating conditions for the set of operations which ensure the DIA construction. The ways of constructing the P-, G-, T-, and I-models by using one-ring DIA of a special type are described. The possibility of formalization of different image representation (models) will help to use the principles of algebraic recognition for image handling.

1. INTRODUCTION

The paper describes new results from investigations into developing mathematical tools for the analysis and estimation of information represented by images. These investigations have been carried out for several years in the Computer Science Laboratory of the Scientific Council on Cybernetics of the Russian Academy of Sciences, where the descriptive approach to image analysis and recognition is being developed and implemented [5, 6]. The fundamental purpose of these investigations is to create a general theory that would encompass various approaches and operations used in image processing. A new image algebra (IA), a descriptive image algebra (DIA), is being developed as such a theory [7, 8, 10, 11]. Its main distinction from the standard IA is that it considers both algorithms and descriptions of input information as algebraic objects. DIA generalizes a standard IA and makes it possible to use basic image models and operations over images (or both) as ring elements. The new algebra tools can be used for automation of development, testing, and evaluation of the quality of algorithms of image processing and analysis. They can also serve as a basis for designing new computational architectures specialized for image processing and analysis. By applying a new algebra, a new language is obtained for comparison and standardization of different algorithms for image analysis, recognition, and processing.

The results presented below are obtained in the framework of general research into structuring the concepts of DIA and augmenting the set of basic operations of DIA with respect to known image algebras. Main efforts were focused on the construction and study of the basic examples of DIA and on defining the conditions for a set of operations ensuring DIA construction.

To this end, the following tasks should be first carried out:

(1) The necessary and sufficient conditions for a set of operations applied to ring elements and generating one-ring DIA should be established.

(2) The possibility of generating special image models with the help of DIA and of using them as operands in DIA should be investigated.

Our theoretical research consists of the following stages:

At the first stage, examples of sets of operations are constructed, which may have (or have not) physical meaning and may lead (or may not) to the construction of one-ring DIA. The standard algebraic operations, as well as specialized operations of image processing and analysis represented in algebraic form, can be used as operations. Images and operations over images are used as ring elements.

In this paper, we provide the following five examples: (i) three examples of basic one-ring DIA with different elements and operations having a certain physical meaning: in Examples 1 and 2, the images are ring elements, and, in Example 3, binary operations over images are ring elements; (ii) Example 4, where the constructed set with the introduced operations forms a group over addition; and (iii) Example 5, where the operations which form groups neither over addition nor over multiplication are shown.

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At the second stage, the results of applying one-ring DIA operations to various sets of elements are collected. The specificity of pattern recognition makes it possible to successfully recognize a new object from incomplete information about it. Thus, we can use image models which contain information on the initial image. In this case, we cannot restore an image from its model; however, its correct classification is possible. The following four classes of models are presented here as image models: P-model (parametric model), G-model (procedural model), T-model (generative model), and image I in its natural form [7]. In the second part of the paper, different ways of constructing image models using DIA are demonstrated.

At the third stage, the conditions for the set of operations which ensure the one-ring DIA construction are formulated. At this stage, the standard IA operations introduced by G. Ritter [14] were studied. By applying these operations to a variety of images, a set of operations was obtained leading to the DIA construction. The necessary and sufficient conditions for a set of operations generating one-ring DIA are determined.

2. DESCRIPTIVE IMAGE ALGEBRAS: DEFINITIONS AND EXAMPLES

2.1. The Purposes and Fundamental Works

The stimulus for DIA construction was one of the classical problems in image analysis: an efficient construction of a sequence of operations for transforming the given image into a desired one and vice versa. It turned out that the general solution of this problem requires a formal system for image representation and transformation. This system should meet the following conditions: (i) every object of transformation should be a hierarchical structure built from the primitive objects by using IA operations; (ii) points, sets, models, operations, and morphisms can be used as objects; and (iii) every transformation is a hierarchical structure built from the set of basic transformations by using IA operations.

An algebraic formalism should provide the following possibilities [10]:

—construction of algebraic structures which allow one to use the methods borrowed from other branches of mathematics for image processing, analysis, and recognition;

—construction of correct and compact image descriptions convenient both for procedure interpretations and for the design of new methods;

—construction of a language to describe image transformations and operations over images as compact sets of simple transformations;

—construction of machine-independent, as well as architecture-dependent, descriptions of image transformations;

—enhancement of software implementation by the matching of algebraic expressions to the program blocks;

—establishment of the correlation between the available programming languages and known algorithms of image processing, analysis, and recognition, or, in other words, selection of the most effective programming languages for the resulting algebraic structures.

An essential drawback of the general theory of algebras is that the character of the problem at hand is neglected when the algebraic methods are applied to information represented by images. In addition, a simple interpretation of the results of application of a theory is not always possible. There are a lot of natural image transformations that are easily understood by the user (e.g., rotation, compression, dilation, color inversion, etc.) which are resistant to standard algebraic operations. Therefore, the IA tools should be combined with a set of natural image transformations. Thus, this new class of IA should cover not only the basic models of images as objects of analysis and recognition but also the basic models of transformations, which lead to effective synthesis and implementation of the basic procedures of image formal description, processing, analysis, and recognition.

The idea of creating a general theory enveloping various approaches and operations for image and signal processing has a long history. It starts with the works of von Neumann and continues with the works of S. Anger, M. Duff, G. Materon, J. Serra, G. Ritter, and others.

The mathematical morphology developed by Materon and Serra [15] became a starting point for a new mathematical trend in image processing and analysis. J. Serra and S. Stenberg [16] were the first to succeed in developing an algebraic theory of image processing and analysis on the basis of mathematical morphology. S. Stenberg was supposedly the first to introduce the modern conception of image algebra: "Image algebra is a representation of the algorithms of the cell computer for image processing in the form of algebraic expressions wherein images are variables and logical or geometric combinations of images are operations" [16]. It should be noted that U. Grenander used the concept of image algebra as early as in 1970; however, a different algebraic construction was meant [3]. A great number of works appeared in the framework of this study concerned with the development of special algebraic constructions that use or reject mathematical morphology.

The following investigations of the 1970s–1980s brought about the development of DIA:

—Zhuravlev algebra (also known as algebra of algorithms). The algebraic methods in the theory of algorithms are directed at the construction of efficient procedures for solving problems with poorly formalized and sometimes contradictory information and problems of discrete optimization which can only be solved by an exhaustive search unrealizable in practice [2].

—Descriptive approach to image analysis and understanding as a specialization of the general algebraic approach to the problems of recognition, classification, and forecasting where the initial data are resented by images (transforming images to a form convenient for recognition, standardization of the synthesis of formal image description, classes of image models) [5, 6].

—General pattern theory of U. Grenander (description of analyzed objects) [3, 4]. The works of Grenander are a basis for creating the most general tools for dealing with patterns of arbitrary nature and representing them as algebraic structures. An essential contribution made by Grenander is the unification of algebraic tools with the concept of Markov random processes by setting probabilistic measures on arbitrary algebraic structures. The fundamental idea of the Grenander theory consists in the possibility of expressing knowledge on patterns in terms of regular structures.

—Extended (standard) IA of G. Ritter [14] has a narrower application purpose: it is aimed at the generalization of known local methods of image analysis, e.g., mathematical morphology, for augmenting their possibilities and eliminating the bottlenecks. Since IA not only generalizes mathematical morphology and linear algebra but also is the most wide and convenient structure, the IA language allows one both to implement known algorithms and to create new ones. The IA can serve as a basis for creating a higher level language for image analysis that uses operations and operands of a standard IA. Also, it is possible to extend the structure of IA by introducing new operations, which is helpful in the cases when morphology and linear algebra do not yield a satisfactory result.

2.2. Algebra Definitions

Definition 2.2.1. [17] A nonempty set G of elements of arbitrary nature (e.g., numbers, mappings, transformations) is called *a group* if the following four conditions are satisfied:

(1) A composition law is given which puts every pair of elements a and b from G in correspondence with the third element from G, which is usually called a composition of the elements a and b and is denoted as ab.

(2) An association law. For any three elements a, b, and c from G, the equality

$$(ab)c = a(bc)$$

holds.

(3) There is a (left) unity e in G, i.e., an element e with the following property:

$$ea = a \text{ for all } a \text{ from } G.$$

(4) For every element a from G, there is (at least) one (left) inverse element a^{-1} in G defined by the relation

$$aa^{-1} = e.$$

If the addition is used instead of multiplication, the group is called *additive.*

If the law of commutativity is also fulfilled, i.e., $ab = ba$, then the group is called an *Abelian group.*

Definition 2.2.2. [17] *A system with double composition* is an arbitrary set of elements a, b,…, wherein, for any pair of elements a and b, the sum $a + b$ and the product ab are uniquely determined and belong to the same set. A system with double composition is called *a ring* if the operations over elements of this system obey the following laws:

(1) Laws of addition:

(a) Associative law: $a + (b + c) = (a + b) + c$.

(b) Law of commutation: $a + b = b + a$.

(c) Solvability of equation $a + x = b$ for all a, b.

(2) Law of multiplication:

(a) Associative law: $a(bc) = (ab)c$.

(3) Distributive laws:

(a) $a + (b + c) = ab + ac$.

(b) $(b + c)a = ba + ca$.

If the multiplication obeys the law of commutation (2b): $ab = ba$, then the ring is called *commutative*.

Definition 2.2.3. [17] A ring is called *a body* if

(a) it contains at least one nonzero element; and

(b) equations $\left.\begin{array}{rcl} ax &=& b, \\ ya &=& b \end{array}\right\}$ are unsolvable for $a = 0$.

If, in addition, a ring is commutative, it is called *a field* or *rational ring*.

Definition 2.2.4. [17] Let us consider the following: (i) a body K whose elements a, b, … are called coefficients or scalars; (ii) an additive Abelian group M whose elements x, y, … are called vectors; and (iii) multiplication of the vectors by scalars xa, which meets the following demands:

(1) xa is in M;

(2) $(x + y)a = xa + ya$;

(3) $x(a + b) = xa + xb$;

(4) $x(ab) = (xa)b$;

(5) $x_1 = x$.

If all these demands are satisfied, M is called a vector space over K or, more precisely, a right K-vector space, since the coefficients are placed to the right of vectors. The left K-vector space is defined similarly. If the body K is commutative, these notions coincide.

Definition 2.2.5. [17] A ring U which is a finite-dimensional vector space over some field P is called *an algebra*. If, in addition, the relation $(au)v = u(av) =$

$a(uv)$ is valid for $\alpha \in P$ and $u, v \in U$, then the algebra is called *associative*.

Properties of the algebra:

—Properties of the field $P(\alpha, \beta, \gamma \in P)$:

(1) $\forall\alpha, \beta \in P, \exists!\ (\alpha + \beta) \in P$:

(a) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$;

(b) $\alpha + \beta = \beta + \alpha$;

(c) $\exists 0 \in P, \forall\alpha \in P, \alpha + 0 = \alpha$;

(d) $\forall\alpha \in P, \exists(-\alpha), \alpha + (-\alpha) = 0$.

(2) $\forall\alpha, \beta \in P, \exists!\ (\alpha\beta) \in P$:

(a) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$;

(b) $\alpha\beta = \beta\alpha$;

(c) $\exists 1 \in P. \forall\alpha \in P, 1\alpha = \alpha$.

(3) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

—Properties of the ring $U(a, b, c \in U)$:

(1) $\forall a, b \in U, \exists!\ (a + b) \in U$:

(a) $a + (b + c) = (a + b) + c$;

(b) $a + b = b + a$;

(c) $\exists 0 \in U, \forall a \in U, a + 0 = a$;

(d) $\forall a \in U \exists(-a), a + (-a) = 0$.

(2) $\forall a, b \in U, \exists!(ab) \in U$:

(a) $a(bc) = (ab)c$.

(3) $(\alpha a + \beta b)c = \alpha ac + \beta bc$.

—Properties of the vector space ($\forall\alpha \in P, a \in U$: αa Œ U):

(1) $\alpha(\beta a) = (\alpha\beta)a$.

(2) $(\alpha + \beta)a = \alpha a + \beta a$.

(3) $\alpha(a + b) = \alpha a + ab$.

Definition 2.2.6. An algebra over the field A is called an image algebra (IA) if the images (sets of points) with their values and characteristics are the elements of its ring [14].

Definition 2.2.7. An algebra over the field A is called a descriptive image algebra (DIA) if either the image models or operations with images (or both) are the elements of its ring. As a model, the image itself or the set of corresponding values and characteristics can be chosen [7, 8].

Definition 2.2.8. DIA is called a basic DIA if its ring consists of either image models or operations over images [7, 8].

2.3. Images According to G. Ritter

For the sake of generality, during the construction of examples we use the definition of image (see Definition 2.3.1) and operations over images introduced by Ritter in standard image algebra [14].

Definition 2.3.1. (Images According to Ritter [14]) Let F be a set of values and X be a set of points. A mapping of the set X into set F (the element of the set F^X): $I = \{(x, a(x)), x \in X, a(x) \in F\}$ is called an image that takes values from F in the set F.

Let $I_1 = \{(x, a(x)), x \in X\}$ and $I_2 = \{(x, b(x)), x \in X\}$.

Operations over images according to Ritter ([14]).

Let the following operations be determined over F: addition, multiplication, maximization, and inverse operations, such as subtraction, division, minimization, and exponential operation ($\forall a(x), b(x) \in F$) $\exists! a(x) + b(x)$, $\exists! a(x)b(x)$, $\exists! a(x) \vee b(x)$, $\exists! a(x) - b(x)$, for $b(x) \neq 0$ $\exists! a(x)/b(x)$, for $a(x) > 0$ $\exists! a(x)^{b(x)}$.

The main operations over images from F^x are pointwise addition, multiplication, and maximization:

$$I_1 + I_2 = \{(x, c(x)), c(x) = a(x) + b(x), x \in X\},$$

$$I_1 I_2 = \{(x, c(x)), c(x) = a(x)b(x), x \in X\},$$

$$I_1 \vee I_2 = \{(x, c(x)), c(x) = a(x) \vee b(x), x \in X\}.$$

Operations of subtraction, division, and minimization are introduced as inverse operations:

$$I_1 - I_2 = \{(x, c(x)), c(x) = a(x) - b(x), x \in X\},$$

$$\frac{I_1}{I_2} = \{(x, c(x)), c(x) = \frac{a(x)}{b(x)}, b(x) \neq 0, c(x) = 0, b(x) \in 0\},$$

$$I_1 \wedge I_2 = \{(x, c(x)), c(x) = a(x) \wedge b(x), x \in X\}.$$

Similarly, we can introduce other operations over images, such as

$$I_1^{I_2} = \{(x, c(x)), c(x) = a(x)^{b(x)}, \text{if } a(x) > 0, \text{otherwise } c(x) = 0, x \in X\}.$$

We can introduce unary operations like multiplication by the element of the real number field ($\alpha \in R$:

$$\alpha I_1 = \{(x, c(x)), c(x) = \alpha a(x), x \in X\}.$$

2.4. Compliance with Requirements for Being a Member of DIA

2.4.1. Examples of a set of algebra-generating operations. Here, we define the types of sets, operations, and their physical interpretation that we use in the examples given below. Let us consider different sets U with operations of addition, multiplication, and multiplication by the real number defined on them.

(1) Elements of the set U:

(1.1) Images defined on the set X which has an arbitrary range of values F with dimensionality coincident with the dimensionality of a set X, i.e., $X, F \subset R^n$;

(1.2) Images defined on the set X which has a range of values X, $X \subset R^n$;

(1.3) Standard binary operations over images [14].

(2) Operations over set elements:

(2.1) Addition;

(2.2) Multiplication;

(2.3) Multiplication by the field element.

(3) Physical meaning of operations:

(3.1) addition (getting the summary brightness of two images), multiplication (pointwise filter), multiplication by the element of the field of real numbers (proportional increase or decrease in image brightness).

(3.2) addition (getting the summary brightness of two images); multiplication (global, nonpointwise, filter and definition of an image in a set defined by another image); multiplication by the element of the field of real numbers (proportional increase or decrease in image brightness).

(3.3) addition (global filter: first, two operations are applied to both images, then, the resulting images are added); multiplication (global filter: the second operation is applied to both images and the resulting image serves as the first and second operands of the first operation); multiplication of the operation by the element of the field of real numbers (image multiplication by a field element, i.e., unary operation over images: standard operation of image multiplication by the element of the field of real numbers [7]).

In what follows we describe the examples of DIA generated by the sets U with defined characteristics (in parentheses, we indicate the type of set elements, operations, and their physical interpretations according to the list).

2.4.1.1. Example 1 (1.1, 2.1, 3.1).

Let the family of functions $\{F_i\}_1^\infty$ be given.

—An operation of addition of two elements from F_i, F_j: $i, j = 1, 2, \ldots$, is introduced as $\forall a(x) \in F_i$, $b(x) \in F_j$: $\exists! a(x) + b(x) \in F_k$, $k = 1, 2, \ldots$, $F_k \subset R^n$, with the following properties:

$$(\forall a(x) \in F_i, b(x) \in F_j, c(x) \in F_y, i, j, y = 1, 2, \ldots)$$

(1.1) $a(x) + (b(x) + c(x)) = (a(x) + b(x)) + c(x)$;

(1.2) $a(x) + b(x) = b(x) + a(x)$;

(1.3) $\forall i$: $\forall a(x) \in F_i$, $\exists 0 \in F_i$: $a(x) + 0 = a(x)$;

(1.4) $\forall i$: $\forall a(x) \in F_i$, $\exists(-a(x)) \in F_i$: $a(x) + (-a(x)) = 0$.

In the simplest case (all $F_i \equiv F$), this operation can be introduced as a pointwise addition of two set elements.

—An operation of multiplication of two elements from F_i, F_j: $i, j = 1, 2, \ldots$, is introduced as $a(x) \in F_i$, $b(x) \in F_j$: $\exists! a(x)b(x) \in F_k$, $k = 1, 2, \ldots$, $F_k \subset R^n$, with the following properties:

$$(\forall a(x) \in F_i, b(x) \in F_j, c(x) \in F_y, i, j, y = 1, 2, \ldots):$$

(1.5) $a(x)(b(x)c(x)) = (a(x)b(x))c(x)$.

—An operation of multiplication by the element of the field of real numbers R is introduced over a set F: $\forall \alpha \in R$, $a(x) \in F$: $\exists! \alpha a(x) \in F$, with the following properties:

$$(\forall a(x) \in F_i, b(x) \in F_j, c(x) \in F_y,$$
$$i, j, y = 1, 2, \ldots, \forall a, b \in R);$$

(1.6) $(\alpha a(x) + \beta b(x))c(x) = \alpha a(x)c(x) + \beta b(x)c(x)$;

(1.7) $\alpha(\beta a(x)) = \alpha\beta a(x)$;

(1.8) $(\alpha + \beta)a(x) = \alpha a(x) + \beta a(x)$;

(1.9) $\alpha(a(x) + b(x)) = \alpha a(x) + \alpha b(x)$.

Statement 2.4.1.1.

Suppose that

—R is a field of real numbers;

—$I = \{(x, f(x)), x \in X, f(x) \in F\}$ ($X, F \subset R^n$, $n \in N$, $F \subset \{F_i\}_1^\infty$}, where F is a set of values taken by image I over set X) are elements of the set U;

—$I_1 = \{(x, a(x)), x \in \mathrm{X}, a(x) \in F_1\}$, $I_2 = \{(x, b(x)), x \in \mathrm{X}, b(x) \in F_2\}$, ($X$, F_1, $F_2 \subset R^n$, $n \in N$, F_1, $F_2 \subset \{F_i\}_1^\infty$).

We introduce

—the operation of addition of two images I_1 and I_2 as

$$I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\};$$

—the operation of multiplication of two images I_1 and I_2 as

$$I_1 I_2 = \{(x, a(x)b(x)), x \in X\}; \text{ and}$$

—the operation of multiplication of image I by the element of the field of real numbers $\alpha \in R$ as

$$\alpha I = \{(x, \alpha f(x)), x \in X\}.$$

Then, a set U with operations of addition, multiplication, and multiplication by the element of the field of real numbers introduced over it is *an algebra over the field of real numbers.*

Proof.

The proof is based on the validation of the algebra's properties (see Definition 2.2.5) of the set U with operations of addition, multiplication, and multiplication by the element of the field of real numbers introduced over it.

Properties of algebra:

—R is a field of real numbers;

—Properties of a ring $U(I_1, I_2, I_3 \in U)$, where

$$I_1 = \{(x, a(x)), x \in X, a(x) \in F_1\},$$
$$I_2 = \{(x, b(x)), x \in X, b(x) \in F_2\},$$
$$I_3 = \{(x, c(x)), x \in X, c(x) \in F_3\},$$

are the following:

(1) $\forall I_1, I_2 \in U$, $\exists!(I_1 + I_2) \in U$

(a) $I_1 + (I_2 + I_3) = I_1 + \{(x, (b(x) + c(x)), x \in X\} = \{(x, a(x) + (b(x) + c(x))), x \in X\}$

$(I_1 + I_2) + I_3 = \{(x, a(x) + b(x)), x \in X\} + I_3 = \{(x, (a(x) + b(x)) + c(x)), x \in X)\}$ = {according to Property 1.1} = $\{(x, a(x) + (b(x) + c(x))), x \in X\}$

$$I_1 + (I_2 + I_3) = (I_1 + I_2) + I_3$$

(b) $I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\}$

$I_2 + I_1 = \{(x, b(x) + a(x)), x \in X\}$ = {according to Property 1.2} = $\{(x, a(x) + b(x)), x \in X\}$

$$I_1 + I_2 = I_2 + I_1$$

(c) $O = \{(x, 0), x \in X\}$

$I_1 + O = \{(x, a(x) + 0), x \in X\}$ = {according to Property 1.3} = $\{(x, a(x)), x \in X\} = I_1$

$$\exists O \in U, \forall I_1 \in U, I_1 + O = I_1$$

(d) $(-I_1) = \{(x, -a(x)), x \in X\}$

$I_1 + (-I_1) = \{(x, a(x) - a(x)), x \in X\}$ = {according to Property 1.4} = $\{(x, 0), x \in X\} = O$

$$\forall I_1 \in U, \exists(-I_1), I_1 + (-I_1) = 0$$

(2) $\forall I_1, I_2 \in U, \exists!(I_1I_2) \in$ U

(a) $I_1(I_2I_3) = I_1(\{(x, b(x)c(x)), x \in X\}) = \{(x, a(x)(b(x)c(x))), x \in X\}$

$(I_1I_2)I_3 = (\{(x, a(x)b(x)), x \in X\})I_3 = \{(x, (a(x)b(x))c(x)), x \in X\}$ = {according to Property 1.5} = $\{(x, a(x)(b(x)c(x))), x \in X)\}$

$$I_1(I_2I_3) = (I_1I_2)I_3$$

(3) $\alpha, \beta \in R$

$(\alpha I_1 + \beta I_2)I_3 = (\{(x, \alpha a(x) + \beta b(x)), x \in X\})I_3 = \{(x, (\alpha a(x) + \beta b(x))c(x)), x \in X\}$ = {according to Property 1.6} = $\{(x, \alpha a(x)c(x) + \beta b(x)c(x)), x \in X\}\}$

$\alpha I_1I_3 + \beta I_2I_3 = \{(x, \alpha a(x)c(x)), x \in X\} + \{(x, \beta b(x)c(x)), x \in X\} = \{(x, \alpha a(x)c(x) + \beta b(x)c(x)), x \in X\}$

$$(\alpha I_1 + \beta I_2)I_3 = \alpha I_1I_3 + \beta I_2I_3.$$

—The properties of a vector space ($\forall\alpha \in P, I \in U$: $\alpha I \in U$) are:

(1) $\alpha(\beta I) = \alpha\{(x, \beta f(x)), x \in X\} = \{(x, \alpha(\beta f(x))), x \in X\}$ = {according to Property 1.7} = $\{(x, \alpha\beta f(x)), x \in X\}$

$(\alpha\beta)I = \{(x, \alpha\beta f(x)), x \in X\}$

$\alpha(\beta I) = (\alpha\beta)I$

(2) $(\alpha + \beta)I = \{(x, (\alpha + \beta)f(x)), x \in X\}$ = {according to Property 1.8} = $\{(x, \alpha f(x) + \beta f(x)), x \in X\}$

$\alpha I + \beta I = \{(x, \alpha f(x)), x \in X\} + \{(x, \beta f(x)), x \in X\} = \{(x, \alpha f(x) + \beta f(x)), x \in X)\}$

$$(\alpha + \beta)I = \alpha I + \beta I$$

(3) $\alpha(I_1 + I_2) = \alpha\{(x, a(x) + b(x)), x \in X\} = \{(x, \alpha(a(x) + b(x))), x \in X\}$ = {according to Property 1.9} = $\{(x, \alpha a(x) + \alpha b(x)), x \in X\}$

$\alpha I_1 + \alpha I_2 = \{(x, \alpha a(x)), x \in X\} + \{(x, \alpha b(x)), x \in X\} = \{(x, \alpha a(x) + \alpha b(x)), x \in X\}$

$$\alpha(I_1 + I_2) = \alpha I_1 + \alpha I_2.$$

All properties of a ring, field, and vector space are valid. The set U with operations of addition, multiplication, and multiplication by the element of the field of real numbers introduced over it is the algebra over the field of real numbers. Q.E.D.

2.4.1.2. Example 2 (1.2, 2.2, 3.2).

Let the set X be given:

—The operation of addition of two elements from X $\forall a(x), b(x) \in X$: $\exists!\, a(x) + b(x) \in X$ is introduced which is characterized by the following properties ($\forall a(x), b(x), c(x) \in X$):

(2.1) $a(x) + (b(x) + c(x)) = (a(x) + b(x)) + c(x)$;

(2.2) $a(x) + b(x) = b(x) + a(x)$;

(2.3) $\forall a(x) \in X, \exists 0 \in X$: $a(x) + 0 = a(x)$;

(2.4) $\forall a(x) \in X, \exists(-a(x)) \in$: $a(x) + (-a(x)) = 0$.

—The operation of superposition of two elements from X is introduced: $\forall a(x), b(x) \in X$: $\exists! a(b(x)) \in X$.

—The operation of multiplication by the element of the field of real numbers R is introduced over set X which is characterized by the following properties ($\forall a(x), b(x), c(x) \in X, \forall\alpha, \beta \in R$):

(2.5) $(\alpha a(x) + \beta b(x))c(x) = \alpha a(c(x)) + \beta b(c(x))$;

(2.6) $\alpha(\beta a(x)) = \alpha\beta a(x)$;

(2.7) $(\alpha + \beta)a(x) = \alpha a(x) + \beta a(x)$;

(2.8) $\alpha(a(x) + b(x)) = \alpha a(x) + \alpha b(x)$.

Statement 2.4.1.2.

Suppose that

—R is a field of real numbers;

—$I = \{(x, f(x)), x \in X, f(x) \in X\}$ ($X \subset R^n, n \in N$) are the elements of the set U; and

—$I_1 = \{(x, a(x)), x \in X, a(x) \in X\}$, $I_2 = \{(x, b(x)), x \in X, b(x) \in X\}$.

We introduce

—the operation of addition of two images I_1 and I_2 as

$$I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\};$$

—the operation of multiplication of two images I_1 and I_2 as

$$I_1I_2 = \{(x, a(b(x))), x \in X\}$$

(thus, we stay within the set); and

—the operation of multiplication of image I by the element of the field of real numbers $\alpha \in R$ as

$$\alpha I = \{(x, \alpha f(x)), x \in X\}.$$

(Conditions (2.1)–(2.4), (2.7), and (2.8) imply that the set X is a vector field over field R.)

Then, a set U with operations of addition, multiplication, and multiplication by the element of the field of real numbers introduced over it is *an algebra over the field of real numbers*.

Proof.

The proof is based on the validation of the algebra's properties (see Definition 2.2.5) of the set U with operations of addition, multiplication, and multiplication by the element of the field of real numbers introduced over it.

Properties of the algebra:

—R is a field of real numbers;

—Properties of a ring $U(I_1, I_2, I_3 \in U)$, where

$$I_1 = \{(x, a(x)), x \in X, a(x) \in X\},$$

$$I_2 = \{(x, b(x)), x \in X, b(x) \in X\},$$

$$I_3 = \{(x, c(x)), x \in X, c(x) \in X\},$$

are the following:

(1) $\forall I_1, I_2 \in U, \exists!(I_1 + I_2) \in U$

(a) $I_1 + (I_2 + I_3) = I_1 + \{(x, b(x) + c(x)), x \in X\} = \{(x, a(x) + (b(x) + c(x))), x \in X\}$

$(I_1 + I_2) + I_3 = \{(x, a(x) + b(x)), x \in X\} + I_3 = \{(x, (a(x) + b(x)) + c(x)), x \in X\}$ = {according to Property 2.1} = $\{(x, a(x) + (b(x) + c(x))), x \in X\}$

$$I_1 + (I_2 + I_3) = (I_1 + I_2) + I_3$$

(b) $I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\}$

$I_2 + I_1 = \{(x, b(x) + a(x)), x \in X\}$ = {according to Property 2.2} = $\{(x, a(x) + b(x)), x \in X\}$

$$I_1 + I_2 = I_2 + I_1$$

(c) $O = \{(x, 0), x \in X\}$

$I_1 + O = \{(x, a(x) + 0), x \in X\}$ = {according to Property 2.3} = $\{(x, a(x)), x \in X\} = I_1$

$$\exists O \in U, \forall I_1 \in U, I_1 + O = I_1$$

(d) $(-I_1) = \{(x, -a(x)), x \in X\}$

$I_1 + (-I_1) = \{(x, a(x) - a(x)), x \in X\}$ = {according to Property 2.4} = $\{(x, 0), x \in X\} = O$

$$\forall I_1 \in U, \exists(-I_1), I_1 + (-I_1) = 0$$

(2) $\forall I_1, I_2 \in U, \exists!(I_1 I_2) \in U$

(a) $I_1(I_2 I_3) = I_1(\{(x, b(c(x))), x \in X\}) = \{(x, a(b(c(x)))), x \in X\}$

$(I_1 I_2)I_3 = (\{(x, a(b(x))), x \in X\})I_3 = \{(x, a(b(c(x)))), x \in X\}$

$$I_1(I_2 I_3) = (I_1 I_2)I_3$$

(3) $\alpha, \beta \in R$

$(\alpha I_1 + \beta I_2)I_3 = (\{(x, \alpha a(x) + \beta b(x)), x \in X\})I_3 = \{(x, (\alpha a(c(x)) + \beta b(c(x)))), x \in X\}$

$\alpha I_1 I_3 + \beta I_2 I_3 = \{(x, \alpha a(c(x))), x \in X\} + \{(x, \beta b(c(x))), x \in X\} = \{(x, (\alpha a(c(x)) + \beta b(c(x)))), x \in X\}$

$$(\alpha I_1 + \beta I_2)I_3 = \alpha I_1 I_3 + \beta I_2 I_3.$$

—Properties of a vector space ($\forall \alpha \in P, I \in U$: $\alpha I \in U$):

(1) $\alpha(\beta I) = \alpha\{(x, \beta f(x)), x \in X\} = \{(x, \alpha(\beta f(x))), x \in X\}$ = {according to Property 2.6} = $\{(x, \alpha\beta f(x)), x \in X\}$

$(\alpha\beta)I = \{(x, \alpha\beta f(x)), x \in X\}$

$$\alpha(\beta I) = (\alpha\beta)I$$

(2) $(\alpha + \beta)I = \{(x, (\alpha + \beta)f(x)), x \in X\}$ = {according to Property 2.7} = $\{(x, \alpha f(x) + \beta f(x)), x \in X\}$

$\alpha I + \beta I = \{(x, \alpha f(x)), x \in X\} + \{(x, \beta f(x)), x \in X\} = \{(x, \alpha f(x) + \beta f(x)), x \in X\}$

$$(\alpha + \beta)I = \alpha I + \beta I$$

(3) $\alpha(I_1 + I_2) = \alpha\{(x, a(x) + b(x)), x \in X\} = \{(x, \alpha(a(x) + b(x))), x \in X\}$ = {according to Property 2.8} = $\{(x, \alpha a(x) + \alpha b(x)), x \in X\}$

$\alpha I_1 + \alpha I_2 = \{(x, \alpha a(x)), x \in X\} + \{(x, \alpha b(x)), x \in X\} = \{(x, \alpha a(x) + \alpha b(x)), x \in X\}$

$$\alpha(I_1 + I_2) = \alpha I_1 + \alpha I_2.$$

All properties of a ring, field, and vector space are valid. The set U with operations of addition, multiplication, and multiplication by the element of the field of real numbers introduced over it is the algebra over the field of real numbers. Q.E.D.

2.4.1.3. Example 3 (1.3, 2.3, 3.3).

Let

—$A, B, C, \ldots$ be images acting from X to X, $X \subset R^n$;

—The following operations over images are introduced [14]:

(1) $A + B = \{(x, c(x)): c(x) = a(x) + b(x), x \in X\}$;

(2) $AB = \{(x, c(x)): c(x) = a(x)b(x), x \in X\}$;

(3) $A \vee B = \{(x, c(x)): c(x) = a(x) \vee b(x), x \in X\}$;

(4) $A \wedge B = \{(x, c(x)): c(x) = a(x) \wedge b(x), x \in X\}$;

(5) $\frac{A}{B} = \{(x, c(x)): c(x) = \frac{a(x)}{b(x)}$, if $b(x) \neq 0$, otherwise $c(x) = 0; x \in X\}$;

(6) $A^B = \{(x, c(x)): c(x) = a(x)^{b(x)}$, if $a(x) > 0$, otherwise $c(x) = 0; x \in X\}$;

(7) $A - B = \{(x, c(x)): c(x) = a(x) - b(x), x \in X\}$.

When the operations of addition and multiplication are thus defined, the set of images $A, B, C, \ldots$ forms a ring (see Example 1 (2.4.1.1)).

—$r_1, r_2, \ldots \in \{+, \times, \vee, \wedge, -, \backslash, A^B]$, i.e., r_1, r_2 are the operations over two images;

—$r(A, B)$ is the resulting image after applying operation r to images A and B.

Statement 2.4.1.3.

Suppose that

—R is a field of real numbers;

—$r_1, r_2, \ldots \in \{+, \times, \vee, \wedge, -, \backslash, A^B\}$ are the elements of the set U, i.e., r_1 and r_2 are the operations over two images [14].

We introduce

—the operation of addition of two operations r_1 and r_2 as

$$(r \oplus r_2)(A, B) = r_1(A, B) + r_2(A, B);$$

—the operation of multiplication of two operations r_1 and r_2 as

$$(r \oplus r_2)(A, B) = r_1(r_2(A, B), r_2(A, B)); \text{ and}$$

—the operation of multiplication of operation r by the element of the field of real numbers $\alpha \in R$ as

$(\alpha r)(A, B) = \alpha r(A, B)$ (the right part denotes multiplication of the image by the element of the field).

Then, a set U with operations of addition, multiplication, and multiplication by the element of the field of real numbers introduced over it is *an algebra over the field of real numbers*.

Proof.

The proof is based on the validation of the algebra's properties (see Definition 2.2.5) of the set U with operations of addition, multiplication, and multiplication by the element of the field of real numbers introduced over it.

Properties of the algebra:

—R is a field of real numbers;

—Properties of a ring $U(I_1, I_2, I_3 \in U)$, where

$$r_1 \longrightarrow r_1(A, B),$$
$$r_2 \longrightarrow r_2(A, B),$$
$$r_3 \longrightarrow r_3(A, B),$$

are the following:

(1) $\forall r_1, r_2 \in U, \exists!(r_1 + r_2) \in U$

(a) $(r_1 \oplus (r_2 \oplus r_3))(A, B) = r_1(A, B) + (r_2(A, B) + r_3(A, B))$ = {which is the property of associativity of image addition (Statement 2.4.1.1)} = $(r_1(A, B) + r_2(A, B)) + r_3(A, B)$

$((r_1 \oplus r_2) \oplus r_3)(A, B) = (r_1(A, B) + r_2(A, B)) + r_3(A, B)$

$$(r_1 \oplus (r_2 \oplus r_3))(A, B) = ((r_1 \oplus r_2) \oplus r_3)(A, B);$$

(b) $((r_1 \oplus r_2)(A, B) = r_1(A, B) + r_2(A, B)$

$(r_2 \oplus r_1)(A, B) = r_2(A, B) + r_1(A, B)$ = {which is Property 1b of the ring (Statement 2.4.1.1)} = $r_1(A, B) + r_2(A, B)$

$$(r_1 \oplus r_2)(A, B) = (r_2 \oplus r_1)(A, B);$$

(c) $O(A, B)$ where I_0 is a zero element of the image ring {which is Property 1c of the ring (Statement 2.4.1.1)};

$(r_1 \oplus O)(A, B) = r_1(A, B) + O(A, B) = r_1(A, B)$

$$\exists O \in U, \forall r_1 \in U, (r_1 \oplus O)(A, B) = r_1(A, B)$$

(d) $(-r_1)(A, B) = -r_1(A, B)$ where $-r_1(A, B)$ is an inverse element of the ring element $r_1(A, B)$

$(r_1 \oplus (-r_1))(A, B) = r_1(A, B) + (-r_1(A, B))$ = {which is Property 1d of the ring (Statement 2.4.1.1)} = $O(A, B)$;

$\forall r_1 \in U\ \exists(-r_1), (r_1 \oplus (-r_1))(A, B) = O(A, B)$

(2) $\forall r_1, r_2 \in U, \exists!(r_1 \otimes r_2) \in U$

(a) $(r_1 \otimes (r_2 \otimes r_3))(A, B) = (r_1 \otimes r_2(r_3(A, B), r_3(A, B))))(A, B) = r_1(r_2(r_3(A, B), r_3(A, B)), r_2(r_3(A, B), r_3(A, B)))$

$((r_1 \otimes r_2) \otimes r_3)(A, B) = ((r_1(r_2(A, B), r_2(A, B))) \otimes r_3)(A, B) = r_1(r_2(r_3(A, B), r_3(A, B)), r_2(r_3(A, B), r_3(A, B)))$

$$(r_1 \otimes (r_2 \otimes r_3))(A, B) = ((r_1 \otimes r_2) \otimes r_3)(A, B)$$

(3) $\alpha, \beta \in R$

$((\alpha r_1 \oplus \beta r_2) \otimes r_3)(A, B) = ((\alpha r_1(A, B) + \beta r_2(A, B)) \otimes r_3)(A, B) = \alpha r_1(r_3(A, B), r_3(A, B)) + \beta r_2(r_3(A, B), r_3(A, B))$

$((\alpha r_1 \otimes r_3) \oplus (\beta r_2 \otimes r_3))(A, B)) = ((\alpha r_1(r_3(A, B), r_3(A, B))) \oplus ((\beta r_2(r_3(A, B), r_3(A, B)))(A, B) = \alpha r_1(r_3(A, B), r_3(A, B)) + \beta r_2(r_3(A, B), r_3(A, B))$

—Properties of a vector space ($\forall \alpha \in P, r \in U$: $\alpha r \in U$):

(1) $(\alpha(\beta r))(\mathrm{A, B}) = (\alpha(\beta r(\mathrm{A, B})))(A, B) = \alpha\beta r(A, B)$;

$((\alpha\beta)r)(A, B) = (\alpha\beta)r(A, B) = \alpha\beta r(A, B)$

$(\alpha(\beta r))(A, B) = ((\alpha\beta)r)(A, B)$

(2) $((\alpha + \beta)r)(A, B) = (\alpha + \beta)r(A, B) = \alpha r(A, B) + \beta r(A, B)$

$(\alpha r \oplus \beta r)(A, B) = \alpha r(A, B) + \beta r(A, B)$

$((\alpha + \beta)r)(A, B) = (\alpha r \oplus \beta r)(A, B)$;

(3) $(\alpha(r_1 \oplus r_2))(A, B) = \alpha(r_1(A, B) + r_2(A, B)) = \alpha r_1(A, B) + \alpha r_2(A, B)$

$(\alpha(r_1 \oplus r_2))(A, B) = (\alpha r_1 \oplus \alpha r_2)(A, B)$.

All properties of a ring, field, and vector space are valid. The set U with operations of addition, multiplication, and multiplication by the element of the field of real numbers introduced over it is the algebra over the field of real numbers. Q.E.D.

2.4.2. Examples of sets of operations that do not generate algebra. Here, like in Section 2.4.1, we consider different sets U with operations of addition, multiplication, and multiplication by the real number defined on them.

(1) Elements of the set U:

(1.4) Images defined on the set X having an arbitrary range of values F with dimensionality coincident with that of a set X, i.e., $X, F \subset R^n$;

(1.5) Images defined on any set X_i having an arbitrary range of values F_i with dimensionality coincident with that of a set X_i, i.e., $X_i, F_i \subset R^n$.

(2) Operations over set elements (2.4, 2.5): addition, multiplication, and multiplication by the field element.

(3) Physical meaning of operations:

(3.4) Addition (getting the summary brightness of two images), multiplication (global nonpointwise filter and defining of one image in the set defined by another one; if this operation is not defined (i.e., $F \not\subset X$) the second operand is considered as the resulting image), multiplication by the real number (proportional increase or decrease in image brightness);

(3.5) Addition (getting the summary brightness of two images at the intersection of the sets wherein the images are given; in the points of a set X wherein only one image is defined, this image is considered to be resulting); multiplication (global, nonpointwise filter; in the points of an image X wherein only one image is defined (the first or second operands), the image itself (the first or second operands, respectively) is considered to be the result of operation of multiplication); multiplication by the element of the field of real numbers, i.e., proportional increase or decrease in image brightness.

In what follows, we describe the examples of constructions generated by the sets U with described characteristics (in parentheses, we indicate the types of set elements, operations, and their physical interpretation, according to the above-mentioned list) which are not algebras.

2.4.2.1. Example 4 (1.4, 2.4, 3.4).

Let the family of functions $\{F_i\}_1^\infty$ be given.

—An operation of addition of two elements from F_i, F_j: $i, j = 1, 2, \ldots$, is introduced as $\forall a(x) \in F_i, b(x) \in F_j$: $\exists! a(x) + b(x) \in F_k$, $k = 1, 2, \ldots, F_k \subset R^n$ with the following properties:

($\forall a(x) \in F_i, b(x) \in F_j, c(x) \in F_y, k = 1, 2, \ldots, y = 1, 2, \ldots$);

(4.1) $a(x) + (b(x) + c(x)) = (a(x) + b(x)) + c(x)$;

(4.2) $a(x) + b(x) = b(x) + a(x)$;

(4.3) $\forall i$: $\forall a(x) \in F_i, \exists 0 \in F_i$: $a(x) + 0 = a(x)$;

(4.4) $\forall i$: $\forall a(x) \in F_i, \exists(-a(x)) \in F_i$: $a(x) + (-a(x)) = 0$.

In the simplest case (all $F_i \equiv F$), this operation can be introduced as a pointwise addition of two set elements.

—An operation of superposition of two elements from F_i, F_j: $i, j = 1,2,\ldots$, is introduced as $\forall a(x) \in F_i$, $b(x) \in F_j$, in the points where $b(x) \in X$: $\exists! a(b(x)) \in F_i$.

—An operation of multiplication by the elements of field R $\forall \alpha \in R$, $a(x) \in F$: $\exists! \alpha a(x) \in F$, is introduced with the following characteristics: ($\forall a(x) \in F_i, b(x) \in F_j, c(x) \in F_y, i, j, y = 1, 2, \ldots \forall \alpha, \beta \in R$):

(4.5) $(\alpha a(x) + \beta b(x))c(x) = \alpha a(x)c(x) + \beta b(x)c(x)$;

(4.6) $\alpha(\beta a(x)) = \alpha\beta a(x)$;

(4.7) $(\alpha + \beta)a(x) = \alpha a(x) + \beta a(x)$;

(4.8) $\alpha(a(x) + b(x)) = \alpha a(x) + \alpha b(x)$.

Statement 2.4.2.1.

Suppose that

—R is a field of real numbers;

—$I = \{(x, f(x)), x \in X, f(x) \in F\}$, ($X, F \subset R^n, n \in N$, $F \subset \{F_i\}_1^\infty$, F is a set of values of image I on the set X) are the elements of the set U; and

—$I_1 = \{(x, a(x)), x \in X, a(x) \in F_1\}$, $I_2 = \{(x, b(x)), x \in X, b(x) \in F_2\}$, ($X, F_1, F_2 \subset R^n, n \in N, F_1, F_2 \subset \{F_i\}_1^\infty$).

We introduce

—the operation of addition of two images I_1 and I_2 as

$$I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\};$$

—the operation of multiplication of two images I_1 and I_2 as

$$I_1 I_2 = \begin{cases} (x, a(b(x))), b(x) \in X \\ (x, b(x)), b(x) \notin X \end{cases} \text{; and}$$

—the operation of multiplication of image I by the element of the field of real numbers $\alpha \in R$ as

$$\alpha I = \{(x, \alpha f(x)), x \in X\}.$$

Then, the construction obtained is not an algebra but *an additive group*.

Proof.

The proof is based on the validation of the algebra's properties (see Definition 2.2.5) of the set U with operations of addition, multiplication, and multiplication by the element of the field of real numbers introduced over it and also on testing the properties of the group of sets U with the operation of addition (see Definition 2.2.1).

Properties of the algebra:

—R is a field of real numbers;

—The properties of a ring $U(I_1, I_2, I_3 \in U)$, where

$$I_1 = \{(x, a(x)), x \in X, a(x) \in F_1\},$$
$$I_2 = \{(x, b(x)), x \in X, b(x) \in F_2\},$$
$$I_3 = \{(x, c(x)), x \in X, c(x) \in F_3\},$$

are the following:

(1) $\forall I_1, I_2 \in U, \exists!(I_1 + I_2) \in U$

(a) $I_1 + (I_2 + I_3) = I_1 + \{(x, b(x) + c(x)), x \in X\} = \{(x, a(x) + (b(x) + c(x))), x \in X\}$

$(I_1 + I_2) + I_3 = \{(x, a(x) + b(x)), x \in X\} + I_3 = \{(x, (a(x) + b(x)) + c(x)), x \in X\}$ = {according to Property 4.1} $= \{(x, a(x) + (b(x) + c(x))), x \in X\}$

$$I_1 + (I_2 + I_3) = (I_1 + I_2) + I_3$$

(b) $I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\}$

$I_2 + I_1 = \{(x, b(x) + a(x)), x \in X\}$ = {according to Property 4.2} $= \{(x, a(x) + b(x)), x \in X\}$

$$I_1 + I_2 = I_2 + I_1$$

(c) $O = \{(x, 0), x \in X\}$

$I_1 + O = \{(x, a(x) + 0), x \in X\}$ = {according to Property 4.3} $= \{(x, a(x)), x \in X\} = I_1$

$$\exists O \in U, \forall I_1 \in U, I_1 + O = I_1$$

(d) $(-I_1) = \{(x, -a(x)), x \in X\}$

$I_1 + (-I_1) = \{(x, a(x) - a(x)), x \in X\}$ = {according to Property 4.4} $= \{(x, 0), x \in X\} = O$

$$\forall I_1 \in U, \exists(-I_1), I_1 + (-I_1) = 0$$

(2) $\forall I_1, I_2 \in U, \exists!(I_1 I_2) \in U$

(a) $$I_1(I_2I_3) = I_1\left(\begin{cases}(x, b(c(x))), c(x) \in X \\ (x, c(x)), c(x) \notin X\end{cases}\right) =$$

$$\begin{cases}\begin{cases}(x, a(b(c(x)))), b(c(x)) \in X \\ (x, b(c(x))), b(c(x)) \notin X\end{cases} \\ (x, c(x)), c(x) \notin X\end{cases} =$$

$$\begin{cases}(x, a(b(c(x)))), b(c(x)) \in X, c(x) \in X \\ (x, b(c(x))), b(c(x)) \notin X, c(x) \in X \\ (x, c(x)), c(x) \notin X\end{cases}$$

$$(I_1I_2)I_3 = \left(\begin{cases}(x, a(b(x))), b(x) \in X \\ (x, b(x)), b(x) \notin X\end{cases}\right)I_3 =$$

$$\begin{cases}\begin{cases}(x, a(b(c(x)))), b(c(x)) \in X, c(x) \in X \\ (x, b(c(x))), b(c(x)) \notin X, c(x) \in X \\ (x, c(x)), c(x) \notin X\end{cases} & b(x) \in X \\ \begin{cases}(x, b(c(x))), c(x) \in X \\ (x, c(x)), c(x) \notin X\end{cases} & b(x) \notin X\end{cases} =$$

$$\begin{cases}(x, a(b(c(x)))), b(x) \in X, b(c(x)) \in X, c(x) \in X \\ (x, b(c(x))), b(c(x)) \notin X, c(x) \in X, b(x) \in X, \\ \text{or } c(x) \in X, b(x) \notin X \\ (x, c(x)), c(x) \notin X.\end{cases}$$

Let us consider the point $x \in X$ such that $b(x) \notin X$, $b(c(x)) \in X$, $c(x) \in X$:

(1) $I_1(I_2I_3) = \{(x, a(b(c(x))))\}$,

(2) $(I_1I_2)I_3 = \{(x, b(c(x)))\}$,

where $I_1(I_2I_3) \neq (I_1I_2)I_3$.

Not every property of a ring is valid for elements of the set U. For the operation of addition, all properties of a group are valid. In view of the given definitions of the operations of addition and multiplication, the only group constructed without additional restrictions was the group over addition. Q.E.D.

2.4.2.2. Example 5 (1.5, 2.5, 3.5).

Let the families of functions $\{F_i\}_1^\infty$ and $\{X_i\}_1^\infty$ be given:

—An operation of addition of two elements from F_i, $F_j \subset R^n$: $i, j = 1, 2, \ldots$, is introduced as $\forall a(x) \in F_i$ for $x \in X$, $b(x) \in F_j$ for $x \in X_j$, for $x \in X_i \cap X_j$ $\exists! a(x) + b(x) \in F_k$, $k = 1, 2, \ldots, F_k \subset R^n$ with the following properties ($\forall a(x) \in F_i$, $b(x) \in F_j$, $c(x) \in F_y$, i, j, $y = 1, 2, \ldots$ in the general set of definitions X):

(5.1) $a(x) + (b(x) + c(x)) = (a(x) + b(x)) + c(x)$;

(5.2) $a(x) + b(x) = b(x) + a(x)$;

(5.3) $\forall i$: $\forall a(x) \in F_i$, $\exists 0 \in F_i$: $a(x) + 0 = a(x)$;

(5.4) $\forall i$: $\forall a(x) \in F_i$, $\exists(-a(x)) \in F_i$: $a(x) + (-a(x)) = 0$.

—An operation of multiplication of two elements from F_i, F_j: $i, j = 1, 2, \ldots$, is introduced as $\forall a(x) \in F_i$, for $x \in X_i$, $b(x) \in F_j$ for $x \in X_j$, for $x \in X_j$, for $x \in X_i \cap X_j$ $\exists! a(x) \times b(x) \in F_k$, $k = 1, 2, \ldots, F_k \subset R^n$ with the following properties ($\forall a(x) \in F_i$, $b(x) \in F_j$, $c(x) \in F_y$, i, j, $y = 1, 2, \ldots$ in the general set of definitions X):

(5.5) $a(x)(b(x)c(x)) = (a(x)b(x))c(x)$.

—An operation of multiplication by the element of the field of real numbers R is introduced over a set F: $\forall \alpha \in R$, $a(x) \in F$: $\exists! \alpha a(x) \in F$ with the following properties ($\forall a(x) \in F_i$, for $x \in X_i$, $b(x) \in F_j$ for $x \in X_j$, $c(x) \in F_y$ for $x \in X_y$, for $x \in X_i \cap X_j \cap X_y$ $i, j, y = 1, 2, \ldots$, $\forall \alpha, \beta \in R^n$):

(5.6) $(\alpha a(x) + \beta b(x))c(x) = \alpha a(x)c(x) + \beta b(x)c(x)$;

(5.7) $\alpha(\beta a(x) = \alpha\beta a(x)$;

(5.8) $(\alpha + \beta)a(x) = \alpha a(x) + \beta a(x)$;

(5.9) $\alpha(a(x) + b(x)) = \alpha a(x) + \alpha b(x)$.

Statement 2.4.2.5.

Suppose that

—R is a field of real numbers;

—$I = \{(x, f(x)), x$ Œ $X, f(x) \in F\}$ ($X, F \subset R^n$, $n \in N$, $F \subset \{F_i\}_1^\infty$, $X \subset \{X_i\}_1^\infty$, where F is a set of values taken by image I over set X) are the elements of the set U; and

—$I_1 = \{(x, a(x)), x \in X_1, a(x) \in F_1\}$, $I_2 = \{(x; b(x)), x \in X_2, b(x) \in F_2\}$, ($X_1, X_2, F_1, F_2 \in R^n$, $F_1, F_2 \subset \{F_i\}_1^\infty$, $X_1, X_2 \subset \{X_i\}_1^\infty$).

We introduce

—the operation of addition of two images I_1 and I_2 as

$$I_1 + I_2 = \begin{cases}(x, a(x) + b(x)), x \in X_1 \cap X_2 \\ (x, a(x)), x \in X_1 \backslash X_2 \\ (x, b(x)), x \in X_2 \backslash X_1\end{cases};$$

—the operation of multiplication of two images I_1 and I_2 as

$$I_1 I_2 = \begin{cases}(x, a(x) \times b(x)), x \in X_1 \cap X_2 \\ (x, a(x)), x \in X_1 \backslash X_2 \\ (x, b(x)), x \in X_2 \backslash X_1\end{cases}; \text{ and}$$

—the operation of multiplication of image I by the element of the field of real numbers $\alpha \in R$ as

$$\alpha I = \{(x, \alpha f(x)), x \in X\}.$$

Then, a set U with operations introduced over it *is neither an algebra nor a group for any of the thus-defined operations*.

Proof.

The proof is based on the validation of the algebra's properties (see Definition 2.2.5) of the set U with operations of addition, multiplication, and multiplication by the element of the field of real numbers introduced over it and also on testing the properties of the group of a set U with addition (see Definition 2.2.1).

Properties of the algebra:

—R is a field of real numbers;

—Properties of a ring $U(I_1, I_2, I_3 \in U)$, where

$$I_1 = \{(x, a(x)), x \in X, a(x) \in X\},$$
$$I_2 = \{(x, b(x)), x \in X, b(x) \in X\},$$
$$I_3 = \{(x, c(x)), x \in X, c(x) \in X\},$$

are the following:

(1) $\forall I_1, I_2 \in U, \exists!(I_1 + I_2) \in U$

(a) $I_1 + (I_2 + I_3) = I_1 +$

$$\begin{cases} (x, b(x) + c(x)), x \in X_2 \cap X_3 \\ (x, b(x)), x \in X_2 \backslash X_3 \\ (x, c(x)), x \in X_3 \backslash X_2 \end{cases} =$$

$$\begin{cases} (x, a(x) + b(x) + c(x)), x \in X_1 \cap X_2 \cap X_3 \\ (x, a(x)), x \in X_1 \backslash \{X_2 \cap X_3\} \\ (x, b(x) + c(x)), x \in \{X_2 \cap X_3\} \backslash X_1 \\ (x, a(x) + b(x)), x \in X_1 \backslash \{X_2 \backslash X_3\} \\ (x, a(x)), x \in X_1 \cap \{X_2 \backslash X_3\} \\ (x, b(x)), x \in \{X_2 \backslash X_3\} \backslash X_1 \\ (x, a(x) + c(x)), x \in X_1 \cap \{X_3 \backslash X_2\} \\ (x, a(x)), x \in X_1 \backslash \{X_3 \backslash X_2\} \\ (x, c(x)), x \in \{X_3 \backslash X_2\} \backslash X_1 \end{cases} =$$

$$\begin{cases} (x, a(x) + b(x) + c(x)), x \in X_1 \cap X_2 \cap X_3 \\ (x, a(x) + b(x)), x \in X_1 \cap \{X_2 \backslash X_3\} \\ (x, b(x) + c(x)), x \in \{X_2 \cap X_3\} \backslash X_1 \\ (x, a(x) + c(x)), x \in X_1 \cap \{X_3 \backslash X_2\} \\ (x, a(x)), x \in X_1 \\ (x, b(x)), x \in \{X_2 \backslash X_3\} \backslash X_1 \\ (x, c(x)), x \in \{X_3 \backslash X_2\} \backslash X_1 \end{cases}$$

$$(I_1 + I_2) + I_3 = \begin{cases} (x, a(x) + b(x)), x \in X_1 \cap X_2 \\ (x, a(x)), x \in X_1 \backslash X_2 \\ (x, b(x)), x \in X_2 \backslash X_1 \end{cases} + I_3 =$$

$$\begin{cases} (x, a(x) + b(x) + c(x)), x \in X_1 \cap X_2 \cap X_3 \\ (x, a(x) + b(x)), x \in \{X_1 \cap X_2\} \backslash X_3 \\ (x, c(x)), x \in X_3 \backslash \{X_1 \cap X_2\} \\ (x, a(x) + c(x)), x \in \{X_1 \backslash X_2\} \cap X_3 \\ (x, a(x)), x \in \{X_1 \backslash X_2\} \backslash X_3 \\ (x, c(x)), x \in X_3 \backslash \{X_1 \backslash X_2\} \\ (x, b(x) + c(x)), x \in \{X_2 \backslash X_1\} \cap X_3 \\ (x, b(x)), x \in \{X_2 \backslash X_1\} \backslash X_3 \\ (x, c(x)), x \in X_3 \backslash \{X_2 \backslash X_1\} \end{cases} =$$

$$\begin{cases} (x, a(x) + b(x) + c(x)), x \in X_1 \cap X_2 \cap X_3 \\ (x, a(x) + b(x)), x \in \{X_1 \cap X_2\} \backslash X_3 \\ (x, b(x) + c(x)), x \in \{X_2 \backslash X_1\} \cap X_3 \\ (x, a(x) + c(x)), x \in \{X_1 \backslash X_2\} \backslash X_3 \\ (x, a(x)), x \in \{X_1 \backslash X_2\} \backslash X_3 \\ (x, b(x)), x \in \{X_2 \backslash X_1\} \backslash X_3 \\ (x, c(x)), x \in X_3. \end{cases}$$

It is obvious that, for the identity

$$I_1 + (I_2 + I_3) = (I_1 + I_2) + I_3$$

to be fulfilled, the following condition is required:

$$X_1 \cap X_2 = X_2 \cap X_3 = X_3 \cap X_1 = \emptyset.$$

This contradicts the definition of the sum, since it is defined on the union of sets of summands and multipliers. Similarly, the multiplication operation contradicts the property of associativity. The set is not a group for any defined operation. Q.E.D.

3.APPLICATION OF ONE-RING DIA TO ELEMENTS OF VARIOUS NATURES

3.1. Definition of Image: Image Models

Definition 3.1.1. *Image I* is information in the form of a square numerical matrix written on some computer carrier and reproducing the properties of the imaged object (scene) and deformations caused by the method and process of image acquisition [13].

Definition 3.1.2. *Models* are different image representations. The models of the following four classes were introduced by I. Gurevich: (i) parametric model (P-model), (ii) procedural model (G-model), (iii) generative model (T-model), and (iv) image in its natural form (I-model).

Definition 3.1.3. The class of *I-models* consists of images proper; e.g., the digital image can be considered as a model image of an analog image, i.e., digital matrix-model of an image.

Definition 3.1.4. *P-model* is an object description by some numerical characteristics, e.g., image representation in the form of a numerical feature vector. (Numerical features are defined below, see Definition 3.2.1.2).

Definition 3.1.5. *G-model* is an abstract image representation, such as

—vector graphics (images are not stored pixel-by-pixel but are described by equations of curves);

—description of image by indicating the arrangement of basic primitives and relations between them;

—various types of coding by using an alphabet of code symbols (codebook), e.g., coding by neural nets; and

—image representation in the form of Boolean functions, particularly, in disjunctive normal form.

Definition 3.1.6. *T-model* represents images as a sequence of transformations of one or several initial images.

It should be noted that the original image could be transformed into a model of any type; however, the transition from one model to another is sometimes very difficult and yields ambiguous results. For instance, in order to restore a digital image from its P-model, the feature vector must have a sufficient number of coordinates. It is possible to completely restore an image if the number of coordinates coincides with the number of pixels; however, in this case, the P-model is not required.

3.2. DIA with a Ring of Numerical Features and Standard Algebraic Operations

3.2.1. Definitions.

Definition 3.2.1.1. A mathematical object used for producing formal image description is called *a feature*. Most often, the numerical features are used.

Definition 3.2.1.2. *A numerical image feature* is a variable representing numerical characteristics of the spatial image properties.

Definition 3.2.1.3. *A formal image description by features* is a set of mathematical objects (e.g., numbers, vectors, matrices) that contain some information on an image and are allowed by the recognition algorithm used for problem solving.

Definition 3.2.1.4. An *operation of calculation of the feature vector* is a mapping $f(I)$ of image I into a set of numerical features. (The mapping can be into the number, vector, or matrix, whose elements can be integer, real-number, and binary values.)

There are different classifications of image features. According to [9], all image features can be conditionally divided into three groups: statistical features, shape-based features, and spectral features.

Statistical features are widely used in different image recognition problems. They are calculated on the basis of the assumption that the analyzed image is a realization of some field of random numbers. The physical meaning of statistical features is the frequency of occurrence of different image subsets or the values of functions defined on these subsets. The most frequently used statistical features are histograms, cooccurence matrices, entropy-based features, and calculated fractal dimensions.

Shape-based features are the sets of primitives with corresponding relations and properties. The types of primitives depend on the class of images. For instance, in analysis and synthesis of artificial scenes, the primitives are standard geometrical figures, such as circles, angles, straight lines, etc. Usually, they are detected in images by matching the image with patterns. This group of features also contains topological features, different moments, coefficients of curve polynomial approximation with the given precision, etc.

When *spectral features* are calculated, the image is regarded as a quantized discrete signal. For image enhancing, e.g., for image filtering, the Fourier transform is often used. Also, this group of features contains an image spectrum, its elements, and different functions based on gradient calculation.

There are other classifications, and the most interesting among them are based on

(1) function (generative, parametric, symbols, specific objects, and procedures for feature extraction);

(2) informational nature (global and local);

(3) way of definition (computable, measurable, extractable, and simulated);

(4) mathematical apparatus (algebraic or structural, combinatorial, logical, arithmetical, statistic, spectral, topological or geometric, and matrix-type);

(5) type of images (binary, grayscale, and texture);

(6) nature of an object that supports feature extraction (contour, segment, skeleton, and point).

3.2.2. Conditions of generation of P-models of images.

Theorem 3.2.2.1. Let a set U consist of *operations for calculating numerical features, i.e., mappings of the form* $f_1(I)$, $f_2(I)$,…, *where* I *is an image*. Let also the *standard algebraic operations of addition, multiplication, and multiplication by the element* of some field P of elements of the set U be introduced, so that U and operations introduced on it are an algebra. Then, for a vector of numerical features to be the *P-model of image* I, it is necessary and sufficient that each element of the vector belong to algebra U.

Proof.

Remark 1. Numerical features can be represented as integer, real-valued, and binary vectors and matrices of various dimensions. According to the conditions of the theorem, the following operations are introduced on the elements of the set U: addition, multiplication, and multiplication by the element of some field. For correct introduction of these operations, one of the two following conditions should hold:

(a) Elements of the set U should be brought to uniformity. Introduction of operations of addition and multiplication of numerical features of different natures (e.g., binary feature of a triangle being present in the image or statistical feature of a cooccurrence matrix) has no physical meaning; therefore, this condition for correctness will not be used in the theorem.

(b) Elements of the set U must be of the same nature; i.e., they must belong to some subset of a set of operations for calculating numerical features. Therefore, next we consider different variants of the set U.

Sufficiency.

(1) Let

—the set U be all numerical features represented as real-valued vectors of dimension n. A set of vectors of integer values belongs to a set of vectors of real values, and, thus, is not considered separately;

—with no loss of generality, R be the field of real numbers; and

—for the elements of the set U ($f_1(I), f_2(I) \in U$), the following operations be introduced:

(a) addition of two operations of calculation of numerical features $f_1(I)$, $f_2(I)$:

$$f_1(I) + f_2(I) \in U$$

(addition is a standard algebraic operation of addition of two vectors having the same dimension; addition of two vector elements is a standard algebraic operation of addition of two real numbers);

(b) multiplication of two operations of calculation of numerical features $f_1(I)$, $f_2(I)$:

$$f_1(I)f_2(I) \in U$$

(multiplication is a standard algebraic operation of termwise multiplication of two vectors of the same dimension; multiplication of two vector elements is a standard algebraic operation of multiplication of two real numbers);

(c) operation of multiplication of an operation of numerical feature calculation $f(I)$ by the element of the field of real numbers $\alpha \in R$:

$$\alpha f(I) \in U$$

(the operation of multiplication by the element of the field corresponds to the standard algebraic operation of multiplication of the vector and scalar; multiplication of every vector element by the element of the real-valued field is a standard algebraic operation of multiplication of two real numbers).

The set U with the operations introduced over it is an algebra. By applying the introduced operations to the set U, we obtain all possible numerical features represented as real-valued vectors of one dimensionality. Each of the obtained features describes image I, i.e., is a part of a P-model of the image I.

(2) Let

—numerical features represented as binary vectors of dimension n be the elements of set U;

—Z^+ be the field of natural values and zero; and

—for the elements of the set U ($f_1(I), f_2(I) \in U$), the following operations be introduced:

(a) addition of two operations of calculation of numerical features $f_1(I)$, $f_2(I)$:

$$f_1(I) + f_2(I) \in U$$

(addition is a standard algebraic operation of addition of two vectors of the same dimension; addition of two vector elements is an operation over elements of a Boolean set, e.g., addition modulo two);

(b) multiplication of two operations of calculation of numerical features $f_1(I)$, $f_2(I)$:

$$f_1(I)f_2(I) \in U$$

(multiplication is a standard algebraic operation of termwise multiplication of two vectors of the same dimension; multiplication of two vector elements is an operation over elements of Boolean set, e.g., conjunction);

(c) operation of multiplication of an operation of calculation of numerical feature $f(I)$ by the element of the field $\alpha \in Z^+$:

$$\alpha f(I) \in U$$

(operation of multiplication by the element of the field corresponds to the standard algebraic operation of multiplication of the vector and scalar; multiplication of a vector element by the element of the field of nonnegative integers corresponds to, e.g., α-multiple addition of a vector element with itself modulo two).

The set U with the operations introduced over it is an algebra. By applying the introduced operations to the set U, we obtain all possible numerical features represented as binary vectors of the same dimension. Each of the obtained features describes image I, i.e., is a part of a P-model of image I.

(3) Let

—numerical features represented as square matrices of real-valued elements with dimensions $n \times n$ be the elements of set U;

—R be the field of natural values; and

—for the elements of the set U ($f_1(I)$, $f_2(I) \in U$), the following operations be introduced:

(a) addition of two operations of calculation of numerical features $f_1(I)$, $f_2(I)$:

$$f_1(I) + f_2(I) \in U$$

(addition is a standard algebraic operation of addition of two matrices of the same dimension; addition of two matrix elements is a standard algebraic operation of two real numbers);

(b) multiplication of two operations of calculation of numerical features $f_1(I)$, $f_2(I)$:

$$f_1(I)f_2(I) \in U$$

(multiplication is a standard algebraic operation of multiplication of two square matrices of the same dimension; multiplication of two matrix elements is a standard algebraic operation of multiplication of two real numbers);

(c) multiplication of an operation of calculation of numerical feature $f(I)$ by the element of the field $\alpha \in Z^+$:

$$\alpha f(I) \in U$$

(operation of multiplication by the element of the field corresponds to the standard algebraic operation of multiplication of the matrix and scalar; multiplication of a matrix element by the element of the field of real numbers corresponds to standard algebraic operation of multiplication of two real numbers).

The set U with the operations introduced over it is an algebra. By applying the introduced operations to the set U, we obtain all possible numerical features represented as square matrices of the same dimension. Each of the obtained numerical features describes the image I, i.e., is a part of a P-model of the image I.

(4) Let

—numerical features of the type not considered in (1)–(3) be the elements of set U;

—P be an arbitrary field; and

—for the elements of the set U ($f_1(I)$, $f_2(I) \in U$), the following operations be introduced:

(a) addition of two operations of calculation of numerical features $f_1(I)$, $f_2(I)$:

$$f_1(I) + f_2(I) \in U;$$

(b) multiplication of two operations of calculation of numerical features $f_1(I)$, $f_2(I)$:

$$f_1(I)f_2(I) \in U;$$

(c) multiplication of an operation of calculation of numerical feature $f(I)$ by the element of the field $\alpha \in P$:

$$\alpha f(I) \in U.$$

The set U with the operations introduced over it is an algebra. By applying the introduced operations to the set U, we obtain all possible numerical features represented as (4).

Necessity.

By definition, P-model is an image described by numerical characteristics. Since the considered type of DIA allows us to obtain any kind of numerical features for the given image, for the vector of numerical characteristics to be P-model, it is necessary that each of its elements belong to some algebra U constructed over the operations of calculation of numerical characteristics of the given image I. Q.E.D.

3.3. DIA with a Ring of Images and Standard Algebraic Operations

3.3.1. Definitions.

By Definition 3.1.1, image I is information presented in the form of a square numerical matrix written on some computer carrier and reproducing the properties of the imaged object (scene) and deformations caused by the method and process of image acquisition.

We distinguish two types of images:

(1) images in the form of a matrix whose rank is equal to the number of rows (columns) of the matrix and

(2) images in the form of a matrix whose rank is less than the number of rows (columns) of the matrix.

Definition 3.3.1.1. Images in the form of a matrix whose rank is less than the number of rows (columns) of the matrix are called *image fragments*.

3.3.2. Conditions of generation of type-1 images.

Theorem 3.3.2.1. Let the type-1 images be the elements of the set U. Let also the standard algebraic operations over elements of the set U be introduced so that the constructed set U along with operations introduced on it is an algebra. Then, the elements of algebra U are type-1 images and not type-2 images (i.e., image fragments; see Definition 3.3.1).

Proof.

By introducing standard algebraic operations over images in such a way that the set U and the operations introduced in it would be algebra, we again obtain images by definition.

Obviously, by applying standard algebraic operations to type-2 images, we can obtain both images of type 2 and type 1. Let us prove that, by using standard algebraic operations, it is impossible to obtain type-2 image (i.e., image fragments) from two type-1 images.

There are four standard algebraic operations: addition, multiplication, division, and subtraction.

(I) By definition of algebra, the operation of addition in the ring U should have the following properties:

Properties of the algebra:

Properties of the ring $U(a, b, c \in U)$:

(1) $\forall a, b \in U, \exists!(a + b) \in U$:

(a) $a + (b + c) = (a + b) + c$;

(b) $a + b = b + a$;

(c) $\exists 0 \in U, \forall a \in U, a + 0 = a$;

(d) $\forall a \in U, \exists(-a), a + (-a) = 0$.

These properties are satisfied by standard operations of addition and multiplication and are not satisfied by standard algebraic operations of subtraction and division, since they are not associative.

(II) By definition of algebra, the operation of multiplication in the ring U should have the following properties:

(2) $\forall a, b \in U, \exists!(ab) \in U$

(a) $a(bc) = (ab)c$;

(3) $(\alpha a + \beta b)c = \alpha ac + \beta bc$.

These properties are satisfied by standard operations of addition and multiplication and are not satisfied by standard algebraic operations of subtraction and division, since they are not associative.

Standard operations of addition and multiplication applied to type-1 images yield only type-1 images. Q.E.D.

3.4. DIA with a Ring of Operations over Images and Standard Algebraic Operations

Let us formulate and prove the theorem of generating G- and T-models of images and fragments of images.

Theorem 3.4.1. Let the following operations of image algebra be the elements of the set U:

(1) r_1, r_2 are unary operations,

(2) r_1, r_2 are binary operations

(such operations can be operations over images introduced by Ritter [14], standard operations over images like rotation, compression, superposition of two images, etc.).

Let also standard algebraic operations of addition, multiplication, and multiplication by the element of the field P be introduced so that the constructed set U along with the operations introduced on it is an algebra. To construct G- and T-models of images or images and fragments of images, it is sufficient to apply operations of algebra U to the elements of the algebra.

Proof.

To prove the theorem, it is required (I) to construct G- and T-models of image, images, or image fragments and (II) to prove that, by applying operations of this algebra to the elements of the algebra, it is impossible to obtain constructions that differ from G- and T-models of image, images, or image fragments.

Let the operations of addition, multiplication, and multiplication by the element of the field satisfying the properties of algebra be described so that $r_1 \oplus r_2 \in U$, $r_1 \otimes r_2 \in U$, $\alpha r \in U$ $(\alpha \in P)$.

(I) Construction of G- and T-models of image, images, or image fragments.

(A) Generation of the G-model.

(1) Generation of the image in the form of equations describing image-constituent curves.

Let operations $r_1, r_2, \dots$ be unary operations that calculate the presence of curves of types 1 and 2 in the image and obtain their equations in the presence of curves. Then, the elements of the constructed algebra U are all possible unary operations of obtaining equations describing the curves constituting image I; i.e., a type 1 G-model of image I is constructed.

(2) Image description by indicating the arrangement of the basic primitives and relations between them.

Let operations $r_1, r_2, \dots$ be unary operations of calculating the presence of the primitives of types 1 and 2 in the image and of determining the position of a primitive if it is present. Then, the elements of the constructed algebra U are all possible unary operations of obtaining the arrangement of various elements and relations between them in the image I; i.e., a type 2 G-model of image I is constructed.

(3) Various types of encoding with the alphabet of the code symbols.

Let operations $r_1, r_2, \dots$ be unary operations of obtaining the code from the image. Then, the elements of the constructed algebra U are all possible operations of obtaining the code from the image I; i.e., a type 3 G-model of image I is constructed.

(4) Image representation in the form of Boolean functions, particularly, in disjunctive normal form (DNF).

Let operations $r_1, r_2, \dots$ be unary operations of calculating the presence of the primitives of types 1 and 2 from the image. Then, the elements of the constructed algebra U are all possible combinations of the presence of the primitives in the image I. If we use disjunction and conjunction as such operations, it is easy to construct a DNF of the initial image, i.e., a type 4 G-model of image I.

Let operations $r_1, r_2, \dots$ be

(1) standard unary operations over images (rotation, scaling, noise addition, etc.);

(2) standard binary operations over images (superimposing, image subtraction, etc.).

(B) Obtaining the T-model.

By applying standard algebraic operations of addition, multiplication, and multiplication by the field element to the set U, we obtain various sequences of transformations of one (for unary operations) or several (for binary operations) image into another, i.e., a T-model (under the condition that these operations meet the conditions of an algebra).

(C) Obtaining images and image fragments.

By applying standard algebraic operations of addition, multiplication, and multiplication by the field element to the set U, we obtain various sequences of transformations of one (for unary operations) or several (for binary operations) images into another (under the condition that these operations meet the conditions of an algebra). If we apply the available operations that have physical meaning to the particular input images, we shall obtain images in their explicit form. We can obtain images of two types: (i) prime image and (ii) fragments of images.

As opposed to Theorem 2, we can obtain fragments of images if one or several operations r_1, r_2, … yield image fragments from the initial image (or images) (see Definition 3.3.1).

(II) Impossiblity to obtain constructions that differ from G- and T-models of images, images and fragments of images.

By definition of algebra, one cannot obtain objects which fall outside the limits of a set by using an operation of algebra. The operations themselves do not lead outside the image set. Therefore, by applying operations of algebra, we can obtain either a description of image generation, i.e., a G-model or T-model, or the image itself in the explicit form (images of types 1 and 2). Q.E.D.

3.5. DIA with a Ring of Standard Algebraic Operations with a Parameter and Operations of Image Algebra

3.5.1. Definitions.

Definition 3.5.1.1. The similarly described operations over images with parameters which change operation while changing themselves are *standard algebraic operations with a parameter.*

For instance, $r_i = (\alpha_i)$ is rotation by angle α_i or the more complex operation $r_j = (\alpha_j, x_j, t_j)$ is rotation by α_j, shift by x_j, and scaling by t_j.

3.5.2. Conditions of generation of G- and T-models of image, images, and fragments of images.

Theorem 3.5.2.1. Let algebraic operations with parameters r_1, r_2, … be the elements of set U and let operations of image algebra over elements of set U be introduced so that the constructed set U together with the operations introduced over it is an algebra. To construct G- and T-models of image, images, or image fragments, it is sufficient to apply operations of algebra U to the elements of the algebra.

Proof.

To prove the theorem, it is necessary (i) to construct G- and T-models of image, images, or image fragments and (ii) to prove that, by applying operations of this algebra to the algebra elements, it is impossible to obtain constructions other than G- and T-models of an image, images, or image fragments.

Let the operations of addition, multiplication, and multiplication by the field element be described such that they satisfy the algebra's properties: $r_1 \oplus r_2 \in U$, $r_1 \otimes r_2 \in U$, $\alpha r \in U$.

The possibility of constructing G- and T-models of an image, images, or image fragments is proved similarly to the previous theorem.

By definition of algebra, one cannot obtain objects which fall outside the limits of a set by using an operation of algebra. The operations with parameters themselves do not lead outside the image set. Therefore, by applying operations of algebra, we can obtain either description of image generation, i.e., a G-model or T-model, or the image itself in the explicit form (images of types 1 and 2). Q.E.D.

3.6. DIA with a Ring of Images and Image Representations and Operations of Image Algebra

Let us formulate and prove the following theorem of generation of the P-, G-, and T-models, images, and image fragments.

Theorem 3.6.1. Let the images and image representations (models) be the elements of the set U and the operations of image algebra be introduced such that the constructed set U together with operations introduced over it is an algebra. Then, P-, G-, and T-models, images, and image fragments are the elements of the given algebra.

Proof.

By definition of algebra, the operations of addition, multiplication, and multiplication by the field element do not lead outside the set U. {the P-, G-, and T-models, images and image fragments} $\in U$. Q.E.D.

3.7. Resulting Table

Theorem 3.7.1. By using different types of DIA, one can obtain any image model.

Corollary. By using two DIAs (of types 1 and 3 or of types 1 and 4), one can obtain the P-, G-, and T-models, images, and image fragments.

The considered algebras which lead to the construction of image models are tabulated in Table 1.

4. ONE-RING DIA: BASIC VARIANTS

Let us look more closely at Definition 2.2.7. Algebra over field A is called descriptive image algebra if the elements of its ring are either image models (including the image itself or the set of image values and characteristics), operations over images, or both simultaneously.

Below, the elements of the ring of DIA are listed.

(1) Operations of calculation of numerical features (Definition 3.2.1.4);

(2) Operations of image algebra (described by Ritter in [14]);

(3) Standard algebraic operations with parameters (Definition 3.5.1.1);

(4) Images (Definitions 2.3.1 and 3.1.1.);

Table 1. Generation of image models by DIA

Types of models	Ring elements	Ring operations	Result
1	Operations of calculation of numerical features	Standard algebraic operations	P-models
2	Images	Standard algebraic operations	Images
3	Operations of image algebra	Standard algebraic operations	G- and T-models, images, fragments
4	Standard algebraic operations with a parameter	Operations of image algebra	G- and T-models, images, fragments
5	Images and image representations	Operations of image algebra	P- G-, and T-models, images, fragments

(5) Image models (Definition 3.1.2).

Ring operations:

(1) Standard algebraic operations (+ and $\times$; division and subtraction are unsuitable, since they are not associative);

(2) Standard operations of image algebra; for example, operations over images introduced by Ritter. On the other hand, one can consider standard operations over images, such as rotation, compression, superimposing of two images, etc.

(3) Complex algebraic operations (combinations of algebraic operations, threshold functions, filters).

Let us consider DIA that can be constructed.

Let P be some field ($\forall\alpha, \beta, \gamma \in P, \exists!(a + b) \in P, \exists!(\alpha\beta) \in P$ satisfying the properties of the field).

(1) Elements of the ring are images I.

Operations introduced on the ring:

(A) Standard algebraic operations.

Let us consider two definitions of image:

—An image is information organized in the form of a numerical matrix (Definition 3.1.1). Standard operations of addition and multiplication of square matrices are used as ring operations. If P is a field of real numbers, then the multiplication by the field element corresponds to the multiplication of the matrix and scalar.

—Image $I = \{(x, a(x)), x \in X, a(x) \in F\}$ (Definition 2.3.1).The operations of addition and multiplication of two images are introduced in [14]. These operations are part of the operations of image algebra.

(B) Operations of image algebra. The most comprehensive description of all possible algebraic operations over images can be found in [14]. Here, DIA is standard image algebra (the operations introduced on the set of images must satisfy the properties of algebra).

(C) Complex algebraic operations.

Let us consider two definitions of an image:

—An image is information organized in the form of a numerical matrix (Definition 3.1.1). Standard operations over square matrices can be used as ring operations. If P is a field of real numbers, then multiplication by the field element corresponds to multiplication of the matrix and scalar.

—Image $I = \{(x, a(x)), x \in X, a(x) \in F\}$ (Definition 2.3.1).The operations of image algebra and their combinations, threshold functions, introduced in [14] can be used as ring operations. These operations are selected according to the definition of an algebra.

The way of testing using examples was shown above.

(2) Elements of the ring are image models I, P, G, and T.

According to the definition of an algebra, special operations for particular models are introduced on the elements of this type.

(3) Elements of the ring are operations of image algebra.

Operation introduced on the ring:

(A) Standard algebraic operations (+, $\times$);

(B) Operations of image algebra can be used only if after their application one also obtains images. Operations are selected according to the algebra definition.

(C) Complex algebraic operations. Operations are selected according to the algebra definition.

(4) Elements of the ring are standard algebraic operations with a parameter.

According to the definition of the standard algebraic operation (Definition 3.5.1.1), its application leads to the image; i.e., all operations for images are possible.

(5) Elements are the operations of calculation of numerical features, i.e., mapping $f(I)$ which can be a scalar, a vector, or a matrix.

Operations which can be introduced on the elements of this nature are the operations with numbers, vectors, and matrices; i.e., the following operations are possible: standard algebraic operations (+, $\times$) and complex algebraic operations under the condition that the properties of the algebra are satisfied.

Theorem 4.1. Let P be a field of arbitrary nature and U be a set of elements of the following natures: (i) operations of calculation of numerical features, (ii) operations of image algebra, (iii) standard algebraic operations with a parameter, (iv) images, and (v) image models. Let also Θ be a set of operations (addition, multiplication, and multiplication by the field element) out of the set of operations: (i) standard algebraic operations, (ii) operations of image algebra, and (iii) complex algebraic operations. Then, a mathematical construction (P, U, Θ) is one-ring DIA under the condition that all elements of U, along with the set of operations Θ introduced over it and the field P, satisfy the conditions of the algebra (Definition 2.2.5).

Theorem 4.2. Let F be a field. Let also images (in accordance with Ritter's definition) be the elements of the ring U; i.e., $I = \{(x, a(x)), x \in X, a(x) \in F\}$, where F is a set of values and X is a set of points. The operations $\oplus$, $\otimes$, and multiplication by the field element are introduced. These operations are from the set of standard image algebra operations. Then, the following conditions are necessary and sufficient for obtaining DIA:

(1) $I \in (R^X)$ or $I \in (R^n)X$, then

—operation $\oplus$ is addition of two images;

—operation $\otimes$ is multiplication of two images; and

—multiplication by the field element is a scalar multiplication of the element of the set F and the image.

(2) $I \in (2^F)^X$, then

—operation $\otimes$ is the operation of union and operation of intersection of two images; and

—multiplication by the field element is a scalar multiplication of the element of the set F and the image.

(3) $I \in (R^2)^X$, then, besides (1),

—operation $\oplus$ is addition of two images;

—operation $\otimes$ is the following operation:

Let γ_1 and γ_2 be binary operations $R^2 \times R^2 \longrightarrow R$ defined as

$$(x_1, x_2)\gamma_1(y_1, y_2) = x_1y_1 - x_2y_2,$$

$$(x_1, x_2)\gamma_2(y_1, y_2) = x_1y_2 + x_2y_1;$$

then, if $I_1, I_2 \in (R^2)X$ are two complex-valued images, the product $I_3 = I_1\gamma I_2$ is a complex product

$$I_3 = \{(x, c(x)), c(x) = (a_1(x)b_1(x) - a_2(x)b_2(x), a_1(x)b_2(x) + a_2(x)b_1(x)), x \in X\};$$

—multiplication by the field element is a scalar multiplication of the element of the set F and the image.

Remark. All operations described by Ritter [14, pp. 20, 22] were considered as operations of standard image algebra.

Proof.

The proof is based on the validation of the algebra's properties (see Definition 2.2.5) of operations of standard image algebra and their combinations. As an operation of multiplication by the element of the field, we consider two operations induced by operations in algebraic system F:

For $k \in F$ and $a \in F^X$,

$$k\gamma a = \{(x, c(x)): c(x) = k\gamma a(x), x \in X)\},$$

$$a\gamma k = \{(x, c(x)): c(x) = a(x)\gamma k, x \in X)\}.$$

Binary operations over images are also induced by operations introduced in algebraic system F. For instance, if γ is a binary operation over the set F and $a, b \in F^X$, then

$$a\gamma b = \{(x, c(x)): c(x) = a(x)\gamma b(x), x \in X)\}.$$

For this operation to be an addition operation in the ring U, it must satisfy Property 1 of the ring and operations of multiplication and multiplication by the element of the field must satisfy Property 3 of the ring and properties of the vector space.

Let us check the fulfilment of Properties 1 and 2 of the ring U for all operations described in [14]:

—Let $I_1, I_2 \in (R^X)$.

Replacing γ by the particular operations $+$, $\times$, $\vee$, (operation of maximization), and $\wedge$ (operation of minimization), we obtain for the real-valued images

$$I_1 + I_2 = \{(x, c(x)): c(x) = a(x) + b(x), x \in X)\}$$

(Operation of addition satisfies Properties 1 and 2 of the ring);

$$I_1I_2 = \{(x, c(x)): c(x) = a(x)b(x), x \in X)\}$$

(Operation of multiplication satisfies Properties 1 and 2 of the ring);

$$I_1 \vee I_2 = \{(x, c(x)): c(x) = a(x) \vee b(x), x \in X)\}$$

(Operation of maximization satisfies Property 2 and does not satisfy Properties 1c and 1d of the ring);

$$I_1 \wedge I_2 = \{(x, c(x)): c(x) = a(x) \wedge b(x), x \in X)\}$$

(Operation of minimization satisfies Property 2 and does not satisfy Properties 1c and 1d of the ring).

—Let $I_1, I_2 \in (2^F)^X$.

Let 2^X be a power set, i.e., a set of all subsets of the set X. Let also the image I be $I: X \longrightarrow 2^F$. In this case, the following binary operations are possible:

$$I_1 \cup I_2 = \{(x, c(x)): c(x) = a(x) \cup b(x), x \in X)\}$$

(Operation of union satisfies Property 2 and does not satisfy Properties 1c and 1d of the ring);

$$I_1 \cap I_2 = \{(x, c(x)): c(x) = a(x) \cap b(x), x \in X)\}$$

(Operation of intersection satisfies Property 2 and does not satisfy Properties 1c and 1d of the ring);

—Let $I_1, I_2 \in (R^{\geq 0})^X$,

$$I_1^{I_2} = \{(x, c(x)): c(x) = a(x)^{b(x)}, x \in X\}$$

(Operation of exponential does not satisfy Properties 1a and 2 of associativity).

—Let $I_1, I_2 \in (R^+)^X$

$$\log_{I_2} I_1 = \{(x, c(x)): c(x) = \log_{b(x)} a(x), x \in X\}$$

(Operation of taking the logarithm does not satisfy Properties 1a and 2 of associativity).

—Let $I_1 \in (F)^X$, $I_2 \in (F)^Y$, where X, Y are the subsets of topological space.

The expansion of image $a \in F^X$ by image $b \in F^Y$ on the set Y, where X and Y are the subsets of a topological space, is denoted by $a|^b$ and defined as

$$a|^b = \begin{cases} a(x) \text{ if } x \in X \\ b(x) \text{ if } x \in Y \backslash X \end{cases}.$$

The operations of concatenation of image series are also introduced as

$$a \in F^{Z_m x Z_k} \text{ and } b \in F^{Z_m x Z_n}:$$

$$(a|b) \equiv a|^{b+(0,k)}.$$

Similarly, a concatenation on the column is introduced by using the notion of matrix transposition:

$$\left(\frac{a}{b}\right) = (a|b)$$

(Operations of concatenation do not satisfy the associativity and commutativity properties);

—Let $I_1, I^2 \in (R^n)^X$

$$I_1 + I_2 = (I_1^1 + I_2^1, \ldots, I_1^n + I_2^n)$$

(Operation of addition satisfies Properties 1 and 2 of the ring);

$$I_1 I_2 = (I_1^1 I_2^1, \ldots, I_1^n I_2^n)$$

(Operation of multiplication satisfies Properties 1 and 2 of the ring);

$$I_1 \vee I_2 = (I_1^1 \vee I_2^1, \ldots, I_1^n \vee I_2^n)$$

(Operation of maximization satisfies Property 2 and does not satisfy Properties 1c and 1d of the ring).

$$I_1 \wedge I_2 = (I_1^1 \wedge I_2^1, \ldots, I_1^n \wedge I_2^n)$$

(Operation of minimization satisfies Property 2 and does not satisfy Properties 1c and 1d of the ring).

Let the binary operation γ be such that γ_j: $R^n \times R^n \longrightarrow R$, where $j = 1, \ldots, n$, defined as

$$I_1 \gamma I_2 = (I_1 \gamma_1 I_2, \ldots, I_1 \gamma_n I_2).$$

For instance, if γ_j: $R^n \times R^n \longrightarrow R$ are defined such that $(x_1, \ldots, x_n)\gamma(y_1, \ldots, y_n) = \max\{x_i \vee y_j\colon 1 \le i \le j\}$, then, for $I_1, I_2 \in (R^n)^X$ and $I_3 = I_1 \gamma I_2$, the components $c(x) = (c_1(x), \ldots, c_n(x))$ have the values

$c_j(x) = a(x)\gamma_j b(x) = \max\{a_i(x) \vee a_j(x)\colon 1 \le i \le j\}$, where $j = 1, \ldots, n$

(Operation does not satisfy the associativity Properties 1a and 2).

Table 2. Basic DIA variants

Ring elements	Ring operations
1. Operations of calculation of numerical features	1) Standard algebraic operations 2) Complex algebraic operations
2. Operations of image algebra	1) Standard algebraic operations 2) Operations of image algebra on the subset of operations 3) Complex algebraic operations
3. Standard algebraic operations with a parameter	1) Standard algebraic operations 2) Operations of image algebra 3) Complex algebraic operations
4. Images	1) Standard algebraic operations 2) Operations of image algebra 3) Complex algebraic operations
5. Images and image representations	Complex algebraic operations

Let us cite another example: suppose that γ_1 and γ_2 are binary operations $R^2 \times R^2 \longrightarrow R$ defined as

$$(x_1, x_2)\gamma_1(y_1, y_2) = x_1 y_1 - x_2 y_2,$$

$$(x_1, x_2)\gamma_2(y_1, y_2) = x_1 y_2 + x_2 y_1;$$

then, if $I_1, I_2 \in (R^2)^X$ are two complex-valued images, the product $I_3 = I_1 \gamma I_2$ is a complex product

$$c(x) = (a_1(x)b_1(x) - a_2(x)b_2(x), a_1(x)b_2(x) + a_2(x)b_1(x))$$

(Operation satisfies Properties 1 and 2 of the ring).

Let us consider other examples of operations ($I_1, I_2 \in (R^n)^X$):

$$I_1 \vee|_j I_2 = \{(x, c(x)\colon c(x) = a(x) \text{ if } a_j(x) \ge b_j(x) \text{ otherwise } c(x) = b(x)\}$$

(Operation of maximization satisfies Property 2 and does not satisfy Properties 1c and 1d of the ring);

$$I_1 \wedge|_j I_2 = \{(x, c(x)\colon c(x) = a(x) \text{ if } a_j(x \le b_j(x) \text{ otherwise } c(x) = b(x)\}$$

(Operation of maximization satisfies Property 2 and does not satisfy Properties 1c and 1d of the ring).

For better presentation of the operations that satisfy Properties 1 and 2, we tabulated them (see Table 3).

Analysis of different combinations of operations shows that only operations listed in the corollary correspond to Property 3 of the ring U and to properties of the vector space over field F. Q.E.D.

Conditions for properties of an algebra:

Let U be a given set and P be a given field.

(I) Θ is a set of operations (addition $\oplus$, multiplication $\otimes$, and multiplication by the element of a field) from the set of following operations:

Table 3. Implementation of the ring properties

	Operations	Ring property 1	Ring property 2
$I_1, I_2, \in (R^X)$	$I_1 + I_2 = \{(x, c(x)): c(x) = a(x) + b(x), x \in X\}$	+	+
	$I_1 \cdot I_2 = \{(x, c(x)): c(x) = a(x) \cdot b(x), x \in X\}$	+	+
	$I_1 \vee I_2 = \{(x, c(x)): c(x) = a(x) \vee b(x), x \in X\}$	–	+
	$I_1 \wedge I_2 = \{(x, c(x)): c(x) = a(x) \wedge b(x), x \in X\}$	–	+
$I_1, I_2, \in (2^F)^X$	$I_1 \cup I_2 = \{(x, c(x)): c(x) = a(x) \cup b(x), x \in X\}$	–	+
	$I_1 \cap I_2 = \{(x, c(x)): c(x) = a(x) \cap b(x), x \in X\}$	–	+
$I_1, I_2, \in (R^n)^X$	$I_1 + I_2 = (I_1^1 + I_2^1, \ldots, I_1^n + I_2^n)$	+	+
	$I_1 \cdot I_2 = (I_1^1 \cdot I_2^1, \ldots, I_1^n \circ I_2^n)$	+	+
	$I_1 \vee I_2 = (I_1^1 \vee I_2^1, \ldots, I_1^n \vee I_2^n)$	–	+
	$I_1 \wedge I_2 = (I_1^1 \wedge I_2^1, \ldots, I_1^n \wedge I_2^n)$	–	+
$I_1, I_2, \in (R^2)^X$	Complex product	+	+
$I_1, I_2, \in (R^n)^X$	$I_1 \vee\|_j I_2 = \{(x, c(x)): c(x) = a(x)$, if $a_j(x) \geq b_j(x)$, otherwise $c(x) = b(x)$	–	+
	$I_1 \wedge\|_j I_2 = \{(x, c(x)): c(x) = a(x)$, if $a_j(x) \leq b_j(x)$, otherwise $c(x) = b(x)$	–	+

(1) standard algebraic operations of addition, multiplication, and standard multiplication by the element of a field;

(2) operations of image algebra:

(a) $I \in (R^X)$ or $I \in (R^n)^X$, then

—operation $\oplus$ is an operation of addition of two images;

—operation $\otimes$ is an operation of multiplication of two images; and

—scalar multiplication of element from the set F and image is multiplication by the field element;

(b) $I \in (2^F)^X$, then

—operation $\otimes$ is an operation of union of two images and an operation of intersection of two images; and

—scalar multiplication of an element from the set F and an image is multiplication by the field element;

(c) $I \in (R^2)^X$, then, in addition to case (a)

—operation $\oplus$ is an operation of addition of two images;

—operation $\otimes$ is complex multiplication; and

—scalar multiplication of element from the set F and image is multiplication by the field element.

Remark. All operations described by Ritter [14, pp. 20, 22] were considered as operations of standard image algebra.

(3) Complex algebraic operations.

(II) The following properties are fulfilled:

—ring properties $U(a, b, c \in U)$

(1) $\forall a, b \in U, \exists!(a \oplus b) \in U$

(a) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;

(b) $a \oplus b = b \oplus a$;

(c) $\exists 0 \in U, \forall a \in U, a \oplus 0 = a$;

(d) $\forall a \in U, \exists(-a), a \oplus (-a) = 0$.

(2) $\forall a, b \in U, \exists!(a \otimes b) \in U$

(a) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$.

(3) $(\alpha a \oplus \beta b) \otimes c = \alpha a \otimes c \oplus \beta b \otimes c$.

—vector space properties $(\forall \alpha \in P, a \in U: \alpha a \in U)$

(1) $\alpha(\beta a) = (\alpha\beta)a$.

(2) $(\alpha + \beta)a = \alpha a \oplus \beta a$.

(3) $\alpha(a \oplus b) = \alpha a \oplus \alpha b$.

Corollary. Let P be a field of arbitrary nature and U be a set of the following elements: (i) operations of calculation of numerical features, (ii) image algebra operations, (iii) standard algebraic operations with parameter, (iv) images, and (v) image models. Θ is a set of operations (addition, multiplication, and multiplication by the element of the field P of the set U). Mathematical construction (P, U, Θ) is descriptive image algebra if and only if the set of operations Θ satisfies the conditions of fulfillment of the algebra's properties.

CONCLUSIONS

In the first part of this paper, examples of sets with different elements and operations introduced on them are considered. They can both belong and not belong to algebras. To construct examples, the standard image operations are used in order to verify that DIA covers mathematical constructions of standard IA that satisfies the algebra's properties. These examples are the first step in systematizing operations introduced on different sets both generating DIA and not.

An image has no natural way of being represented as a set of features used in classical image recognition problems. The possibility of formalizing different image representations (models) will help to extend the algebraic concept of recognition on images. The second part of the paper describes the ways of constructing P-, G-, T-, and I-models by the use of one-ring DIA of special type.

The method for formulating and verifying the conditions for a set of operations that ensure DIA construction is a basis for obtaining a mathematically valid criterion for choosing operations in order to generate efficient algorithms for image analysis and recognition. The third part of the paper presents necessary and sufficient conditions imposed on the set of operations that ensure one-ring DIA construction.

We plan to proceed with the study of the one-ring DIA and construct new examples of DIA by using image-processing operations both realizable and unrealizable physically. DIA with operations that have no physical interpretation are interesting because of the fine mathematical constructions they produce. We plan to study examples of DIA with image models as ring elements. Further research will be concentrated on studying DIA with several rings. The experimental part of our research concerns the construction, investigation, and implementation of algebraic schemes intended for simulated and applied problems of analysis and estimation of information represented by images.

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