Algorithms for Estimate Calculations Designed for the Case of 2D Support Sets. Part 1: Rectangular Support Sets[1]

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Abstract—Most of the advanced data processing and analysis technologies designed for solving domain-specific problems employ the automation and optimization techniques of decision-making based on "real" (incomplete, indirect, heterogeneous, inconsistent, erroneous, etc.) information. The methods of mathematical theory of pattern recognition play an important role here. To carry out image recognition, we need an image representation that corresponds to the requirements of the efficient recognition algorithm chosen for the task. A vast majority of the efficient image recognition algorithms only work with image descriptions or models. To completely use the information contained in images, it is necessary to overcome the principal discrepancy between the nature of images and the data-extraction techniques based on symbol models of images. Thus, there is a practical need for an efficient recognition algorithm that directly deals with images and their fragments. Moreover, the algorithm should provide the possibility of posing and solving the problem of choosing the best recognition algorithm. This class of algorithms—algorithms of estimate calculations based on 2D information (2D-AEC)—was defined by I. Gurevich as a special type of the classical model of the recognition algorithms based on estimate calculations (AEC) introduced by Yu. Zhuravlev. Generally, the AEC model can cope with the spatial (2D) image structure. The principal feature of the 2D-AEC is the use of the proximity of objects in spatial support sets, i.e., in images and their fragments. The range of the problems of 2D-AEC includes the enumeration and investigation of spatial support sets as well as definition of the subclasses of algorithms (corresponding to the types of the support sets) which allow one to produce efficient formulas that model the work of the algorithms. In this work, we find these formulas for the particular subclass of 2D-AEC—algorithms of estimate calculations with rectangular support sets.

INTRODUCTION

Most of the current data technologies for information processing and analysis designed to solve domain-specific content-driven problems employ the automation and optimization techniques of operative decision-making based on "real" (incomplete, indirect, heterogeneous, inconsistent, erroneous, etc.) information. These information technologies are widely used in medical and technical diagnoses, nondestructive testing, ecological monitoring, natural disaster and emergency forecasting (like technogenic catastrophes, earthquakes, floods, forest fires), information and copyright protection, security systems, scientific research automation, smart weapons, remote earth sensing, criminal law, and control. The methods of mathematical theory of pattern recognition are the basic tools for solving all of these problems. In most cases, when initial data are completely or partly represented as images, it is necessary to use the methods of analysis and estimation of information represented by images.

A vast majority of problems which arise during image analysis are, naturally, pattern recognition problems. At the same time, the problems of image recognition *per se* are formulated and solved much more seldom than is required by practical needs. The reasons are quite evident.

At the informational level, the main difficulties are connected to the following two problems:

(i) image description (modeling) and

(ii) development and optimization of the choice of mathematical methods for image transformations.

The solution to the image recognition problem implies that (i) there is an image representation and an efficient recognition algorithm and (ii) this representation corresponds to the requirements which the algorithm imposes on the initial data [4, 5]. Generally, in recognition problems, there are only two ways of data representation:

(i) as direct spatial information (e.g., by pixels or a local neighborhood of the second order that consists of pixel arrays) and

(ii) as a system of objects and relations extracted in images.

Before applying a recognition algorithm, it is necessary to present initial data in the form convenient for recognition.

In the first case, a recognition algorithm should allow the image itself or its fragments to be processed; here, the procedures of transforming the initial data in the form convenient for recognition are reduced to choosing the shape of the fragments whereby the rec-

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ognized object is matched to the template, to fragmenting the image, etc.

In the second case, the procedures of transforming the initial data in the form convenient for recognition should yield a mathematical model of an image. This model should reflect the inner structure and content of the image as an outcome of operations that construct the image from the subimages and other objects of simpler nature, i.e., from the primitives and objects extracted in the image at different stages of processing. During image recognition, one should use information reflecting the way of pattern formation, i.e., of the image as a whole, and of the objects presented in the image.

Three types of information characterize an image:

(i) identifiable objects with a well-defined structure;

(ii) identifiable objects with an ill-defined structure; and

(iii) nonidentifiable objects.

To allow for an image structure means to extract subimages (objects) in an image, to define the possible elementary level for them, and to define the relations between these objects and elements. As a result, the hierarchical structural information of an image may be explicitly presented and utilized. An image is described by a system of objects, each object is described by simpler objects, etc. The structural information can be introduced into recognition process in two ways.

First, according to classical pattern recognition theory, we can use the list of features as a main formalization principle and

(a) two types of features are introduced in the description; they reflect a two-dimensional character of the object to be recognized:

—characteristics which reflect the properties of some local image fragment (the distribution of pixel values in this fragment, the presence or absence of a certain geometrical object in this fragment, the type of the object's shape, etc.);

—characteristics of relations of separate objects and features;

(b) the weights are assigned to the features, which indicate the degree of their importance for image description;

(c) separate features are combined into a system of features and treated as a single feature.

The second way of introducing structural information into recognition process is based on a regularity—a property which is immanent to such information (and to the real world) and manifests itself in different orderings and structures. By using structural methods of recognition, we can obtain a practically unlimited diversity of descriptions from the limited set of primitives and rules of their combination by endlessly applying these rules to the initial primitives and to the results of some combinatorial transformations.

The overwhelming majority of the computationally efficient image recognition algorithms are designed for working with feature descriptions or image models only. To maximally use information contained in images, we should overcome the conflict between the image nature and the information-extraction techniques based on using symbolic models of images. Images are visual, informationally compact, and contain a vast amount of complementary and sometimes redundant information. Thus, the semantic nature of images allows us to use various context data (forbidden order relations, partial order relations, and other constraints common to the physical and logic structure of the real world) during recognition and analysis. At the same time, most of the image recognition techniques are purely heuristic, and their success is determined by their ability to reflect a pictorial character of images by nonpictorial means. As a result, image analysis and recognition rest on transformations which do not depend on presenting information in the form of images.

The above reasoning implies that there is a need for efficient recognition algorithms directly applied to images and their fragments. In addition, these algorithms should provide a possibility for choosing the extreme algorithm according to functional of recognition quality. This means that the algorithm is a model, i.e., a set of variables, objects, functions, parameters, and ranges of their variations. By fixing a set of certain variables, objects, parameters and function types, we can choose a particular algorithm in the model under consideration.

This class of the algorithms—algorithms based on estimate calculation by using two-dimensional information (2D-AEC)—was defined in [2–4] as a specification of the classical model of recognition algorithms based on estimate calculation (AEC), introduced by Zhuravlev [16]. A principle of partial precedence underlines AEC working. The proximity between the parts of the description of the objects already classified and the object presented for recognition is analyzed. The proximity is a partial precedent estimated according to some predefined rule (numerical estimation). A set of proximity measures yields a general estimate of an object to be recognized for a class. This estimate is a value of the function of a membership of the object to the corresponding class.

Experience shows that discriminating information is not contained in separate features, but in their combinations. The AEC class carries this idea to its logical conclusion. Since it is not always known which of the feature combinations is most informative, the proximity measure is calculated by matching all possible (or the particular, if the feature combinations of the maximum discriminating power are known) feature combinations in object description. In AEC, the proximity measures of objects are calculated by simple analytical formulas, which allow us to avoid exhaustive search during recognition (at the stage of tuning the algorithm's parameters while learning). In addition, the AEC makes it possible to take account of differences in the information content (discriminative power) among individual features and their combinations and in representativeness among some objects of learning sample.

Therefore, in general, the AEC model can process information related to a spatial (two-dimensional) image structure. The main distinction of the 2D-AEC class is the use of spatial (two-dimensional) support sets, i.e., images and their fragments, for calculating proximity measures of objects. The range of the problems of 2D-AEC includes enumeration and investigation of spatial support sets as well as the definition of the subclasses of algorithms (corresponding to the types of the support sets) which allow one to produce efficient formulas that model the work of the algorithms. One of these subclasses of 2D-AEC is described in [2–4] for a case of rectangular support sets. In this work, we find the formulas for the particular subclass of 2D-AEC—algorithms of estimate calculations with rectangular support sets.

Since a lot of research was made on the AEC class, we start with reviewing the main results of different researchers concerning different ways of specifying the systems of 1D support sets while constructing efficient formulas for estimate calculations.

In Section 1, the AEC model and its parameters are described and general definitions and examples are given.

In Section 2, the problem of efficient estimate calculations is formulated and the results obtained during its solution (for the AEC class) are given.

In Section 3, support sets for the considered subclass of 2D-AEC are introduced.

In Section 4, a method for efficient estimate calculations in this subclass with rectangular support sets is described.

In Section 5, the main results of the research are outlined. We prove that the problem thus formulated is equivalent to the problem of constructing an efficient procedure of searching for some spatial generating element on a binary raster. The problem of searching for a spatial generative element on a binary raster is formulated. We suggest a formalism that describes the multistep search procedures and introduce a natural criterion for efficiency (computational complexity) evaluation of these procedures. An efficient two-step search procedure is suggested for a generating element (a rectangle), and its optimality in a subclass of all two-step procedures of search for a rectangle is proven.

In Section 6, the results of the analysis of the information content of initial data by the introduced method are exemplified by the solution of the hemoblastoses classification problem.

1. GENERAL CHARACTERISTICS OF THE AEC CLASS

The model of algorithms based on calculation of the estimates (AEC) was successfully used for solving many problems of pattern recognition [16]. The model describes the structure of recognition algorithm and parameters necessary for choosing particular algorithm in the model. In the framework of a model, the algorithms differ by their parameters and, therefore, by the way of their classification of the given objects. The results of applying the algorithm of a model to the test sample show the adequacy of this algorithm to the problem at hand. Thus, all of the algorithms of a given model can be supplied with a quality functional.

The choice and/or synthesis of the algorithm, extreme according to the quality functional, presents the main problem in implementing AEC in practice. This problem is closely connected to the reduction of the computational complexity of AEC. The algorithms of acceptable computational complexity are based on efficient formulas which model the algorithm's performance; these are formulas for calculating proximity estimates of objects under recognition and precedents. The complexity of formulas for estimate calculations substantially depends on the AEC parameters, such as the system of support sets and the type of proximity function. A recognition algorithm uses a system of support sets as a system of feature subsets for matching object descriptions. A proximity function defines whether the matched objects are "close."

By now, the most comprehensive study concerned the problems of deriving efficient formulas for AEC in the case where all available objects are described by the one-dimensional feature vectors and the support sets encode the parts of these one-dimensional descriptions. This problem was solved for the main subclasses of AEC in [14–16].

When the efficient formulas for estimate calculations are constructed, the optimal (for the given model) algorithm can be chosen by one of the classical optimization techniques or by modifying these techniques. A lot of research was devoted to this problem [7, 10, 12, 14, 18–22, 24–26].

Although the efficient formulas are constructed almost for every AEC model of practical interest, the problem of constructing such formulas in the case where the objects are images, object descriptions are 2D matrices, and support sets are spatial (2D) objects still needs to be solved. As was noted above, the AEC class was specialized to operate with 2D object descriptions called a class of algorithms of estimate calculation from two-dimensional information (2D-AEC) [2–4]. Note that, by now, the efficient formulas are constructed for one 2D-AEC subclass only: for a subclass with a square as a generative element of the system of support sets [2–4].

Here, we propose the method for constructing the efficient algorithms for the 2D-AEC subclass with two-dimensional support sets generated by a rectangle. The idea underlying this procedure consists in transforming the rectangle with the sides $R_1 \times R_2$ into the unit square by compressing the plane along the one side R_1 times and along the other R_2 times.

Let us recall the basic objects and properties of the AEC model [11–13, 16]. Generally, a recognition algorithm contains a recognizing operator and a decision rule [13]. In AEC, a recognizing operator converts a standard description of object S, subject to recognition into a set of numerical estimates $(\Gamma_1(S), \Gamma_2(S), \ldots, \Gamma_l(S))$, where l is a number of classes. A decision rule helps us to construct the information vector $(\alpha_1, \alpha_2, \ldots, \alpha_l)$, $\alpha_j \in \{0, 1, \Delta\}$, from this set. Here, $\alpha_j = 0$ if an algorithm does not assign object S to jth class; $\alpha_j = 1$ if an algorithm assigns object S to jth class; and $\alpha_j = \Delta$ if an algorithm cannot classify object S.

To define a recognizing operator, it is necessary to assign a system of support sets, proximity function, feature weights, and precedent weights. Let us consider these parameters in detail.

1. System of support sets.

A system of support sets is a totality of nonempty subsets of the feature set $N = \{1, 2, \ldots, n\}$; the object is described by the values of these features. A system of support sets is denoted by Ω_A.

Below, we list the examples of support sets.

1.1. $\Omega_A = 2^N$; i.e., a system of support sets is a class of all (nonempty) subsets of feature set N.

1.2. $\Omega_A = \{\Omega | \Omega \subseteq N, |\Omega| = k\}$, where k is an integer and $1 \le k \le n$; i.e., a system of support sets consists of all of the subsets of the set N which have a predefined power k, e.g., $\Omega_A = \{\{1\}, \{2\}, \ldots, \{n\}\}$ for $k = 1$ and $\Omega_A = \{N\}$ for $k = n$.

The following relation connects the systems of support sets of 1.1 and 1.2:

$$2^N = \bigcup_{k=1}^{n} \{\Omega | \Omega \subseteq N, |\Omega| = k\}.$$

1.3. $\Omega_A = \{\Omega | \Omega \subseteq N, |\Omega| \le k\}$, where k is an integer and $1 \le k \le n$; i.e., Ω_A consists of all of the subsets of the set N which have a power no more than the predefined one.

1.4. $\Omega_A = \{\Omega | \Omega \subseteq N, |\Omega| \in \{k_1, k_2, \ldots, k_u\}\}$, where $k_1, k_2, \ldots, k_u$ are integers and $1 \le k_i \le n$, $i = \overline{1, u}$.

Any support set Ω can be encoded by the binary vector $\tilde{\omega}$ of the length n in the following way: the ith coordinate $\tilde{\omega}$ is equal to one if and only if the ith feature is contained in Ω. The thus-constructed vector $\tilde{\omega}$ is called a characteristic vector of the support set Ω. It is obvious that a set Ω and its characteristic vector $\tilde{\omega}$ are connected by a one-to-one correspondence.

Sometimes, it is convenient to consider a system of support sets as a set of characteristic vectors that encode the support sets of the algorithm. In those cases, we consider Ω_A to be a set of vertices $\{\tilde{\omega}\}$ of n-dimensional Boolean cube E^n.

As usual, we denote the norm (weight) of the binary vector $\|\tilde{\omega}\|$ equal to the number of its unitary coordinates by $\tilde{\omega}$. The set of all binary vectors of the weight k is called the kth layer of the Boolean cube and denoted by E_n^k.

We call the Boolean function $f_A(\tilde{\omega})$ a characteristic function of the system of support sets Ω_A if $f_A(\tilde{\omega}) = 1 \Leftrightarrow \tilde{\omega} \in \Omega_A$. Obviously, a system of support sets of the algorithm is unambiguously described by its characteristic function.

The characteristic function of a system of support sets of (1.1)-type vanishes only if all its variables vanish.

For a system of support sets of (1.2)-type, the characteristic function is equal to unity in the whole layer of Boolean cube and only there.

For a system of support sets of (1.3)-type, the characteristic function is equal to unity in the whole first, second, …, kth layers of Boolean cube and only there.

2. Proximity function.

Let $I(S) = (a_1, a_2, \ldots, a_n)$ be a standard (feature) description of object S, $\Omega = \{i_1, i_2, \ldots, i_k\}$, and let $\tilde{\omega}$ be a characteristic vector Ω. We denote a subdescription of the object S represented in the form $(a_{i_1}, a_{i_2}, \ldots, a_{i_k})$ by the symbols $\tilde{\omega} I(S)$ or $\tilde{\omega}(S)$.

The proximity function $B_{\tilde{\omega}}(S, S')$ depends on $\tilde{\omega}$-subdescriptions of objects S, S' and takes two values: 0 if the objects are not close and 1 otherwise.

Most often, the following proximity functions are considered:

2.1.

$$B_{\tilde{\omega}}(S, S') = \begin{cases} 1, & \tilde{\omega}S = \tilde{\omega}S' \\ 0, & \tilde{\omega}S \ne \tilde{\omega}S'. \end{cases} \tag{1.1}$$

2.2. Let the metric (or semimetric) $\rho_i(x, y)$, $i = \overline{1, n}$, be defined on the range of definition of the ith feature. Let $\tilde{\omega}S = (a_{i_1}, a_{i_2}, \ldots, a_{i_k})$, $\tilde{\omega}S' = (b_{i_1}, b_{i_2}, \ldots, b_{i_k})$, and the quantities $\varepsilon_i \ge 0$, $i = \overline{1, n}$, $\varepsilon \ge 0$, where ε is integer, be set. Consider a system of inequalities

$$\rho_{i_1}(a_{i_1}, b_{i_1}) \le \varepsilon_{i_1}, \quad \rho_{i_2}(a_{i_2}, b_{i_2}) \le \varepsilon_{i_2}, \ldots, \rho_{i_k}(a_{i_k}, b_{i_k}) \le \varepsilon_{i_k} \tag{1.2}$$

and denote the number of unsatisfied inequalities in this system by γ. Then,

$$B_{\tilde{\omega}}(S, S') = \begin{cases} 1, & \gamma \le \varepsilon \\ 0, & \gamma > \varepsilon. \end{cases} \quad (1.3)$$

The parameters of this function are the vector $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ and the quantity ε (maximally admissible number of unsatisfied inequalities in system (1.2)). If $\varepsilon_i = 0$, $i = \overline{1, n}$, and $\varepsilon = 0$, this proximity function is identically equal to the proximity function determined in 2.1.

2.3 If in the conditions of the previous point, we set two integers ε^1 and ε^2 ($\varepsilon^1, \varepsilon^2 \ge 0$) instead of ε, then

$$B_{\tilde{\omega}}(S, S') = \begin{cases} 1, & \|\tilde{\omega}\| - \gamma \ge \varepsilon^1, \ \gamma \le \varepsilon^2 \\ 0, & \text{otherwise.} \end{cases} \quad (1.4)$$

Vector $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ and quantities ε^1 and ε^2 (minimal accessible number of satisfied inequalities in system (1.2) and maximal accessible number of unsatisfied inequalities in system (1.2), respectively) are parameters of the function. For $\varepsilon^1 = 0$ and $\varepsilon^2 = \varepsilon$, we get the (2.2)-type proximity function.

Suppose once more that $I(S) = (a_1, a_2, \ldots, a_n)$, $I(S') = (b_1, b_2, \ldots, b_n)$, the metric (or semimetric) $\rho_i(x, y)$ is defined in the range of definition of the ith feature, and the values $\varepsilon_i \ge 0$, $i = \overline{1, n}$ are set. The binary vector $\tilde{\delta} = \tilde{\delta}(S, S') = (\delta_1, \delta_2, \ldots, \delta_n)$ defined as follows:

$$\delta_i = \begin{cases} 1, & \rho_i(a_i, b_i) \le \varepsilon_i \\ 0, & \rho_i(a_i, b_i) > \varepsilon_i, \end{cases} \quad (1.5)$$

where $i = \overline{1, n}$, is called a characteristic vector of the proximity of objects S and S'.

By using a characteristic vector of proximity, we can rewrite the expression for (2.2)-type proximity function as

$$B_{\tilde{\omega}}(S, S') = \begin{cases} 1, & (\tilde{\delta}', \tilde{\omega}) \le \varepsilon \\ 0, & (\tilde{\delta}', \tilde{\omega}) > \varepsilon, \end{cases} \quad (1.6)$$

where $\tilde{\delta}'$ is a binary vector obtained by the coordinatewise negation of vector $\tilde{\delta}$ and (α, β) is a scalar product of the vectors α and β which is equal to the sum of their coordinatewise multiplications.

In a similar way, for the (2.3)-type proximity function,

$$B_{\tilde{\omega}}(S, S') = \begin{cases} 1, & (\tilde{\delta}, \tilde{\omega}) \ge \varepsilon^1, \ (\tilde{\delta}', \tilde{\omega}) \le \varepsilon^2 \\ 0, & \text{otherwise.} \end{cases} \quad (1.7)$$

We can introduce vector $\tilde{\delta}(S, S')$ while ignoring the metric ρ_i and quantities ε_i in the following way:

$$\delta_i = \begin{cases} 1, & a_i = b_i \\ 0, & a_i \ne b_i. \end{cases} \quad (1.8)$$

In this case, the expression for the (2.1)-type proximity function can be rewritten in the following way:

$$B_{\tilde{\omega}}(S, S') = \begin{cases} 1, & (\tilde{\delta}, \tilde{\omega}) = \|\tilde{\omega}\| \\ 0, & (\tilde{\delta}, \tilde{\omega}) < \|\tilde{\omega}\|. \end{cases} \quad (1.9)$$

3. Feature weights.

Feature weights are set by the vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, $p_i > 0$, $i = \overline{1, n}$.

Let $\{i_1, i_2, \ldots, i_k\}$ be a set of the indices of all unit coordinates of the characteristic vector $\tilde{\omega}$. The weight of the support set Ω with a characteristic vector $\tilde{\omega}$ is denoted by $p(\tilde{\omega})$; $p(\tilde{\omega}) = p_{i_1} + p_{i_2} + \ldots + p_{i_k}$.

4. Precedent weights.

Precedent weights are defined by the vector $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m)$, where $\gamma_q = \gamma(S_q) > 0$, $q = \overline{1, m}$, and m is a total number of precedents. This point concludes the list of the parameters of the recognition operator of AEC.

The estimate $\Gamma_j(S)$ of the object S over the jth class is defined by the following formula:

$$\Gamma_j(S) = \frac{1}{K}\frac{1}{|W_j|}\sum_{S' \in W_j} \gamma(S') \sum_{\tilde{\omega} \in \Omega_A} p(\tilde{\omega}) B_{\tilde{\omega}}(S, S'), \quad j = \overline{1, l}, \quad (1.10)$$

where K is a normalized coefficient and W_j is a set of precedents of the jth class.

Sometimes we use the formulas to assess an estimate $\Gamma_j(S)$ that differ from Eq. (1.10). In any case, however, the semantics of the initial formulas for Γ_j is the same; i.e., over all of the support sets, the value of proximity function (and/or of its negation) is calculated for the given object S and for each object S' from the learning set. Each time, the weights of the features and the precedents are equally accounted of.

We finish the definition of recognition algorithm by setting the decision rule (see [13, 16]).

Note one important property of the estimate (1.10). The following equality is true:

$$\Gamma_j^S(\Omega_A^1 \cup \Omega_A^2) = \Gamma_j^S(\Omega_A^1) + \Gamma_j^S(\Omega_A^2) - \Gamma_j^S(\Omega_A^1 \cap \Omega_A^2),$$

where $\Gamma_j^S(\Omega_A^t)$ is an estimate $\Gamma_j(S)$, defined according the system of support sets Ω_A^t, $t = 1, 2$.

This equality admits the generalization for the case of a union of any finite number of the support set systems.

The estimate $\Gamma_j^S\left(\bigcup_{t=1}^{u}\Omega_A^t\right)$ over the union of mutually disjoint systems of support sets is merely the sum of estimates $\Gamma_j^S(\Omega_A^t)$ over all $t = \overline{1, u}$. Therefore, the estimate $\Gamma_j(S)$ is additive with respect to the union of disjoint systems of the support sets.

This property allows us to easily obtain the estimate $\Gamma_j^S\left(\bigcup_{t=1}^{u}\Omega_A^t\right)$ if the estimates $\Gamma_j^S(\Omega_A^t)$ ($\Omega_A^{t_1} \cap \Omega_A^{t_2} = \varnothing$, $t_1 \neq t_2$) are known. Suppose, for example, that Ω_{2^N} is a (1.1)-type system of support sets and Ω_k is (1.2)-type system of support sets with a given power k of N subsets. Then,

$$\Gamma_j^S(\Omega_{2^N}) = \sum_{k=1}^{n}\Gamma_j^S(\Omega_k).$$

Hereinafter, for the convenience, we omit the multiplier $\frac{1}{K}\frac{1}{|W_j|}$ in Eq. (1.10):

$$\Gamma_j(S) = \sum_{S' \in W_j}\gamma(S')\sum_{\tilde{\omega} \in \Omega_A} p(\tilde{\omega})B_{\tilde{\omega}}(S, S'), \qquad j = \overline{1, l}. \tag{1.11}$$

2. EFFICIENT FORMULAS FOR ESTIMATE CALCULATION IN A CLASS OF AEC: MAIN RESULTS

In some application problems, there are large quantities of precedents (hundreds and thousands) and support set systems that have a high power (e.g., $2^n - 1$, C_n^k for n of the order of a thousand and $2 \leq k \leq n$). Then, the estimate calculations according to Eq. (1.11) become time-consuming and, sometimes, impracticable ($|W_j|$ $|\Omega_A|$ summands are subject to calculation). The main difficulty here is the calculation of the sum

$$\Phi_{\Omega_A}(S, S') = \sum_{\tilde{\omega} \in \Omega_A} p(\tilde{\omega})B_{\tilde{\omega}}(S, S'). \tag{2.1}$$

This quantity depends on the choice of the support set system Ω_A and the proximity function $B_{\tilde{\omega}}(S, S')$.

Therefore, a problem arises of getting the efficient formulas for estimate calculation, i.e., the formulas that allow us to avoid the exhaustive search in Eq. (2.1) and, thus, to reduce combinatorial complexity of Eq. (1.11) to the complexity proportional to the size of a learning table (i.e., to the number of elements of the matrix

$\vdots$, where S_1, S_2, …, S_m are the objects of the learning sample).

2.1. Combinatorial Systems of Support Sets

The authors of [16] obtained efficient formulas for estimate calculations for (1.2)-type system of support sets and (2.2)-type proximity function as well as for some other cases.

Let us omit the multiplier $p(\tilde{\omega})$ in Eq. (1.11) (the feature weights are not considered) and set $k > \varepsilon$ (otherwise, $B_{\tilde{\omega}}(S, S') = 1$ for any S, S' and Eq. (1.11) is elementarily simplified).

Theorem 2.1[16].

$$\Gamma_j(S) = \sum_{S' \in W_j}\gamma(S')\sum_{t=0}^{\varepsilon} C_{r(S,S')}^{k-t}C_{n-r(S,S')}^{t}, \qquad j = \overline{1, l}, \tag{2.2}$$

where $r(S, S')$ is a number of satisfied inequalities in the system $\rho_i(x, y) \leq \varepsilon_i$, $i = \overline{1, n}$. Hereinafter, we additionally suppose that C_m^n for $n > m$.

To prove the theorem, we perform the direct calculation of the number of support sets where two arbitrary objects S and S' are close to each other for fixed k, ε, and known $r(S, S')$.

Note that for (1.2)-type system of support sets, the omitting of feature weights means that the weights of all the features should be the same. If we set feature weights to p, then a multiplier pk appears before the first summation sign in Eq. (2.2). The identity of weights of all the features does not substantially affect the recognition ability of AEC.

Corollary 2.1 [16]. For the same system of support sets and (2.1)-type proximity function, the following expression is valid:

$$\Gamma_j(S) = \sum_{S' \in W_j}\gamma(S')C_{r(S,S')}^{k}, \quad j = \overline{1, l}. \tag{2.3}$$

To prove the corollary, it is sufficient to put $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_n = 0$, $\varepsilon = 0$.

Corollary 2.2 [16]. For the (1.1)-type system of support sets and (2.1)-type proximity function,

$$\Gamma_j(S) = \sum_{S' \in W_j}\gamma(S')(2^{r(S,S')} - 1), \quad j = \overline{1, l}. \tag{2.4}$$

The proof is based on additivity of estimate $\Gamma_j(S)$ over the union of mutually disjoint systems of support sets (see example at the end of Section 1).

For the (1.1)-type system of support sets, (2.2)-type proximity function, and features with unity weights, the following assertion is valid.

Assertion 2.1 [16].

$$\Gamma_j(S) = \sum_{S' \in W_j} \gamma(S')2^{r(S, S') - 1} r(S, S'), \quad j = \overline{1, l}. \quad (2.5)$$

If instead of $p(\tilde{\omega})$, we place $a^{p(\tilde{\omega})}$ (a is positive constant) under the second summation sign of Eq. (1.3), other conditions being equal, we can make the second assertion:

Assertion 2.2 [16].

$$\Gamma_j(S) = \sum_{S' \in W_j} \gamma(S')((1 + a)^{r(S, S')} - 1), \quad j = \overline{1, l}. \quad (2.6)$$

The authors of [16] also obtained the efficient formulas for the case when the probability measures were used as feature estimates in object description (for (1.1)-type system of support sets and the proximity function of a special type). They considered a case with the gaps in the object's description denoting the absence of information about the values of some features. For the (1.1)- and (1.2)-type systems of support sets, the representations of proximity function were defined and efficient formulas were derived.

2.2. Intervals of the Boolean Cube

The efficient formulas of estimate calculations for the case when the system of support sets is an interval of the Boolean cube were obtained in [6]. They may afford a basis for efficient formulas for the systems of support sets representing the union of disjoint intervals. The authors also considered the possibility of deriving the efficient formulas for systems of support sets representing the union of intersecting intervals. Note that the efficient formulas of the subclass of 2D-AEC stem from this result. Here, a square image fragment was used as a generative element of a support set system [2–4].

Let a system of support sets in a Boolean cube be represented as an interval. Then, a characteristic function of a system of support sets can be written as

$$f_A(x_1, x_2, \ldots, x_n) = x_{i_1}^{\sigma_{i_1}} x_{i_2}^{\sigma_{i_2}} \ldots x_{i_k}^{\sigma_{i_k}},$$

$$\text{where } \sigma_{i_j} \in \{0, 1\}, \quad x^\sigma = \begin{cases} x, & \sigma = 1 \\ \bar{x}, & \sigma = 0. \end{cases} \quad (2.7)$$

Let $\{i_1, i_2, \ldots, i_k\} = \{\mu_1, \mu_2, \ldots, \mu_u\} \cup \{\nu_1, \nu_2, \ldots, \nu_v\}$, where $\sigma_{\mu_j} = 1, j = \overline{1, u}$ and $\sigma_{\nu_j} = 0, j = \overline{1, v}$. Using the introduced notations, we can describe the system of support sets in the following way: every support set of the system (i) does contain features with the numbers $\mu_1, \mu_2, \ldots, \mu_u$, (ii) does not contain features with the numbers $\nu_1, \nu_2, \ldots, \nu_v$, and (iii) probably contain some features with the numbers from the set $N \backslash \{i_1, i_2, \ldots, i_k\} = \{l_1, l_2, \ldots, l_{n-k}\}$.

Let a (2.2)-type proximity function be set with $\varepsilon = 0$, $\tilde{\omega} S = (a_{t_1}, a_{t_2}, \ldots, a_{t_q})$, and $\tilde{\omega} S' = (b_{t_1}, b_{t_2}, \ldots, b_{t_q})$. If the system of equations

$$\rho_{t_1}(a_{t_1}, b_{t_1}) \le \varepsilon_{t_1},$$
$$\rho_{t_2}(a_{t_2}, b_{t_2}) \le \varepsilon_{t_q}, \ldots, \rho_{t_q}(a_{t_q}, b_{t_q}) \le \varepsilon_{t_q}$$

contains at least one unsatisfied inequality with the number from the set $\{\mu_1, \mu_2, \ldots, \mu_u\}$, then it is obvious that $B_{\tilde{\omega}}(S, S') = 0$.

Definition 2.1 [6]. Object S' is called inefficient for S if $B_{\tilde{\omega}}(S, S') = 0$.

Theorem 2.2 [6].

$$\Gamma_j(S) = \sum_{S' \in (W_j \backslash W_j^S)} \gamma(S')2^{r(S, S')}, \quad j = \overline{1, l}, \quad (2.8)$$

where W_j^S is a set of all objects from W_j inefficient for S and $r(S, S')$ is a number of satisfied inequalities in a system $\rho_{l_i}(a_{l_i}, b_{l_i}) \le \varepsilon_{l_i}$, $i = \overline{1, n-k}$. In the model outlined here, we do not consider feature weights.

The theorem is proven by the direct calculation of the number of support sets which are used for determination of the proximity of two arbitrary objects S and S', the number $r(S, S')$ being known.

By using the property of additivity of the estimate $\Gamma_j(S)$ over the union of mutually disjoint systems of the support sets, Theorem 2.2 can be generalized to the case when a system of support sets is represented in a Boolean cube as the union of disjoint intervals.

A generalization of Theorem 2.2 implies that efficient formulas for estimate calculations may be constructed on the basis of DNF of the characteristic function of a system of support sets where all elementary conjunctions are mutually orthogonal. This DNF can be constructed for any Boolean function which is not identically equal to zero. If, however, the constructed DNF with mutually orthogonal conjunctions is too long (i.e., the intervals corresponding to elementary conjunctions contain one or several vertices only), then the complexities of corresponding formulas will be the same as the complexity of exhaustive search and, therefore, they will be inefficient. The formulas constructed according to this method essentially reduce the search only if they are constructed along the relatively short DNF with orthogonal conjunctions—their complexity is in direct proportion to the length of the DNF used. However, synthesis of the short DNF with orthogonal conjunctions obviously has a complexity similar to the complexity of the synthesis of the shortest DNF.

The author of [8] found another way of constructing efficient formulas for AEC with a system of the support sets presented as a union of the arbitrary intervals of a Boolean cube. He considered a proximity function of (2.2)-type with $\varepsilon = 0$. Unlike paper [6], the feature weights were taken into account there. The formulas for efficient calculation of Eq. (2.1) were obtained. The complexity of the obtained formulas and, therefore, their efficiency depend on the number of conjunctions in DNF which underlines the construction of these formulas.

Definition 2.2 [8]. Vector $\tilde{\omega}_I = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ defined by the rule

$$\gamma_i = \begin{cases} 0, & \gamma_i^0 = \gamma_i^1 = 0 \\ 1, & \gamma_i^0 = \gamma_i^1 = 1, \qquad i = \overline{1, n} \\ 2, & \gamma_i^0 = 0, \ \gamma_i^1 = 1, \end{cases} \tag{2.9}$$

is called a characteristic vector of the interval $I = \{\tilde{\gamma} \in E^n | \tilde{\gamma}^0 \le \tilde{\gamma} \le \tilde{\gamma}^1\}$, $\tilde{\gamma}^0 = (\gamma_1^0, \gamma_2^0, \ldots, \gamma_n^0) \in E^n$, $\tilde{\gamma}^1 = (\gamma_1^1, \gamma_2^1, \ldots, \gamma_n^1) \in E^n$ of the Boolean cube.

A number of twos in the characteristic vector is called a dimensionality of an interval.

It is obvious that the interval and its characteristic vector are connected by one-to-one correspondence.

The author considered the proximity function of (2.2)-type, $\varepsilon = 0$, objects S and S' and their characteristic proximity vector $\tilde{\delta}(S, S') = (\delta_1, \delta_2, \ldots, \delta_n)$.

Theorem 2.3 (about the reduction to the problem with the unity vector) [8]. Let the system of support sets Ω_A be the interval I in E^n. If $\tilde{\delta}(S, S') \ne (1, 1, \ldots, 1)$, then either $\Phi_I(S, S') = 0$ or we can discard some of the features in the description of the objects S and S' and proceed to the feature space of dimensionality $n^* < n$, where the system of support sets Ω_A is an interval I^* in E^{n^*}, and $\Phi_I(S, S') = \Phi_{I^*}(S, S') > 0$ and $\tilde{\delta}^*(S, S') = (1, 1, \ldots, 1)$.

The proof of the theorem is based on the fact that if $(\tilde{\delta})_i = 0$, then $\Phi_I(S, S') = 0$ for $(\tilde{\gamma})_i = 1$ (where $\tilde{\gamma}$ is a characteristic vector of the interval I); if $(\tilde{\gamma})_i = 0$, we can discard the ith component in the descriptions of objects S and S', in the Boolean representation of the system of support sets Ω_A, and in the vector ε, which is a parameter of a proximity function. Thus, the dimensionality of the initial classification problem and the dimensionality of the AEC parameters are decreased by one. The equality $\Phi_I(S, S') = \Phi_{I^*}(S, S')$ is still valid, but the value of $\Phi_{I^*}(S, S')$ is calculated for a new problem and with new parameters of AEC of lower dimensionality. The case of $(\tilde{\gamma})_i = 2$ is reduced to the considered cases.

A similar theorem is valid for the case of support set system being represented by a union of arbitrary intervals of a Boolean cube.

During the construction of efficient formulas for $\Phi_{\Omega_A}(S, S')$, Theorem 2.3 and its generalization make it possible to consider the case of $\tilde{\delta}(S, S') = (1, 1, \ldots, 1)$ only.

Theorem 2.4 [8]. Let the system of support sets Ω_A be the interval I in E^n with characteristic vector $\tilde{\omega}_I$ and dimensionality m. Then, the following equation is valid:

$$\Phi_I(S, S') = 2^m \sum_{i:\, \gamma_i = 1} p_i + 2^{m-1} \sum_{i:\, \gamma_i = 2} p_i \tag{2.10}$$

for $\tilde{\delta}(S, S') = (1, 1, \ldots, 1)$.

Here, two intervals I_1 and I_2 in E^n with characteristic vectors $\tilde{\omega}_{I_1} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, and $\tilde{\omega}_{I_2} = (\beta_1, \beta_2, \ldots, \beta_n)$, respectively, are considered.

Definition 2.3 [8]. We define operation of multiplication $\circ$ of two characteristic vectors as follows:

(1) the product $\tilde{\omega}_{I_1} \circ \tilde{\omega}_{I_2}$ is not defined if $\exists i \in N$: $\alpha_i = 0$ and $\beta_i = 1$ or $\alpha_i = 1$ and $\beta_i = 0$; otherwise

(2) $\tilde{\omega}_{I_1} \circ \tilde{\omega}_{I_2} = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, where

$$\gamma_i = \begin{cases} 0, & \alpha_i = \beta_i = 0, \text{ or } \alpha_i = 0 \\ & \beta_i = 2, \text{ or } \alpha_i = 2, \ \beta_i = 0 \\ 1, & \alpha_i = \beta_i = 1, \text{ or } \alpha_i = 1 \\ & \beta_i = 2, \text{ or } \alpha_i = 2, \ \beta_i = 1 \\ 2, & \alpha_i = 2, \ \beta_i = 2, \end{cases}$$

$$i = \overline{1, n}.$$

Obviously, the product $\tilde{\omega}_{I_1} \circ \tilde{\omega}_{I_2}$ is a characteristic vector of interval $I_1 \cap I_2$ (if such interval exists, i.e., if $I_1 \cap I_2 \ne \varnothing$). The introduced operation $\circ$ is commutative and associative.

Let the intervals $I_1, I_2, \ldots, I_k$ be given with characteristic vectors $\tilde{\omega}_{I_1}, \tilde{\omega}_{I_2}, \ldots, \tilde{\omega}_{I_k}$, respectively, and $I = I_1 \cap I_2 \cap \ldots \cap I_k \ne \varnothing$. Then, $\tilde{\omega}_{I_1} \circ \tilde{\omega}_{I_2} \circ \ldots \circ \tilde{\omega}_{I_k} = \tilde{\omega}_I$.

Theorem 2.5 [8]. Let the system of support sets Ω_A be the intersection of the intervals $I_1, I_2, \ldots, I_k$ in E^n with characteristic vectors $\tilde{\omega}_{I_1}, \tilde{\omega}_{I_2}, \ldots, \tilde{\omega}_{I_k}$, respectively, and $I = I_1 \cap I_2 \cap \ldots \cap I_k \ne \varnothing$ and $\tilde{\omega}_I = (\gamma_1, \gamma_2, \ldots, \gamma_n)$. Then, for $\tilde{\delta}(S, S') = (1, 1, \ldots, 1)$,

$$\Phi_{I_1 \cap I_2 \cap \ldots \cap I_k}(S, S') = 2^{m_{1,2,\ldots,k}} \sum_{i:\, \gamma_i = 1} p_i + 2^{m_{1,2,\ldots,k} - 1} \sum_{i:\, \gamma_i = 2} p_i, \tag{2.11}$$

where $m_{1, 2, \ldots, k}$ is the dimensionality of the interval $I_1 \cap I_2 \cap \ldots \cap I_k$.

The proof of this theorem follows from Theorem 2.4 and Definition 2.3.

The matrix of intervals $I_1, I_2, \ldots, I_k$ with characteristic vectors $\tilde{\omega}_{I_1}, \tilde{\omega}_{I_2}, \ldots, \tilde{\omega}_{I_k}$, respectively, is considered, where $\tilde{\omega}_{I_u} = (\alpha_1^u, \alpha_2^u, \ldots, \alpha_n^u)$, $u = \overline{1, k}$, i.e.,

$$A_{k \times n} = \begin{bmatrix} \alpha_1^1 & \ldots & \alpha_t^1 & \ldots & \alpha_n^1 \\ \alpha_1^2 & \ldots & \alpha_t^2 & \ldots & \alpha_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_1^k & \ldots & \alpha_t^k & \ldots & \alpha_n^k \end{bmatrix}. \tag{2.12}$$

The column vector of this matrix $\tilde{t} = (\alpha_t^1, \alpha_t^2, \ldots, \alpha_t^k)$, called the interval vector of feature t over intervals I_1, $I_2, \ldots, I_k$, corresponds to each feature $t \in N$.

Theorem 2.6 [8]. Let the system of support sets Ω_A be the union of the intervals $I_1, I_2, \ldots, I_k$ in E^n. Then, for $\tilde{\delta}(S, S') = (1, 1, \ldots, 1)$,

$$\Phi_{I_1 \cup I_2 \cup \ldots \cup I_k}(S, S') = \sum_{t=1}^{n} p_t \left[\sum_{j=1}^{k} (-1)^{j+1} \right. \tag{2.13}$$
$$\left. \times \left(\sum_{\substack{1 \le i_i < \ldots < i_j \le k \\ E(t) \cap \{i_1, \ldots, i_j\} \ne \varnothing}} f_{i_1 \ldots i_j} + \sum_{\substack{1 \le i_i < \ldots < i_j \le k \\ H(t) \cup \{i_1, \ldots, i_j\} = H(t)}} f'_{i_1 \ldots i_j} \right) \right],$$

where

$$E(t) = \{ i | \alpha_t^i = 1, 1 \le i \le k \},$$

$$H(t) = \{ i | \alpha_t^i = 2, 1 \le i \le k \},$$

$$f_{i_1 \ldots i_j} = \begin{cases} 0, & \exists t_0 \longrightarrow \tilde{t}_0 = (\alpha_{t_0}^1, \alpha_{t_0}^2, \ldots, \alpha_{t_0}^k)^T \\ & \exists k_1, k_2 \in \{1, \ldots, j\}: \alpha_{t_0}^{i_{k_1}} = 0, \ \alpha_{t_0}^{i_{k_2}} = 1 \\ 2^{m_{i_1 \ldots i_j}}, & \text{otherwise,} \end{cases}$$

$$f'_{i_1 \ldots i_j} = f_{i_i \ldots i_j}/2.$$

Suppose we know that any q intervals ($2 \le q \le k$) from the intervals $I_1, I_2, \ldots, I_k$ do not intersect. Then, $f_{i_1, i_2, \ldots, i_j} = f'_{i_1, i_2, \ldots, i_j} = 0$, $\forall j \ge q$ and Eq. (2.13) may be rewritten in the following form:

$$\Phi_{I_1 \cup I_2 \cup \ldots \cup I_k}(S, S') = \sum_{t=1}^{n} p_t \left[\sum_{j=1}^{q-1} (-1)^{j+1} \right. \tag{2.14}$$
$$\left. \times \left(\sum_{\substack{1 \le i_i < \ldots < i_j \le k \\ E(t) \cap \{i_1, \ldots, i_j\} \ne \varnothing}} f_{i_1 \ldots i_j} + \sum_{\substack{1 \le i_i < \ldots < i_j \le k \\ H(t) \cup \{i_1, \ldots, i_j\} = H(t)}} f'_{i_1 \ldots i_j} \right) \right].$$

The following recurrence relation may be used for adding a new interval to the system of the support sets in Eq. (2.13):

$$\Phi_{I_1} = 2^{m_1} \sum_{t:\, \alpha_t^1 = 1} p_t + 2^{m_1 - 1} \sum_{t:\, \alpha_t^1 = 2} p_t, \tag{2.15}$$
$$\text{where } k = 1,$$

$$\Phi_{I_1 \cup I_2 \cup \ldots \cup I_k} = \Phi_{I_1 \cup I_2 \cup \ldots \cup I_{k-1}} + \sum_{t=1}^{n} p_t \left[\sum_{j=1}^{k} (-1)^{j+1} \right. \tag{2.16}$$
$$\left. \times \left(\sum_{\substack{1 \le i_i < \ldots < i_j = k \\ E(t) \cap \{i_1, \ldots, i_j\} \ne \varnothing}} f_{i_1 \ldots i_j} + \sum_{\substack{1 \le i_i < \ldots < i_j = k \\ H(t) \cap \{i_1, \ldots, i_j\} = H}} f'_{i_1 \ldots i_j} \right) \right],$$

where $k \ge 2$.

We use the results outlined in [8] to propose a new scheme of efficient calculation of the estimate $\Gamma_j(S)$ in the case when a system of support sets is a union of the intervals of the Boolean cube. The current object S' from the learning sample is subjected to the procedure of reducing dimensionality of the initial problem and of AEC parameters (a generalization of the procedure described in the proof of Theorem 2.3). This procedure may yield $\Phi_{\Omega_A}(S, S') = 0$; otherwise, the dimensionality is successfully reduced; for a new problem an equality $\tilde{\delta}^*(S, S') = (1, 1, \ldots, 1)$ is valid and Eq. (2.13) may be used to calculate the values of $\Phi_{\Omega_A}(S, S')$.

The reasoning made after Theorem 2.2 is also actual for the computational complexity of Eq. (2.13) with the only correction: Eq. (2.13) is valid for the case of the union of arbitrary intervals (the conjunctions corresponding to the intervals are not necessarily be orthogonal).

2.3 Symmetrical Proximity Functions

A general approach to deriving efficient formulas of estimate calculations is based on the following representation of Eq. (1.3) [1, 13]:

$$\Gamma_j(S) = \sum_{S' \in W_j} \gamma(S') \sum_{t=1}^{n} p_t V_t(S, S'), \quad j = \overline{1, l}, \quad (2.17)$$

where $V_t(S, S')$ is the number of the support sets Ω from Ω_A containing feature $t \in N$ such that $B_{\tilde{\omega}}(S, S') = 1$. As a result, instead of efficient calculation of the value of Eq. (2.1), we should efficiently calculate the value of

$$\sum_{t=1}^{n} p_t V_t(S, S'). \quad (2.18)$$

Suppose that the function $V_t(S, S')$ takes k different values ($1 \le k \le n$) for varied parameter t and fixed objects S, S'. Then, feature set N is separated into k disjoint sets of features $N_1, N_2, \ldots, N_k$, such that $V_t(S, S') = V^u$ for $t \in N_u$, $1 \le u \le k$. To calculate the value of V^u, an arbitrary feature i_u is selected from the group of features N_u, then, searching in all support sets $\Omega \in \Omega_A$ ($i_u \in \Omega$) is performed and the value of proximity function $B_{\tilde{\omega}}(S, S')$ is calculated.

Thus, knowing that the function V_t takes k different values, to calculate the value of Eq. (2.18), we may confine ourselves to the support sets that contain at least one feature from the set $\{i_1, i_2, \ldots, i_k\}$. Therefore, when the number k of different values of the function V_t is smaller than n and, moreover, the number of the support sets that do not contain any feature from the set $\{i_1, i_2, \ldots, i_k\}$ is relatively large, then the search in the support sets for calculating the value of Eq. (2.18) can be sufficiently reduced.

Some types of the proximity functions and so-called regular systems of the support sets with a small number of different values of $V_t(S, S')$ were considered in [13].

Definition 2.4 [13]. A system of support sets Ω_A is considered to be regular if the conditions $\Omega \in \Omega_A$ and $|\Omega| = k$, $1 \le k \le n - 1$ imply that all support sets of power k belong to Ω_A.

In a Boolean cube, a regular system of support sets is represented as a union of some of its layers. All systems of support sets considered in Section 1 ((1.1)–(1.4)-types) are regular.

The value of function V_t depends on the type of proximity function.

Definition 2.5 [13]. Proximity function $B_{\tilde{\omega}}(S, S')$, which takes the values 0, 1, is called symmetrical if $\forall \tilde{\delta}$, $\forall \tilde{\omega}_1$, $\tilde{\omega}_2 \in \Omega_A$, the condition $\|(\tilde{\omega}_1 \cdot \tilde{\delta})\| = \|(\tilde{\omega}_2 \cdot \tilde{\delta})\|$ implies that $B_{\tilde{\omega}_1}(\tilde{\delta}) = B_{\tilde{\omega}_2}(\tilde{\delta})$. Here, $(\alpha \cdot \beta)$ is a vector obtained by coordinatewise product of vectors α and β.

Let us introduce two assumptions:

(1) $B_{\tilde{\omega}}(S, S') = B_{\tilde{\omega}}(\tilde{\delta}(S, S'))$; i.e., if the way of the construction of characteristic proximity vector $\tilde{\delta}$ is preassigned, then the proximity function does not depend on particular values of features in object descriptions, but on the type of the corresponding proximity vector and, formally, is a function of two n-dimensional binary vectors $\tilde{\omega}$ and $\tilde{\delta}$. This assumption concerning proximity function is rather natural—all proximity functions considered in Section 1 ((2.1)–(2.3)-types) obey it.

(2) There are such a learning sample and such an object S that function $\tilde{\delta}(S, S')$ takes all values from E^n (except for zero value) for varied S'.

Definition 2.5 implies that

(i) a symmetrical function only depends on the number of features used for establishing the subdescription closeness of two objects and is independent of, e.g., the indices of these features;

(ii) generally, a symmetry of a proximity function depends on the system of the support sets: thus, e.g., (2.3)-type proximity function is symmetrical if (1.2)-type of the support set system is chosen and is not symmetrical if we choose the support set system of (1.1)-type.

Theorem 2.7 [13]. Let the system of support sets be regular and the proximity function be symmetrical. Then, $V_t(S, S')$ takes no more than two different values for $t = \overline{1, n}$, $\forall S, S'$.

The proof of the theorem implies that all features that correspond to zero (unit) coordinates in the vector $\tilde{\delta}(S, S')$ enter into the equal number V^0 (V^1) of support sets $\Omega \in \Omega_A$ such that $B_{\tilde{\omega}}(S, S') = 1$. Then, according to the conditions of Theorem 2.7, Eq. (2.18) can be rewritten as

$$(\tilde{\delta}', p)V^0 + (\tilde{\delta}, p)V^1,$$

where $\tilde{\delta}'$ is a vector obtained by coordinate negation of $\tilde{\delta}$.

Theorem 2.7 is also valid for the particular case of (2.2)-type of the proximity function and (1.2)-type of the support sets.

Assertion 2.3 [13]. The following expressions are valid for the (1.2)-type system of support sets and the (2.2)-type proximity function:

$$V^0(S, S') = \sum_{u=1}^{\varepsilon} C^{u-1}_{n-r(S,S')-1} C^{k-u}_{r(S,S')}, \quad (2.19)$$

$$V^1(S, S') = \sum_{u=0}^{\varepsilon} C^{u}_{n-r(S,S')-1} C^{k-u-1}_{r(S,S')-1}, \quad (2.20)$$

where $r(S, S') = \|\tilde{\delta}(S, S')\|$.

Thus, the value V^1 is equal to the number of subsets N of power k containing the feature t, which corresponds to the unity coordinate of vector $\tilde{\delta}(S, S')$, and no more than ε features, which correspond to the zero coordinate of vector $\tilde{\delta}(S, S')$.

The notion of proximity function symmetrical over the partitioning is a generalization of the notion of a symmetrical proximity function.

Let a partitioning R of the feature set N into subsets $N_1, N_2, \ldots, N_v$, be set, i.e., $N = \bigcup_{i=1}^{v} N_i$.

Definition 2.6 [13]. The support sets Ω_1 and Ω_2 with the characteristic vectors $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are equivalent over the partitioning R (or R-equivalent) if $\|(\tilde{\omega}_1 \cdot \tilde{\omega}^i)\| = \|(\tilde{\omega}_2 \cdot \tilde{\omega}^i)\|$, $i = \overline{1, v}$, where $\tilde{\omega}^i$ is a characteristic vector of the subset N_i. We denote R-equivalency of the support sets Ω_1 and Ω_2 (characteristic vectors $\tilde{\omega}_1$ and $\tilde{\omega}_2$) by the following notation: $\Omega_1 \overset{R}{\sim} \Omega_2$ ($\tilde{\omega}_1 \overset{R}{\sim} \tilde{\omega}_2$).

Definition 2.7 [13]. The proximity function $\mathrm{B}_{\tilde{\omega}}(S, S')$ is called symmetrical over the partitioning R if $\forall \tilde{\delta}$, $\forall \tilde{\omega}_1, \tilde{\omega}_2 \in \Omega_A$, the condition $\tilde{\omega}_1 \overset{R}{\sim} \tilde{\omega}_2$ implies that $B_{\tilde{\omega}_1}(\tilde{\delta}) = B_{\tilde{\omega}_2}(\tilde{\delta})$.

It is evident that

(i) any proximity function is symmetrical over the partitioning R of the form $N = \{1\} \cup \{2\} \cup \ldots \cup \{n\}$, because for any $\tilde{\omega}_1$ and $\tilde{\omega}_2$ $\tilde{\omega}_1 \overset{R}{\sim} \tilde{\omega}_2 \Leftrightarrow \tilde{\omega}_1 = \tilde{\omega}_2$;

(ii) any symmetrical proximity function is symmetrical over the arbitrary partitioning R of the set N;

(iii) there are proximity functions symmetrical over some partitioning R which are not simply symmetrical (examples can be found in [6]).

Definition 2.8 [13]. Features t_1 and t_2 are called equivalent over the partitioning R of the set N if they simultaneously enter (or do not enter) into each subset N_i, $i = \overline{1, v}$.

Theorem 2.8 [13]. Let the system of support sets be regular and the proximity function be symmetrical over some partitioning R. Then, the function $V_t(S, S')$ assumes no more than $2r(R)$ different values for $t = \overline{1, n}$, $\forall S, S'$, $r(R)$ is a number of classes of R-equivalence, $1 \le r(R) \le n$.

Corollary 2.3 [13]. Suppose that in the conditions of Theorem 2.8, the partitioning R consists of v disjoint subsets. Then, $r(R) = v$ and, therefore, a number of different values of function $V_t(S, S')$ does not exceed $2v$.

Thus, the estimates of the number of different values $V_t(\tilde{\delta})$—the function of the argument t and parameter $\tilde{\delta}$—were obtained in [13] during the construction of the efficient formulas of estimate calculations on the base of Eq. (2.18) for the regular systems of support sets and proximity functions having a special property (symmetry, symmetry over the partitioning).

2.4. Ranks of the Systems of Support Sets

The change of the number of different values $V_t(\tilde{\delta})$ when different isometric substitutions and set-theoretic operations were applied to Ω_A was investigated in [1] for the arbitrary system of support sets Ω_A and (2.3)-type proximity function. In addition, two new characteristics were introduced which generalized the characteristic $V_t(\tilde{\delta})$. The change in these characteristics during isometric substitution and set-theoretic operations were also studied.

It is obvious that the (2.3)-type proximity function $B_{\tilde{\omega}}(S, S')$ can be considered as function $B_{\tilde{\omega}}(\tilde{\delta}, \varepsilon^1, \varepsilon^2)$ and, therefore, $V_t(\tilde{\delta}) = V_t(\tilde{\delta}, \varepsilon^1, \varepsilon^2)$, i.e.,

$$B_{\tilde{\omega}}(\delta, \varepsilon^1, \varepsilon^2) = \begin{cases} 1, & |\Delta| - \|\delta^1 + \omega^1\| \ge \varepsilon^1, \ \|\delta^2 + \omega^2\| \le \varepsilon^2 \\ 0, & \text{otherwise,} \end{cases} \tag{2.21}$$

where $\Delta = \{i_1, i_2, \ldots, i_u\}$ is a set of the indices of all unit coordinates of vector $\tilde{\delta}$, $N \backslash \Delta = \{i_{u+1}, i_{u+2}, \ldots, i_n\}$ and, therefore, every vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ can be expressed as $\alpha = (\alpha^1, \alpha^2)$, where $\alpha^1 = (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_u})$ and $\alpha^2 = (\alpha_{i_{u+1}}, \alpha_{i_{u+2}}, \ldots, \alpha_{i_n})$.

Definition 2.9 [1]. A system of support sets Ω_A has Δ-rank k if $\forall \tilde{\delta}, \varepsilon^1, \varepsilon^2$, the number of different values of the function $V_t(\tilde{\delta}, \varepsilon^1, \varepsilon^2)$ (of argument t) does not exceed k and the number of different values of this function is k for some $\tilde{\delta}$, ε_1, and ε_2. We denote Δ-rank of the system of support sets Ω_A as $R_\Delta(\Omega_A)$.

If in Eq. (2.21), Δ is an arbitrary subset of N, then $B_{\tilde{\omega}}(\tilde{\delta}, \varepsilon^1, \varepsilon^2) = B_{\tilde{\omega}}(\tilde{\delta}, \Delta, \varepsilon^1, \varepsilon^2)$ and, therefore, $V_t(\tilde{\delta}) = V_t(\tilde{\delta}, \Delta, \varepsilon^1, \varepsilon^2)$. In addition, Eq. (2.21) loses its previous content (see definition of (2.3)-type proximity function).

Definition 2.10 [1]. A system of support sets Ω_A has the rank k if $\forall \tilde{\delta}, \Delta, \varepsilon^1, \varepsilon^2$, a number of different values of the function $V_t(\tilde{\delta}, \Delta, \varepsilon^1, \varepsilon^2)$ does not exceed k and the number of different values of this function is k for some $\tilde{\delta}$, Δ, ε^1, and ε^2.

The rank of a system of support sets Ω_A is denoted as $\mathrm{R}(\Omega_A)$. It is obvious that $1 \le \mathrm{R}_\Delta(\Omega_A) \le \mathrm{R}(\Omega_A)$.

Definition 2.11 [1]. A closure of the set $\{\sigma_{ij}\}_{i,j=1}^{n} \cup \{\pi_i\}_{i=1}^{n}$ relative to the operation of composition of mappings is called a group Π of isometric permutations. Here, $\sigma_{ij}: E^n \longrightarrow E^n$ is the permutation of the ith and jth coordinates of the Boolean cube's vectors ($i, j \in N$) and $\pi_i: E^n \longrightarrow E^n$ is a replacement of the ith coordinate of the Boolean cube's vectors by its negation, $i \in N$.

This group exhausts all permutations φ in E^n such that $\|\alpha + \beta\| = \|\varphi(\alpha) + \varphi(\beta)\|$.

Definition 2.12 [1]. A set $G(\Omega_A) = \{\varphi | \varphi \in \Pi, \varphi(\tilde{\omega}) \in \Omega_A, \forall \tilde{\omega} \in \Omega_A\}$ with an operation of composition of mappings is called a group of symmetry of the set Ω_A.

$G(\Omega_A)$ is a maximum subgroup of the group Π which leaves the set Ω_A static (i.e., converts Ω_A into itself).

Let us consider a group $S_{N_1} \times S_{N_2} \times \ldots \times S_{N_k}$ induced by the partitioning R of the length k, where R is a partitioning of the set N into disjoint subsets $N_1, N_2, \ldots, N_k$: $N = \bigcup_{t=1}^{k} N_t$, $N_i \cap N_j = \varnothing$, $i \neq j$, and S_{N_t} is a subgroup in Π generated by the closure of the set $\{\sigma_{ij}\}_{i,j \in N_t}^{n}$ relatively to the composition of the mappings.

Definition 2.13 [1]. A system of support sets Ω_A has a symmetrical rank k if a group of symmetries $G(\Omega_A)$ has a subgroup induced by some partitioning R of the length k and does not contain a subgroup induced by any partitioning R' of the length smaller than k. A symmetrical rank of the system of support sets Ω_A is denoted as $\mathrm{SR}(\Omega_A)$.

Definition 2.13 implies that

(i) for any system of support sets Ω_A, $1 \le \mathrm{SR}(\Omega_A) \le n$, because $G(\Omega_A)$ always has a unit subgroup induced by the partitioning of a set N of the form $\{1\} \cup \{2\} \cup \ldots \cup \{n\}$; and

(ii) the smaller the rank of symmetry of the system of support sets Ω_A, the greater the symmetry of the representation of Ω_A in a Boolean cube.

The following theorem proves a connection between a rank and a symmetry rank of the system of support sets.

Theorem 2.9 [1]. For any system of support sets Ω_A, the following inequality is valid: $\mathrm{R}(\Omega_A) \le 4\mathrm{SR}(\Omega_A)$.

The proof of the theorem is based on the observation that function $V_t(\tilde{\delta}, \Delta, \varepsilon^1, \varepsilon^2)$ can possess no more than four different values at each block N_i from some partitioning of the length $\mathrm{SR}(\Omega_A)$.

Corollary 2.4 [1]. For any system of support sets Ω_A, the following inequality is valid: $\mathrm{R}_\Delta(\Omega_A) \le 2\mathrm{SR}(\Omega_A)$.

This corollary directly follows from the proof of Theorem 2.9 because function $V_t(\tilde{\delta}, \varepsilon^1, \varepsilon^2)$ can possess no more than two different values at each block N_i from some partitioning of the length $\mathrm{SR}(\Omega_A)$.

Theorem 2.10 [1]. Let the group $S_{N_1} \times S_{N_2} \times \ldots \times S_{N_k}$ be induced by the partitioning $N_1, N_2, \ldots, N_k$ of the length k. Then, there is a system of support sets Ω_A such that $\mathrm{SR}(\Omega_A) = k$ and $S_{N_1} \times S_{N_2} \times \ldots \times S_{N_k} \subseteq G(\Omega_A)$.

The proof of the theorem is constructive. The desired system of support sets may have the following form: $\Omega_A = \{\tilde{\omega}^1, \tilde{\omega}^2, \ldots, \tilde{\omega}^k\}$, where $\tilde{\omega}^j = (\omega_1^j, \omega_2^j, \ldots, \omega_n^j)$ and $\omega_i^j = 1 \Leftrightarrow N_1 \cup N_2 \cup \ldots \cup N_j$, $i = \overline{1, n}$, $j = \overline{1, k}$.

The following theorem defines a change in the rank of a system of support sets during its transformation by isometric permutations.

Theorem 2.11 [1]. Let $\varphi \in \Pi$. Then, if $\varphi \in S_{\{1, 2, \ldots, n\}}$, then $\mathrm{R}_\Delta(\Omega_A) = \mathrm{R}_\Delta(\varphi(\Omega_A))$ and $\mathrm{R}(\Omega_A) = \mathrm{R}(\varphi(\Omega_A))$. If $\varphi \notin S_{\{1, 2, \ldots, n\}}$, then $0.5 \le \mathrm{R}(\Omega_A)/\mathrm{R}(\varphi(\Omega_A)) \le 2$.

Corollary 2.5 [1]. Let the group $S = S_{N_1} \times S_{N_2} \times \ldots \times S_{N_k}$ be induced by the partitioning of the length k, and let there be $\varphi \in \Pi$ such that $\varphi S \varphi^{-1} \subseteq G(\Omega_A)$. Then, if $\varphi \in S_{\{1, 2, \ldots, n\}}$, then $\mathrm{R}_\Delta(\Omega_A) \le 2k$ and $\mathrm{R}(\Omega_A) \le 4k$. If $\varphi \notin S_{\{1, 2, \ldots, n\}}$, then $\mathrm{R}(\Omega_A) \le 8k$.

Theorem 2.12 [1]. Let the group $S = S_{N_1} \times S_{N_2} \times \ldots \times S_{N_k}$ be induced by the partitioning of the length k, and let there be $\varphi \in \Pi$ such that $\varphi = \pi\sigma$, where $\sigma \in S$ and π only specifies negations. Then, if $\varphi S \varphi^{-1} \subseteq G(\Omega_A)$, then $\mathrm{SR}(\Omega_A) \le 2k$.

Theorem 2.13 [1]. For the arbitrary systems of support sets Ω_A^1 and Ω_A^2, the following relations are true:

$$(\mathrm{SR}(\Omega_A^1 \cup \Omega_A^2), \mathrm{SR}(\Omega_A^1 \cap \Omega_A^2)) \le \mathrm{SR}(\Omega_A^1)\mathrm{SR}(\Omega_A^2),$$

$$\mathrm{SR}(\Omega_A^1) = \mathrm{SR}(E^n \backslash \Omega_A^1).$$

Corollary 2.6 [1]. For the arbitrary systems of support sets Ω_A^1 and Ω_A^2, the following inequalities are true:

$$(\mathrm{R}(\Omega_A^1 \cup \Omega_A^2), \mathrm{R}(\Omega_A^1 \cap \Omega_A^2)) \le 4\mathrm{SR}(\Omega_A^1)\mathrm{SR}(\Omega_A^2),$$

$$(\mathrm{R}_\Delta(\Omega_A^1 \cup \Omega_A^2), \mathrm{R}_\Delta(\Omega_A^1 \cap \Omega_A^2)) \le 2\mathrm{SR}(\Omega_A^1)\mathrm{SR}(\Omega_A^2),$$

$$\mathrm{R}(E^n \backslash \Omega_A^1) \le 4\mathrm{SR}(\Omega_A^1), \quad \mathrm{R}_\Delta(E^n \backslash \Omega_A^1) \le 2\mathrm{SR}(\Omega_A^1).$$

Theorem 2.14 [1]. A projection of a system of support sets onto an interval increases its rank of symmetry by no more than two. When a system of support sets is projected from E^n to E^{n+q} by adding q dummy variables, its symmetry rank is increased by no more than unity.

The proof is based on the following reasoning.

The projection Ω_A on interval $N^{\sigma_1, \sigma_2, \dots, \sigma_u}_{i_1, i_2, \dots, i_u}$ in E^n-set $\mathrm{pr}(\Omega_A) = \{\tilde{\omega} \mid \tilde{\omega} \in \Omega_A, \omega_{i_1} = \sigma_1, \omega_{i_2} = \sigma_2, \dots, \omega_{i_u} = \sigma_u\}$ and the partitioning of the form $N = M_0 \cup M_1 \cup (N_1 \backslash M) \cup (N_2 \backslash M) \cup \dots \cup (N_k \backslash M)$, $\mathrm{SR}(\Omega_A) = k$, $N_1 \cup N_2 \cup \dots \cup N_k = N$, $S_{N_1} \times S_{N_2} \times \dots \times S_{N_k} \subseteq G(\Omega_A)$, $M = \{i_1, i_2, \dots, i_u\}$, $M^\alpha = \{i_j \mid i_j \in M, \sigma_j = \alpha\}$, $\alpha \in \{0, 1\}$ are considered. Since the product of the groups induced by this partitioning is contained in $\mathrm{G}(\mathrm{pr}(\Omega_A))$, the rank of symmetry of this projection is no more than $k + 2$.

By adding q new variables to $\tilde{\omega} \in \Omega_A$, i.e., by transforming $\Omega_A \subseteq E^n$ into $\Omega'_A = \{(\tilde{\omega}, \tilde{\gamma}) \mid \tilde{\omega} \in \Omega_A, \tilde{\gamma} \in E^q\} \subseteq E^{n+q}$, we may establish that if $\mathrm{SR}(\Omega_A) = k$, then $\mathrm{SR}(\Omega'_A) \le k + 1$.

Theorems 2.11–2.14 can be useful in assessing the ranks of the systems of support sets obtained from some basic systems of support sets via operations of union, intersection, complementation, addition of dummy variables, and isometric permutations.

Below, we give the examples of rank evaluation for several systems of support sets.

(1) (1.4)-type system of support sets. $G(\Omega_A) = S_{\{1, 2, \dots, n\}}$; therefore, $\mathrm{R}_\Delta(\Omega_A) \le 2$ and $\mathrm{R}(\Omega_A) \le 4$.

(2) A sphere of a Boolean cube with a center at the point $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and radius r, i.e., $\Omega_A = \{\tilde{\omega} \mid \tilde{\omega} \in E^n, \|\tilde{\omega} + \alpha\| \le r\}$. In this case, $S_{N_0} \times S_{N_1} \subseteq G(\Omega_A) = \varphi^{-1} S_{\{1, 2, \dots, n\}} \varphi$, where $N_\beta = \{i \mid \alpha_i = \beta, 1 \le i \le n\}$, $\beta \in \{0, 1\}$, and φ is an isometric permutation such that if $i \in N_0$, then $(\varphi(\alpha))_i = \alpha_i$ and if $i \in N_1$, then $(\varphi(\alpha))_i = \bar{\alpha}_i$. Therefore, $\mathrm{R}_\Delta(\Omega_A) \le 4$ and $\mathrm{R}(\Omega_A) \le 8$.

(3) An interval of the Boolean cube $\Omega_A = N^{\sigma_1, \sigma_2, \dots, \sigma_u}_{i_1, i_2, \dots, i_u}$, $\sigma_i \in \{0, 1\}$. Here, $S_N \subseteq \mathrm{G}(\Omega_A)$, where $N = \{1, 2, \dots, n\} \backslash \{i_1, i_2, \dots, i_u\}$, therefore, $\mathrm{SR}(\Omega_A) \le 3$, $\mathrm{R}_\Delta(\Omega_A) \le 6$, and $\mathrm{R}(\Omega_A) \le 12$.

2.5 Absolutely Symmetrical Systems of Support Sets

A class of systems of support sets Ω_A such that $\mathrm{R}_\Delta(\Omega_A) \le 2$ was completely described in [9].

A system $\Omega_A = \{\tilde{\omega}_1, \tilde{\omega}_2, \dots, \tilde{\omega}_r\}$ was considered. For this system, a matrix

$$C_{r \times n}(\Omega_A) = \begin{bmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \vdots \\ \tilde{\omega}_r \end{bmatrix} = \begin{bmatrix} \alpha_1^1 & \alpha_2^1 & \dots & \alpha_n^1 \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \dots & \dots & \dots & \dots \\ \alpha_1^r & \alpha_2^r & \dots & \alpha_n^r \end{bmatrix} \qquad (2.22)$$

was defined accurate to the order of magnitude in elements enumeration.

Definition 2.14 [9]. Vector $\hat{\hat{t}} = \hat{\hat{t}}(C) = (\alpha_t^1, \alpha_t^2, \dots, \alpha_t^r)^T$ is called a characteristic vector of the feature $t \in N$ over matrix C.

Definition 2.15 [9]. Features $i, j \in N$ are called features–twins if $\forall \tilde{\omega} \in \Omega_A$ $(\tilde{\omega})_i = (\tilde{\omega})_j$.

Note that features i and j are features–twins if and only if $\forall C \in D(\Omega_A)$ $\hat{\hat{i}}(C) = \hat{\hat{j}}(C)$, where $D(\Omega_A)$ is a set of (2.22)-type matrices for all possible ways of enumeration of characteristic vectors of a system of support sets.

Definition 2.16 [9]. A matrix

$$B_{\tilde{\delta}}(C) = \begin{bmatrix} \tilde{\omega}_{i_1} \\ \tilde{\omega}_{i_2} \\ \vdots \\ \tilde{\omega}_{i_k} \end{bmatrix}, \qquad (2.23)$$

where $\tilde{\omega}_{i_1}, \tilde{\omega}_{i_2}, \dots, \tilde{\omega}_{i_k}$ are all characteristic vectors from Ω_A such that $B_{\tilde{\omega}_{i_j}}(\tilde{\delta}) = 1$, is called an saction of the proximity function $B_{\tilde{\omega}}(\tilde{\delta})$ on the system of support sets. If $B_{\tilde{\omega}_i}(\tilde{\delta}) = 0$, $i = \overline{1, r}$, then the action of the proximity function $B_{\tilde{\omega}}(\tilde{\delta})$ on the system of support sets is not determined.

It is significant that $\tilde{b}_{\tilde{\delta}}(C_1) = \tilde{b}_{\tilde{\delta}}(C_2)$, $\forall C_1, C_2 \in D(\Omega_A)$, $\tilde{b}_{\tilde{\delta}}(C)$ is a vector equal to the coordinatewise sum of the vectors $\tilde{\omega}_{i_1}, \tilde{\omega}_{i_2}, \dots, \tilde{\omega}_{i_k}$: $\tilde{b}_{\tilde{\delta}}(C) = \tilde{\omega}_{i_1} + \tilde{\omega}_{i_2} + \dots + \tilde{\omega}_{i_k}$.

Definition 2.17 [9]. Vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in R^n$ is called 2-ideal if $|\{\alpha_1, \alpha_2, \dots, \alpha_n\}| \le 2$.

Definition 2.18 [9]. A system of support sets Ω_A is called 2-ideal if $\exists C \in D(\Omega_A)$ such that vector $\tilde{b}_{\tilde{\delta}}(C)$ is 2-ideal $\forall \tilde{\delta}, \varepsilon^1, \varepsilon^2$). Ω_A is 2-ideal if and only if $\mathrm{R}_\Delta(\Omega_A) \le 2$.

Assertion 2.4 [9]. Any system of support sets $\Omega_A \subseteq E_n^1(E_n^{n-1})$ is 2-ideal.

Definition 2.19 [9]. A system of support sets $\Omega_A \subseteq E^n$ is called reducible to the system of support sets $\Omega_A^* \subseteq E^{n*}$, $n* < n$, if $\exists C_{r \times n} \in D(\Omega_A)$, $\exists C^*_{r \times n*} \in D(\Omega_A^*)$, $\exists i_1, i_2, \dots, i_{n*} \in N$: $|\{i_1, i_2, \dots, i_{n*}\}| = n*$, $\hat{\hat{i}}_1(C) = \hat{\hat{1}}(C*)$, $\hat{\hat{i}}_2(C) = \hat{\hat{2}}(C*)$, $\dots$, $\hat{\hat{i}}_{n*}(C) = \hat{\hat{n}}*(C*)$, $\hat{\hat{j}}(C) \in \{\hat{\hat{1}}(C*), \hat{\hat{2}}(C*), \dots, \hat{\hat{n}}*(C*)\}$ $\forall j \in N \backslash \{i_1, i_2, \dots, i_{n*}\}$.

Definition 2.20 [9]. A matrix $C_1(\Omega_A)$ is called reducible to the matrix $C_2(\Omega_A^*)$ if a system of support sets Ω_A is reducible to the system of support sets Ω_A^*.

This definition implies that matrix C_1 is reducible to the matrix C_2 if and only if matrix C_2 can be reduced from the matrix C_1 by the following transformations:

(i) row permutation (renumbering of the support sets);

(ii) column deletion under condition that the matrix still contains the columns equal to the deleted (deletion of the features–twins);

(iii) column permutation (renaming of the features).

Inversely, matrix C_1 can be reduced from C_2 by row permutation, by adding the columns equal to those that the matrix C_2 already has, and by column permutation.

A system of support sets reducible to a 2-ideal system is 2-ideal itself.

Definition 2.21 [9]. A system of support sets Ω_A is called internal if $\{(0, 0, \ldots, 0), (1, 1, \ldots, 1)\} \cap \Omega_A = \varnothing$.

Definition 2.22 [9]. An internal system of support sets Ω_A with the matrix of the support sets C is called absolutely reducible if either

$$\|\hat{\tilde{t}}(C(\Omega_A))\| \geq 1, \quad \forall t \in N, \tag{2.24}$$

or

$$\|\hat{\tilde{t}}(C(\Omega_A))\| \leq |\Omega_A| - 1, \quad \forall t \in N. \tag{2.25}$$

If $\exists C \in D(\Omega_A)$ such that either condition (2.24) or (2.25) is fulfilled, then the same condition will be fulfilled $\forall C \in D(\Omega_A)$.

An absolutely reducible system of support sets is reducible to $\Omega_A^* \subseteq E_n^1$ if condition (2.24) is fulfilled, or to $\Omega_A^* \subseteq E_n^{n-1}$, if condition (2.25) is fulfilled. All this and assertion 2.4 imply that absolutely reducible system of support sets is 2-ideal.

Definition 2.23 [9]. (1.4)-type system of support sets is called absolutely symmetrical.

In the above-considered example, we show that Δ-rank of absolutely symmetrical system of support sets does not exceed 2; i.e., an absolutely symmetrical system of support sets is 2-ideal.

Absolute reducibility and absolute symmetry are independent properties of the system of support sets. Let us consider it on the following example. For $n = 4$,

$$C(\Omega_A^1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C(\Omega_A^2) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

System Ω_A^1 is absolutely reducible but not absolutely symmetrical, and system Ω_A^2 is absolutely symmetrical but not absolutely reducible.

It turned out that a class of all 2-ideal systems of support sets is exhausted by absolutely reducible and absolutely symmetrical systems.

Theorem 2.15 [9]. Let Ω_A be a 2-ideal system of support sets. Then, either Ω_A is absolutely symmetrical or $\Omega_A \backslash \{(0, 0, \ldots, 0), (1, 1, \ldots, 1)\}$ is absolutely reducible.

2.6. Atomic Systems of Support Sets

In [29], the systems of support sets called atomic were considered. For atomic systems of support sets and a (2.2)-type proximity function, an expression was obtained for calculating the estimate of function $V_t(\tilde{\delta})$.

Definition 2.24 [29]. Atomic set $A(N_0; N_1; (M_1, k_1); (M_2, k_2); \ldots; (M_\nu, k_\nu))$ is a set of all vectors $\tilde{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)$ from E^n such that

(1) $\omega_i = 0, \forall i \in N_0$,

(2) $\omega_i = 1, \forall i \in N_1$, and

(3) $\sum\limits_{i \in M_j} \omega_i = k_j, j = \overline{1, \nu}$

are valid for the partitioning of the set N of the form $N = N_0 \cup N_1 \cup \left(\bigcup\limits_{i=1}^{\nu} M_i\right)$, $N_0 \cap N_1 = N_0 \cap M_i = N_1 \cap M_i = M_i \cap M_j = \varnothing$, $\forall i, j \in \{1, 2, \ldots, \nu\}$, $i \neq j$, where each subset M_i corresponds to the integer k_i: $0 < k_i < |M_i|$, $i = \overline{1, \nu}$.

Obviously, any atomic set is a part of a vector layer of a Boolean cube of the weight $|N_1| + \sum\limits_{i=1}^{\nu} k_i$.

Here are the examples of atomic sets:

—The kth layer of a Boolean cube is atomic set $A((N, k))$ $(N_0 = N_1 = \varnothing, M_1 = N, k_1 = k)$;

—Atomic set $A(N_0; N_1; (M_1, k))$, where $N_0 = \{i_1, i_2, \ldots, i_s\}$, $N_1 = \{j_1, j_2, \ldots, j_t\}$, $N_0 \cap N_1 = \varnothing$, $M_1 = N \backslash (N_0 \cup N_1)$, is the intersection of the interval specified by conjunction of $\bar{x}_{i_1}\bar{x}_{i_2}\ldots\bar{x}_{i_s}x_{j_1}x_{j_2}\ldots x_{j_t}$ with the cube layer $E_n^{k+|N_1|}$.

An expansion of a sphere, ball, and several other sets in E^n into atomic subsets is considered. By using efficient formulas for estimate calculation over some atomic subsets, this expansion allows one to obtain efficient formulas of estimate calculations for the given system of support sets.

Table

System of support sets	Proximity function, additional restrictions	Result
E_n^k	(2.2)-type, feature weights equal	Formulas for $\Gamma_j(S)$
$E^n\backslash\{(0, 0, \dots, 0)\}$	(2.2)-type, feature weights equal to 1	
Absolutely symmetrical	(2.3)-type	$R_\Delta(\Omega_A) \le 2$
Atomic set	(2.2)-type for $\varepsilon = 0$	Formula for $V_t(\tilde{\delta})$

Theorem 2.16 [29]. Let a system of support sets be atomic set $A(N_0; N_1; (M_1, k_1); (M_2, k_2); \dots; (M_\nu, k_\nu))$. Then,

$$V_t(\tilde{\delta}) = \begin{cases} 0, \ t \in N_0 \\ \sum\limits_{\varepsilon_0 \in C_1} \prod\limits_{u=1}^{\nu} C_{m_u - d(M_u)}^{\varepsilon_u^0} C_{d(M_u)}^{k_u - \varepsilon_u^0}, \ t \in N_1 \\ \sum\limits_{\varepsilon_0 \in C_2(p)} \dfrac{\varepsilon_p^0}{m_p - d(M_p)} \prod\limits_{u=1}^{\nu} C_{m_u - d(M_u)}^{\varepsilon_u^0} C_{d(M_u)}^{k_u - \varepsilon_u^0} \\ t \in M_p, \ 1 \le p \le \nu, \ (\tilde{\delta})_t = 0 \\ \sum\limits_{\varepsilon_0 \in C_3(p)} \dfrac{k_p - \varepsilon_p^0}{d(M_p)} \prod\limits_{u=1}^{\nu} C_{m_u - d(M_u)}^{\varepsilon_u^0} C_{d(M_u)}^{k_u - \varepsilon_u^0} \\ t \in M_p, \ 1 \le p \le \nu, \ (\tilde{\delta})_t = 1, \end{cases} \tag{2.26}$$

where (a) vector $\tilde{\delta}$ is partitioned into subvectors $\tilde{\delta}(N_0)$, $\tilde{\delta}(N_1)$, $\tilde{\delta}(M_1)$, $\tilde{\delta}(M_2)$, …, $\tilde{\delta}(M_\nu)$; $\tilde{\delta}(X)$ is a vector with components whose indices are contained in X, $d(X) = \|\tilde{\delta}(X)\|$, $|N_1| = s$, $|M_i| = m_i$, $i = \overline{1, \nu}$, $\varepsilon_{\text{eff}} = \varepsilon - s + d(N_1)$ (ε is a parameter of the (2.2)-type proximity function) and $\varepsilon^0 = (\varepsilon_1^0, \varepsilon_2^0, \dots, \varepsilon_\nu^0)$; and

$$\text{(b) } C_1 = \Big\{\varepsilon^0 \Big| \sum_{u=1}^{\nu} \varepsilon_u^0 \le \varepsilon_{\text{eff}},$$

$$\max(0, k_u - d(M_u)) \le \varepsilon_u^0 \le \min(k_u, m_u - d(M_u)),$$

$$u = \overline{1, \nu}\Big\},$$

$$C_2(p) = \Big\{\varepsilon^0 \Big| \sum_{u=1}^{\nu} \varepsilon_u^0 \le \varepsilon_{\text{eff}}, \ \varepsilon_p^0 \ge 1,$$

$$\max(0, k_u - d(M_u)) \le \varepsilon_u^0 \le \min(k_u, m_u - d(M_u)),$$

$$u = \overline{1, \nu}\Big\},$$

$$C_3(p) = \Big\{\varepsilon^0 \Big| \sum_{u=1}^{\nu} \varepsilon_u^0 \le \varepsilon_{\text{eff}}, \ \varepsilon_p^0 \le k_p - 1,$$

$$\max(0, k_u - d(M_u)) \le \varepsilon_u^0 \le \min(k_u, m_u - d(M_u)),$$

$$u = \overline{1, \nu}\Big\}.$$

The proof of the theorem consists in the consideration of three cases $t \in N_1$, $t \in M_p$, and $(\tilde{\delta})_t = 0$, $t \in N_1$, $(\tilde{\delta})_t = 1$, and in the direct calculation of the value $V_t(\tilde{\delta})$ for each of the cases.

Some of the sets in E^n reducible into atomic subsets are considered. A notion of a ball sector introduced in Section 1 is used for constructing these expansions. For other details, see [29].

(1) A sphere with a center α and radius r are set. Sections of this sphere with the cube's layers of the weigh k, $k = \overline{\max(\|\alpha\| - r, 0), \|\alpha\| + r}$. The layers of this type are called sectors of the sphere. $M_1 = \{i | i \in N, (\alpha)_i = 1\}$, $M_2 = N\backslash M_1$. A sector of sphere located in the kth layer is prove to be an atomic set $A((M_1, k_1); (M_2, k_2))$ where $k_1 = (k + \|\alpha\| - r)/2$ and $k_2 = (k - \|\alpha\| + r)/2$. If at least one of the numbers $(k + \|\alpha\| - r)/2$, $(k - \|\alpha\| + r)/2$ is not an integer, then the intersection of the sphere and the kth layer is an empty set.

Since different sectors of the sphere are mutually disjoint, Theorem 2.16 and the additivity of the estimate $\Gamma_j(S)$ over the union of disjoint systems of the support sets are used for constructing efficient formulas of estimate calculations over the system of support sets in the form of a sphere in a Boolean cube.

(2) A ball. The expansion into the atomic subsets is reduced to the expansion into the sectors of the spheres constituting this ball. Then, obtained expansion consists of mutually disjoint atomic subsets.

(3) Nonempty intersection of two arbitrary balls.

(4) Nonempty intersection of m balls of the same radius with the center α_i in one layer, such that scalar product $(\alpha_i, \alpha_j) = 0$ for any $i \ne j$, $1 \le j \le m$ (orthogonality of centers).

(5) Nonempty intersection of the two groups of balls of the same radius; the balls in every group satisfy condition (4); the centers of the balls of different groups are also mutually orthogonal.

Thus, analysis of the works concerning deriving efficient formulas for estimate calculation shows that

—the system of support sets and the type of proximity function are the AEC parameters defining complexity of formulas for estimate calculation

—reduction of computational complexity of Eq. (1.11) for the estimate $\Gamma_j(S)$ is defined by the choice of AEC parameters which set particular recognition algorithm or a family of such algorithms.

Main results of setting restrictions on the system of support sets are presented in the table in the order of presentation.

By using these results, the efficient formulas for estimate calculations based on the system of support sets may be obtained which are the unions of disjoint systems of the support sets shown in the table.

Some systems of support sets considered above are generalizations, or they can be obtained by combining other systems of support sets. These relations are shown in Fig. 2.1.

The notation $A \longrightarrow B$ in the figure means that a set of the form A can be described in the form B. Thus, for example, the kth layer of the Boolean cube is a sphere with the center at the point (0, 0, …, 0) and radius k. It is obvious, that relation $\longrightarrow$ is transitive; i.e., a ball can be obtained by the union of spheres.

Figure 2.1 shows that, generally, formulas for estimate calculations for the atomic systems of support sets may serve as a basis for constructing formulas for estimate calculation for the systems of the support sets of any type represented in the figure.

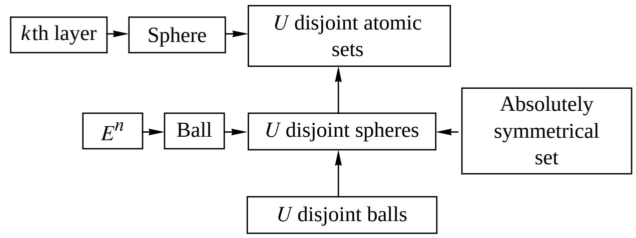

Fig. 2.1. Relations between different systems of support sets.

3. SUPPORT SETS IN THE ALGORITHMS OF ESTIMATE CALCULATIONS USING 2D INFORMATION

Most of the computationally efficient algorithms of image recognition are designed for the work with feature descriptions or image models. To maximally use information contained in images, it is necessary to overcome the principal discrepancy between the image nature and the techniques for information extraction based on symbol models of images. This dictates a practical need in efficient recognition algorithms directly applied to images and their fragments.

This class of algorithms—algorithms based on estimate calculations using two-dimensional information (2D-AEC)—was defined as a special case of AEC. The model of AEC, in the general case, allows processing the information which reflects a spatial (two-dimensional) image structure. The principal property of the 2D-AEC class is the use of two-dimensional support sets, i.e., images and their fragments, for calculation of proximity estimates of images under recognition. The range of the problems of 2D-AEC includes enumeration and investigation of the spatial support sets and definition of the subclasses of the algorithms (corresponding to the types of the support sets), which allow one to produce efficient formulas that model the work of the algorithms.

In contrast to AEC, in 2D-AEC, a matrix of feature values (pixels) is a primary description of the object and a set of double indices, which code places in the description matrix, is a support set.

In practice, the choice of support sets adequate to the problem at hand is of great importance here. In most cases, the use of support sets popular in AEC models does not make sense, particularly, due to the extremely high computational complexity of the correspondent procedures. Thus, matching two images according to all possible k pixels is *a priori* senseless whatever the proximity function may be. Both the informational nature of the wide class of images encountered in recognition problems and the content of these problems make the matching of images or their fragments by separate arbitrary isolated pixels impractical. On the other hand, the informational regions (segments, contours, lines, etc.) are naturally extracted in images. It is reasonable to perform the matching on the level of these regions. During construction of the support sets, these regions can be considered as a sort of image primitives which are unreasonable or impossible to divide.

Thus, 2D-AEC are based on the region-matching principle—images are matched by some local neighborhoods (connected group of pixels) and not by the separate individual pixels. In this context, a local neighborhood, connected group of pixels of power m ($m \geq 2$), is considered as a minimum potential information-carrier for matching information images.

Let the description of the image be a rectangular matrix $u \times v$, $u > 1$, $v > 1$. Similar to the one-dimensional case, the two-dimensional support set Ω can be represented by the characteristic matrix $\hat{\omega} = (\omega_{ij})_{u \times v}$,

$$\omega_{ij} = \begin{cases} 1, & (i, j) \in \Omega \\ 0, & (i, j) \notin \Omega. \end{cases}$$

The characteristic matrix $\hat{\omega}$ can be considered as a binary image on the raster $u \times v$, where zeros correspond to the color of a background (e.g., white) and unities, to the color of figures (e.g., black). For the convenience, we call the characteristic matrix $\hat{\omega}$ a support set.

A technique of extracting and matching local neighborhoods determines the following necessary stages of constructing two-dimensional support sets:

(1) The number k ($k \geq 1$) of connected components of the support set $\hat{\omega}$ is specified.

(2) In $\hat{\omega}$, k rectangles are fixed; i.e., an upper left position is chosen along with the side length of each of the k rectangles.

(3) A connected component tangent to all sides is inscribed in each rectangle; several connected components should not constitute a new component.

As the connected components, we may choose arbitrary geometrical figures (circles, ellipses, triangles, squares, etc.) and lines.

This description also allows one to structure an admissible two-dimensional support set. This set consists of k connected components, and the position of each connected component is determined by the position of the upper left vertex of the circumscribed rectangle.

Definition 3.1. Configuration made by all connected components of the support set $\hat{\omega}$ is called a generative element of the support set $\hat{\omega}$ (GESS). It is a submatrix $M^{i_1, i_2, \dots, i_k}_{j_1, j_2, \dots, j_s}$ of $k \times s$ matrix $\hat{\omega}$ and is generated by the elements of this matrix located at the intersections of the rows $i_1, i_2, \dots, i_k$ and the columns $j_1, j_2, \dots, j_s$.

If $(i_1, j_1), (i_1, j_2), \dots, (i_k, j_k)$ are the positions of the k connected components of the support set ω and $p_1 = \min\limits_{t = 1, 2, \dots, k} i_t$, $p_2 = \max\limits_{t = 1, 2, \dots, k} i_t$, $q_1 = \min\limits_{t = 1, 2, \dots, k} j_t$, and $q_2 = \max\limits_{t = 1, 2, \dots, k} j_t$, then GESS generating this support set is described by the submatrix $\hat{\omega}$ of the form $M^{p_1, p_1+1, \dots, p_2}_{q_1, q_1+1, \dots, q_2}$.

Definition 3.2. The position (p_1, q_1) of the upper left vertex of the circumscribed rectangle is called a position of GESS on the raster.

Definition 3.3. A number of black points of GESS is called its area.

Among all possible systems of support sets, the systems are distinguished which are generated by one GESS and differ only in its position on the matrix $u \times v$. Correspondingly, different systems of two-dimensional support sets are generated by different GESS (each by its own).

In addition, there is one more possibility to specify a system of support sets generated by GESS in a different way—by GESS-close support sets. These support sets are formed by the following procedure.

Let us fix k connected components on the raster $u \times v$ and transfer each component along the raster, so that they do not intersect. All allowable transfers of such kind generate a system of two-dimensional support sets, of no more than k connected components per set. This family of two-dimensional support sets can be considered as an analog of the family of one-dimensional support sets representing the kth layer of the Boolean cube.

4. EFFICIENT FORMULAS OF ESTIMATE CALCULATIONS USING 2D INFORMATION

The problem of constructing efficient formulas for estimate calculations using 2D information is caused, first of all, by the irrationality of using 2D-AEC in the application problems without adequate formulas. There are two important factors connected with this problem:

—in 2D-AEC, the type of formula (1.11) defining estimate $\Gamma_j(S)$ is not changed;

—systems of support sets considered in the model of 2D-AEC have a great power and substantially differ from the systems of one-dimensional support sets which are already supplied with efficient formulas.

Generally, an arbitrary system of two-dimensional support sets $\hat{\omega} = (\omega_{ij})_{u \times v}$ may be represented as a set of vertices of the Boolean cube E^{uv}. Knowing the expansion of the system of support sets into intervals in E^{uv}, we can use Eq. (2.13) to calculate the estimate $\Gamma_j(S)$. However, as was said before, Eq. (2.13) is efficient not for any expansion of the system of support sets

The simplest way of specifying the 2D-AEC class is the following:

In a standard classification problem, let a set of allowable objects be a set of $u \times v$ matrices with the elements from the set D; i.e.,

$$S = I(S) = (a_{ij})_{u \times v}, \quad S' = I(S') = (b_{ij})_{u \times v}.$$

The parameters of the 2D-AEC family are defined as follows:

(1) A system of support sets Ω_A is a collection of rectangles $\Pi^{ij}_{R_1 \times R_2} = \{(i, j), (i, j+1), \dots, (i, j+R_2-1), (i+1, j), \dots, (i+1, j+R_2-1), \dots, (i+R_1-1, j+R_2-1)\}$ with the sides R_1, R_2, $1 \le R_1 \le u$, $1 \le R_2 \le v$, $2 \le R_1 R_2$.

(2) Let a metric (semimetric) $\rho(x, y)$ be defined in the set D, the values $\varepsilon_{ij} > 0$, $(i = \overline{1, u}, j = \overline{1, v}$, $\varepsilon \ge 0$, ε is an integer), and $\Omega = \Pi^{ij}_{R_1 \times R_2}$ are specified. Consider the following system of inequalities:

$$\rho(a_{ij}, b_{ij}) \le \varepsilon_{ij},$$

$$\rho(a_{i, j+1}, b_{i, j+1}) \le \varepsilon_{i, j+1}, \dots,$$

$$\rho(a_{i+R_1-1, j+R_2-1}, b_{i+R_1-1, j+R_2-1}) \le \varepsilon_{i+R_1-1, j+R_2-1},$$

and denote the number of unsatisfied inequalities by γ. Suppose that

$$B_{\hat{\omega}}(S, S') = \begin{cases} 1, & \gamma \le \varepsilon \\ 0, & \gamma > \varepsilon. \end{cases} \tag{4.1}$$

This proximity function is the analog of the (2.2)-type proximity function for two-dimensional object descriptions.

(3) The feature weights (defined by the matrix $P = (p_{ij})_{u \times v}$, $p_{ij} > 0$), the precedent weights, and the decision rule are all arbitrary.

This concludes the description of the 2D-AEC family.

Similar to the characteristic proximity vector $\tilde{\delta}(S, S')$, the characteristic proximity matrix $C = C(S, S') = (c_{ij})_{u \times v}$:

$$c_{ij} = \begin{cases} 1, & \rho(a_{ij}, b_{ij}) \le \varepsilon_{ij} \\ 0, & \rho(a_{ij}, b_{ij}) > \varepsilon_{ij}. \end{cases} \tag{4.2}$$

is introduced.

As before, a problem of efficient calculation of the value of Eq. (2.1) can be reduced to the problem of efficient calculation of the value of the following expression:

$$\sum_{i=1}^{u}\sum_{j=1}^{v} p_{ij} V_{ij}(S, S'), \tag{4.3}$$

where $V_{ij}(S, S')$ is the number of support sets Ω from Ω_A such that $B_{\hat{\omega}}(S, S') = 1$.

The author of [17] suggested the estimate calculation technique in AEC for (2.2)-type proximity function and for the system of support sets being a collection of rectangles of definite size. The algorithm implemented by operators A_1 and A_2 was proposed for calculating the values of $V_{ij}(S, S')$. First, the case was considered when $\varepsilon = 0$ in Eq. (4.1). The following procedures were executed for each position (i, j) of matrix C such that $c_{ij} = 1$:

(1) Using position (i, j) of matrix the C, operator A_1 constructs a figure $\Phi(i, j)$ which is a union of all maximum rectangles comprised from the unities of the matrix C that contain unity in the position (i, j). Here, a maximum rectangle is a rectangle with the sides that cannot be increased.

(2) Using figure $\Phi(i, j)$, operator A_2 calculates the value of $V_{ij}(S, S')$.

Now, let $0 < \varepsilon < R_1R_2$. The following procedures are executed for each position (i, j) of matrix C:

(1) Operator A_1 constructs a figure $\Phi(i, j)$ which is a union of all maximum rectangles comprised from zeros and unities of the matrix C that contain zero and unity in the position (i, j); here, $\Phi(i, j)$ contains no more than ε zeros.

(2) Using figure $\Phi(i, j)$, operator A_2 calculates the value of $V_{ij}(S, S')$.

The operators A_1 and A_2 are implemented for the cases when $\varepsilon = 0$ and $\varepsilon > 0$. However, the efficiency of the proposed technique was not estimated.

Now, we proceed to the new approach to the efficient estimate calculation in 2D-AEC for the system of rectangular support sets and proximity function (4.1) for $\varepsilon = 0$.

Let the weights of all features be the same and equal to $p > 0$. Then,

$$\Phi_{\Omega_A}(S, S') = pR_1R_2 \sum_{\hat{\omega} \in \Omega_A} B_{\hat{\omega}}(S, S'). \tag{4.4}$$

The value of the sum in Eq. (4.4) is no more than the number of rectangles of the size $R_1 \times R_2$ formed by the unities of the matrix $C(S, S')$.

The essence of this approach consists in providing the efficient (sped up) calculation of value (4.4) by preliminary calculation of some characteristics of the matrix C

	1	2	3	4	5	$h_{1,i}$
1	1	1	1		1	1
2		$\underline{\mathbf{1}}$	1	1	1	1
3	1	$\underline{\mathbf{1}}$	1	1	1	1
4		1		1		
5		1				
$h_{2,j}$		1	1	1	1	

$H_1 = 1, H_2 = 1,$
$H_3 = 2$

Fig. 4.1. Matrix C and corresponding values of H_1, H_2, H_3, $u = v = 5$, $R_1 = R_2 = 3$, empty sells are zeros.

which allow us to check whether the necessary conditions are fulfilled: the matrix contains the rectangles of the given size that contain unities.

Suppose

$$h_{1,i} = \bigvee_{j=1}^{v - R_2 + 1} c_{i,j} \cdot c_{i,j+1} \cdot \ldots \cdot c_{i,j+R_2-1}, \quad i = \overline{1, u}, \tag{4.5}$$

$$h_{2,j} = \bigvee_{i=1}^{u - R_1 + 1} c_{i,j} \cdot c_{i+1,j} \cdot \ldots \cdot c_{i+R_1-1,j}, \quad j = \overline{1, v}. \tag{4.6}$$

Obviously, $h_{1,i} = 1$ ($h_{2,j} = 1$) if and only if there is a continuous sequence of at least R_2 (R_1) unities in the ith row (jth column) of matrix C (see Fig. 4.1).

Let

$$H_1(C(S, S')) = H_1(S') = \left(\bigvee_{i=1}^{u} h_{1,i}\right)\left(\bigvee_{j=1}^{v} h_{2,j}\right). \tag{4.7}$$

$H_1 = 1$ if and only if the following conditions for the matrix C are simultaneously fulfilled (see Fig. 4.1):

—at least one row contains a continuous sequence of at least R_2 unities;

—at least one column contains a continuous sequence of at least R_1 unities.

Therefore, if $H_1(S') = 0$, matrix C *a priori* does not have the rectangle that contains unities with the sides R_1 and R_2; thus, $\Phi_{\Omega_A}(S, S') = 0$.

The following assertion is proven:

Assertion 4.1.

$$\Gamma_j(S) = pR_1R_2 \sum_{\substack{S' \in W_j \\ H_1(S') = 1}} \gamma(S') \sum_{\hat{\omega} \in \Omega_A} B_{\hat{\omega}}(S, S'), \quad j = \overline{1, l}. \tag{4.8}$$

Let

$$H_2(C(S, S') = H_2(S')$$

$$= \left(\bigvee_{i=1}^{u-R_1+1} h_{1,i} \cdot h_{1,i+1} \cdot \ldots \cdot h_{1,i+R_1-1}\right) \tag{4.9}$$

$$\times \left(\bigvee_{j=1}^{v-R_2+1} h_{2,j} \cdot h_{2,j+1} \cdot \ldots \cdot h_{2,j+R_2-1}\right).$$

$H_2 = 1$ if and only if the following conditions for the matrix C are simultaneously fulfilled (see Fig. 4.1):

—there is a sequence of R_2 unities situated in R_1 adjacent rows at least;

—there is a sequence of R_1 unities situated in R_2 adjacent columns at least.

Therefore, if $H_2(S') = 0$, then $\Phi_{\Omega_A}(S, S') = 0$. Note also that (i) if $H_2(S') = 1$, then $H_1(S') = 1$ and (ii) if $H_1(S') = 0$, then $H_2(S') = 0$.

Thus, the following assertion is proven:

Assertion 4.2.

$$\Gamma_j(S) = pR_1R_2 \sum_{\substack{S' \in W_j \\ H_2(S') = 1}} \gamma(S') \sum_{\hat{\omega} \in \Omega_A} B_{\hat{\omega}}(S, S'), \tag{4.10}$$
$$j = \overline{1, l}.$$

Consider matrices $C_1 = (c_{ij}^1)_{(u-R_1+1)\times(v-R_2+1)}$ and $C_2 = (c_{ij}^2)_{(u-R_1+1)\times(v-R_2+1)}$ defined as follows: $c_{ij}^1 = c_{i,j} \cdot c_{i,j+1} \cdot \ldots \cdot c_{i,j+R_2-1}$, $c_{ij}^2 = c_{i,j} \cdot c_{i+1,j} \cdot \ldots \cdot c_{i+R_1-1,j}$, $i = \overline{1, u-R_1+1}$, $j = \overline{1, v-R_2+1}$. If $c_{ij}^1 = 1$ ($c_{ij}^2 = 1$), then it means that in the ith row (jth column) of matrix C, a sequence of at least R_2 (R_1) unities begins.

Let $C' = (c'_{ij})_{(u-R_1+1)\times(v-R_2+1)} = C_1C_2$, $H_3(C'(S, S')) = H_3(S') = \|C'\|$, where C_1C_2 is a coordinatewise product of matrices C_1 and C_2 and $\|C'\|$ is a number of unities in the matrix C'. Let $c'_{ij} = 1$; this means that in the position (i, j) of the matrix C, a vertex of a right angle is situated with the sides located in the ith row and jth column (the length of the side is no less than R_1 unities). For the matrix C presented in Fig. 4.1, the matrix C' contains unities in positions (2, 2) and (3, 2).

Hence, if $H_3(S') = 0$, then $\Phi_{\Omega_A}(S, S') = 0$. Therefore, the following assertion is valid.

Assertion 4.3.

$$\Gamma_j(S) = pR_1R_2 \sum_{\substack{S' \in W_j \\ H_2(S') = 1 \\ H_3(S') \geq 1}} \gamma(S') \sum_{\hat{\omega} \in \Omega_A} B_{\hat{\omega}}(S, S'), \tag{4.11}$$
$$j = \overline{1, l}.$$

If $H_2(S') = 1$, $H_3(S') \geq 1$, then for calculating the value of Eq. (4.4) (which may be zero), it is sufficient to search for $\hat{\omega} = \Pi_{R_1 \times R_2}^{ij} \in \Omega_A$, such that $c'_{ij} = 1$.

Thus, a method of fast calculation of estimate (1.11) consists in the following steps:

1. Construct the matrix $C = C(S, S')$ for the current object $S' \in W_j$.

2. Calculate the value $H_1(C)$; if $H_1(C) = 0$, proceed to the next object $S' \in W_j$.

3. Calculate the value $H_2(C)$; if $H_2(C) = 0$, proceed to the next object $S' \in W_j$.

4. Construct the matrix C'; calculate the value $H_3(C')$; if $H_3(C') = 0$, proceed to the next object $S' \in W_j$; otherwise, for calculating the value of (4.4), sum up all $\hat{\omega} = \Pi_{R_1 \times R_2}^{ij} \in \Omega_A$, such that $c'_{ij} = 1$.

5. SEARCH PROBLEM: MULTISTEP SEARCH PROCEDURE

Let us consider a subclass of DAEC with the following parameters:

(1) Some GESS Φ of the area μ is set. A system of support sets $\Omega_A = \{\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_k\}$ is generated by k parallel transitions of GESS Φ along the raster $u \times v$. Let (i_t, j_t) be a position of GESS Φ in the support set $\hat{\omega}_t$, $t = \overline{1, k}$.

(2) A proximity function has a form of Eq. (4.1) for $\varepsilon = 0$.

(3) Feature weights are equal to some quantity $p > 0$.

(4) Object weights and a decision rule are arbitrary.

This concludes the description of 2D-AEC class.

We have

$$\Phi_{\Omega_A}(S, S') = p\mu \sum_{\hat{\omega} \in \Omega_A} B_{\hat{\omega}}(S, S'). \tag{5.1}$$

The value of $\sum_{\hat{\omega} \in \Omega_A} B_{\hat{\omega}}(S, S')$ is the number of occurrences of GESS Φ in the characteristic proximity matrix $C(S, S')$ such that a position of GESS (i, j) belongs to the set $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$. If Φ does not enter into the matrix C or $(i, j) \in \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$, then $\sum_{\hat{\omega} \in \Omega_A} B_{\hat{\omega}}(S, S') = 0$.

Thus, in the considered subclass of 2D-AEC, the problem of calculating the value of the function (5.1) is equivalent to the problem of the search for GESS Φ in the matrix C. This means that the task of developing a technique of efficient calculation of the value of the function (5.1) is equivalent to the task of constructing an efficient search procedure in the framework of some formalism that describes such procedures.

Consider the statement of the problem of search for GESS on the raster and the formalism describing the procedure of searching for the solution to this problem.

Denote a set of all matrices of the size $u \times v$ (u, v are natural numbers) with the elements from the set $\{0, 1\}$ by $E^{u \times v}$. Let $C = (c_{ij}) \in E^{u \times v}$.

Consider the matrix GESS $\Phi = (\varphi_{pq}) \in E^{R_1 \times R_2}$, where $1 \le R_1 \le u$, $1 \le R_2 \le v$. There are numbers p_1, p_2, q_1, and q_2, such that $\varphi_{p_1 1} = \varphi_{p_2 R_2} = \varphi_{1 q_1} = \varphi_{R_2 q_2} = 1$. S is an area of GESS: $S = |M_\Phi|$, where $M_\Phi = \{(p, q)|_{\varphi_{pq}} = 1, 1 \le p \le R_1, 1 \le q \le R_2\}$.

Definition 5.1. GESS Φ is correctly superimposed on the matrix C in the position (i, j), $1 \le i \le u - R_1 + 1$, $1 \le j \le v - R_2 + 1$, if $c_{i+p-1, j+q-1} = 1$, $\forall (p, q) \in M_\Phi$.

Figures 5.1a and 5.1b exemplify the matrix C and GESS Φ. Cells with unities are colored black, and cells with zeros, white. In this example, GESS Φ is correctly superimposed on the matrix C in the position (2, 2) only.

Definition 5.2. To find a GESS Φ on the matrix C means to put a pair (C, Φ) into correspondence with the matrix $\tilde{C} = \tilde{C}(C, \Phi) = (\tilde{c}_{ij}) \in E^{(u - R_1 + 1) \times (v - R_2 + 1)}$, where $\tilde{c}_{ij} = 1$, if a GESS Φ is correctly superimposed on the matrix C in the position (i, j) and $\tilde{c}_{ij} = 0$ otherwise.

Note that for $R_1 = 1$ and $R_2 = 1$, $\tilde{C} = C$; therefore, hereinafter we consider $R_1 R_2 > 1$ (and, therefore, $S \ge 2$).

In our example, the matrix $\tilde{C}(C, \Phi)$ has the form presented in Fig. 5.1c.

Let us fix a natural number n and consider a sequence of matrices

$$C_0 = (c_{ij}^0)_{u_0 \times v_0},$$
$$C_1 = (c_{ij}^1)_{u_1 \times v_1}, \ldots, C_n = (c_{ij}^n)_{u_n \times v_n}, \tag{5.2}$$

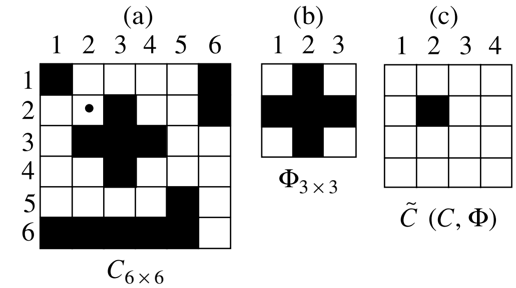

Fig. 5.1. Matrices C, Φ, and $\tilde{C}\,(C, \Phi)$.

where $C_0 \in E^{u_0 \times v_0}$, $C_k \in E^{u_k \times v_k}$, $C_k = f_k(C_{k-1})$, and the function f_k is defined by the relation

$$c_{ij}^k = \mathop{\&}_{(p, q) \in S(i, j, k)} c_{pq}^{k-1}, \tag{5.3}$$

$S(i, j, k) \subseteq \{1, 2, \ldots, u_{k-1}\} \times \{1, 2, \ldots, v_{k-1}\}$, $i = \overline{1, u_k}$, $j = \overline{1, v_k}$, $k = \overline{1, n}$.

Definition 5.3. A set $S(i, j, k)$ is called a domain of dependence of the (i, j)th element of the matrix C_k or the (i, j)th connection of the matrix C_k, and the power of this set $|S(i, j, k)|$ is called the power of the (i, j)th connection of the matrix C_k.

Obviously, the function f_k is completely defined by specifying a set of connections $\{S(i, j, k)| \; i = \overline{1, u_k}, j = \overline{1, v_k}\}$ of the matrix C_k.

Further, we suppose that $u_0 = u$, $v_0 = v$.

Definition 5.4. A set of functions $F^n = (f_1, f_2, \ldots, f_n)$ is called an n-step procedure of the search for GESS Φ if the following conditions are specified:

(1) $C_n = \tilde{C}(C_0, \Phi)$, $\forall C_0 \in E^{u_0 \times v_0}$;

(2) $|S(i, j, k)| \ge 2$, $i = \overline{1, u_k}$, $j = \overline{1, v_k}$, $k = \overline{1, n}$;

(3) $S(i_1, j_1, k) \not\subseteq S(i_2, j_2, k)$ for all admissible (i_1, j_1), (i_2, j_2): $(i_1, j_1) \ne (i_2, j_2)$, $k = \overline{1, n}$;

(4) $\bigcup_{i=1}^{u_k} \bigcup_{j=1}^{v_k} S(i, j, k) = \{1, 2, \ldots, u_{k-1}\} \times \{1, 2, \ldots, v_{k-1}\}$, $k = \overline{2, n}$.

Condition (1) implies that

(a) $u_n = u - R_1 + 1$, $v_n = v - R_2 + 1$; and

(b) $\bigcup_{i=1}^{u_1} \bigcup_{j=1}^{v_1} S(i, j, 1) = \bigcup_{i=1}^{u - R_1 + 1} \bigcup_{j=1}^{v - R_2 + 1} \{(i + p - 1, j + q - 1)| \; (p, q) \in M_\Phi\}$; i.e., the connections of the matrix C_1 should cover only those positions of the matrix C_0, where the unities of GESS Φ may occur.

Thus, the n-step procedure of searching for F^n performs a search for a given GESS Φ on the arbitrary matrix C_0 of the size $u_0 \times v_0$. The procedure is specified by the set of parameters $((u_1, v_1), (u_2, v_2), \ldots, (u_{n-1}, v_{n-1}), (S(i, j, 1))_{u_1 \times v_1}, (S(i, j, 2))_{u_2 \times v_2}, \ldots, (S(i, j, n))_{(u-R_1+1)\times(v-R_2+1)}$, where $(S(i, j, k))_{u_k \times v_k}$ denotes a matrix of the size $u_k \times v_k$ whose elements are the sets $S(i, j, k)$.

Definition 5.5. The quantity

$$|f_k| = \sum_{i=1}^{u_k}\sum_{j=1}^{v_k}(|S(i, j, k)| - 1) \tag{5.4}$$

is called a complexity of the function f_k.

The complexity of the function f_k coincides with the number of conjunctions necessary for its implementation and is a measure of its computational complexity.

Definition 5.6. The quantity

$$|F^n| = \sum_{k=1}^{n}|f_k| \tag{5.5}$$

is called a complexity of the procedure F^n.

Definition 5.7. A procedure F of searching for GESS Φ is called optimal in the class K of the searching procedures Φ if it has the least complexity among all procedures of the class K. A procedure F of searching for GESS Φ is called optimal if it is optimal in the class of all procedures of searching for Φ.

Let us consider some properties of searching procedures.

The one-step procedure of searching for arbitrary GESS Φ is uniquely defined by

$$\begin{gathered} S(i, j, 1) = \{(i+p-1, j+q-1) | (p, q) \in M_\Phi\}, \\ i = \overline{1, u-R_1+1}, \quad j = \overline{1, v-R_2+1}. \end{gathered} \tag{5.6}$$

Here,

$$|F^1| = |f_1| = (u-R_1+1)(v-R_2+1)(S-1). \tag{5.7}$$

If the area of the GESS is equal to 2, then only one-step procedure exists for its search.

Theorem 5.1. The following equations are valid for an n-step procedure F^n of the searching GESS Φ:

(1) $\underline{S}(i, j, n) = \{(i+p-1, j+q-1) | (p, q) \in M_\Phi\}$, $i = \overline{1, u-R_1+1}$, $j = \overline{1, v-R_2+1}$ where $\underline{S}(i, j, n) =$

$$\bigcup_{(i_{n-1}, j_{n-1}) \in S(i,j,n)} \; \bigcup_{(i_{n-2}, j_{n-2}) \in S(i_{n-1}, j_{n-1}, n-1)} \ldots \bigcup_{(i_1, j_1) \in S(i_2, j_2, 2)} S(i_1, j_1, 1);$$

(2) $|S(i, j, k)| \le S$, $i = \overline{1, u_k}$, $j = \overline{1, v_k}$, $k = \overline{1, n}$; $|S(i, j, k)| = S \Leftrightarrow n = 1$;

(3) $\dfrac{u_{k-1}v_{k-1}}{S} \le u_k v_k$, $k = \overline{1, n}$; and

(4) $n \le S - 1$.

Proof.

The relations (1)–(3) obviously follow from the definition of F^n. To prove relation (4), note that, in carrying out a transition from the matrix C_k to matrix C_{k+1}, $1 \le k \le n-1$, at least one position of the matrix C_0 falls in the domain of dependence of each element c_{ij}^k, whereas in carrying out a transition from the matrix C_0 to matrix C_1, minimum two positions of the matrix C_0 fall in the domain of dependence of the element c_{ij}^1. Hence, $2 + (n-1) \le S$.

The ultimate aim in the considered case of the two-dimensional support sets is a construction of an optimal procedure of searching for GESS which is a rectangle $R_1 \times R_2$. The first result in this respect was the construction of the two-step procedure of the special search. It is shown that the complexity of this procedure is less than that of one-step searching procedure implementing the exhaustive search for rectangle and, thus, its efficiency is established. The optimality of this procedure is proven for some class K_1 of searching procedures, which is a subset of class $\{F^2\}$ of all two-step procedures of searching for a rectangle.

Now, let us proceed to the description of this two-step procedure of searching for a rectangle and give an account of the results connected to its efficiency and optimality.

Consider a problem of searching for GESS $\Phi = (\varphi_{pq})_{R_1 \times R_2}$, $\varphi_{pq} = 1$, $p = \overline{1, R_1}$, $q = 1, R_2$ on the raster $u \times v$, and

$$1 < R_1 \le u, \quad 1 < R_2 \le v. \tag{5.8}$$

Let us define a one-step procedure F of searching for Φ:

$$\begin{gathered} c_{ij}^1 = \mathop{\&}_{(p, q) \in M_\Phi} c_{i+p-1, j+q-1}, \\ i = \overline{1, u-R_1+1}, \quad j = \overline{1, v-R_2+1}. \end{gathered} \tag{5.9}$$

The complexity of the procedure F is

$$|F| = (u-R_1+1)(v-R_2+1)(R_1R_2-1). \tag{5.10}$$

Definition 5.8. An operator $f_1(r_1)$ which transfers the matrix $C_{u \times v}$ into the matrix $C_1 = (c_{ij}^1)_{(u-r_1+1)\times v}$, where $c_{ij}^1 = c_{i,j} \cdot c_{i+1,j} \cdot \ldots \cdot c_{i+r_1-1}$, $1 < r_1 \le u$, is called an operator of columns compression.

Definition 5.9. An operator $f_2(r_2)$ which transfers a matrix $C_{u \times v}$ into the matrix $C_2 = (c_{ij}^2)_{u \times (v-r_2+1)}$, where $c_{ij}^2 = c_{i,j} \cdot c_{i,j+1} \cdot \ldots \cdot c_{i+r_2-1}$, $1 < r_2 \le v$, is called an operator of rows compression.

Figure 5.2 shows the example where the operator of columns compression is applied to the matrix C and, then, the operator of rows compression is applied to the obtained result. An example shows that the introduced operators may be used for constructing two-step procedures of searching for a rectangle.

Let us define two two-step operations of searching for Φ by setting

$$F_1 = (\mathrm{f}_1(R_1), \mathrm{f}_2(R_2)), \;\; F_2 = (\mathrm{f}_2(R_2), \mathrm{f}_1(R_1)). \tag{5.11}$$

It is evident that

$$\begin{aligned} |F_1| &= (u - R_1 + 1)v(R_1 - 1) \\ &+ (u - R_1 + 1)(v - R_2 + 1)(R_2 - 1), \end{aligned} \tag{5.12}$$

$$\begin{aligned} |F_2| &= u(v - R_2 + 1)(R_2 - 1) \\ &+ (u - R_1 + 1)(v - R_2 + 1)(R_1 - 1). \end{aligned} \tag{5.13}$$

Note that $|F_2| - |F_1| = (R_1 - 1)(R_2 - 1)(v - R_2 - u + R_1) > 0 \Leftrightarrow v - R_2 > u - R_1$, i.e., for $v - R_2 > u - R_1$, F_1 is a less complex procedure than F_2.

Let $|F_1| < |F_2|$. Then,

$$\begin{aligned} &|F| - |F_1| \\ &= (u - R_1 + 1)(R_1 - 1)(R_2 - 1)(v - R_2). \end{aligned} \tag{5.14}$$

Eq. (5.14) indicates the difference in the computational complexities of the two procedures: the one-step procedure of searching for the rectangle F and the proposed two-step procedure F_1. This difference remains nonnegative even for $|F_1| > |F_2|$.

Now, let us show that for $u - R_1 < v - R_2$ ($u - R_1 > v - R_2$), the procedure F_1 (F_2) is optimal in a certain class $K_1 \subset \{F^2\}$ of the two-step searching procedures.

Let us define this class. Let indices $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_s$: $1 \le i_1 < i_2 < \ldots < i_k \le u_0$, $1 \le j_1 < j_2 < \ldots < j_s \le v_0$ be chosen. A matrix of the size $k \times s$ made up of the elements of the matrix C_0, situated at the intersection of the rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_s$, is a submatrix $M^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_s}$ of the matrix C_0.

Let us fix the integers x and y such that

$$1 \le x \le R_1, \;\; 1 \le y \le R_2, \;\; 1 < xy < R_1 R_2, \tag{5.15}$$

and define a two-step procedure $F^2 = (f_1, f_2)$ in the following way. Let us construct the cover of the submatrix $M^{1, 2, \ldots, R_1}_{1, 2, \ldots, R_2}$ of the matrix C_0 by rectangles $x \times y$ so that it is minimal with respect to the number of the rectangles $x \times y$ used. Each rectangle we associate with a connection in a matrix C_1. In the same manner, we construct the cover of the submatrix $M^{1, 2, \ldots, R_1}_{2, 3, \ldots, R_2 + 1}$ by the rectangles $x \times y$ and define new connections in the matrix C_1. For the rest of the submatrices $M^{i, j+1, \ldots, i + R_1 - 1}_{j, j+1, \ldots, j + R_2 - 1}$ of the matrix C_0, we make the similar construction: the minimal coverings of these submatrices by the rectangles $x \times y$ are performed, and new connections in the matrix C_1 are defined. The function f_1 is now defined.

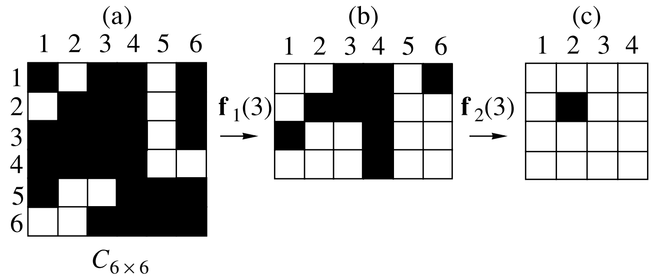

Fig. 5.2. Operators of column and row compression.

Thus,

$$|S(i, j, 1)| = xy, \;\; i = \overline{1, u_1}, \;\; j = \overline{1, v_1}. \tag{5.16}$$

If the inequalities

$$x - 1 \le u - R_1, \;\; y - 1 \le v - R_2 \tag{5.17}$$

are fulfilled, then the matrix C_0 is covered by rectangles $x \times y$ and the number of rectangles is $(u - x + 1)(v - y + 1)$. Therefore, $u_1 v_1 = (u - x + 1)(v - y + 1)$.

The function f_2 is defined in the following way. Connection $S(i, j, 2)$ of the matrix C_2 consists of all positions (and only of them) of the matrix C_1 which are connected with the rectangles $x \times y$ covering the submatrix $M^{i, j+1, \ldots, i + R_1 - 1}_{j, j+1, \ldots, j + R_2 - 1}$ of the matrix C_0.

Hence,

$$\begin{aligned} &|S(i, j, 2)| = (R_1 - x + 1)(R_2 - y + 1), \\ &i = \overline{1, u - R_1 + 1}, \;\; j = \overline{1, v - R_2 + 1}. \end{aligned} \tag{5.18}$$

The procedure $F^2 = (f_1, f_2)$ is completely defined. Integers x and y which satisfy the conditions (5.15) and (5.17) are the parameters of this procedure. Note that if $x = R_1$, $y = 1$, then $F^2 = F_1$ and if $x = 1$, $y = R_2$, then $F^2 = F_2$.

The complexity of the procedure F^2 is

$$\begin{aligned} |F^2| &= |f_1| + |f_2| \\ &= \sum_{i=1}^{u_1} \sum_{j=1}^{v_1} (|S(i, j, 1)| - 1) + \sum_{i=1}^{u_2} \sum_{j=1}^{v_2} (|S(i, j, 2)| - 1) \\ &= (u - x + 1)(v - y + 1)(xy - 1) + (u - R_1 + 1) \\ &\times (v - R_2 + 1)((R_1 - x + 1)(R_2 - y + 1) - 1). \end{aligned} \tag{5.19}$$

The variation of the values of x and y in the range of admissible values defined by conditions (5.15) and (5.17) determines a class K_1 of the two-step procedures of searching for a rectangle $R_1 \times R_2$.

Theorem 5.2. Let $u - R_1 \leq v - R_1$ ($u - R_1 \geq v - R_2$). Then, the procedure with $x = R_1$ and $y = 1$ ($x = 1$ and $y = R_2$) is optimal in the set K_1 of searching procedures.

Proof. Let $u - R_1 \leq v - R_2$. Consider the function $f(x, y) = (u - x + 1)(v - y + 1)(xy - 1) + (u - R_1 + 1)(v - R_2 + 1)((R_1 - x + 1)(R_2 - y + 1) - 1)$ of continuous arguments x and y defined on a set $D_f = [1, R_1] \times [1, R_2] \backslash \{(1, 1), (R_1, R_2)\}$. Let us prove that the point $(R_1, 1)$ is a point of minimum of this function in the area of its definition. Note that

$$\frac{\partial^2 f}{\partial x^2} = -2y(v - y + 1) < 0,$$

$$\frac{\partial^2 f}{\partial y^2} = -2x(u - x + 1) < 0, \quad \forall (x, y) \in D_f.$$

Therefore, at the fixed value of some of its arguments, $f(x, y)$ is a convex function of another argument. Thus, a point of function minimum is among the points $(1, R_2)$, $(1, 2)$, $(2, 1)$, $(R_1, 1)$, $(R_1, R_2 - 1)$, and $(R_1 - 1, R_2)$.

(1) Let us prove that $f(R_1, 1) \leq f(1, R_2)$.

We have

$$f(R_1, 1) = (u - R_1 + 1)v(R_1 - 1)$$
$$+ (u - R_1 + 1)(v - R_2 + 1)(R_2 - 1);$$

$$f(1, R_2) = u(v - R_2 + 1)(R_2 - 1)$$
$$+ (u - R_1 + 1)(v - R_2 + 1)(R_1 - 1).$$

Then,

$$f(1, R_2) - f(R_1, 1)$$
$$= (u - R_1 + 1)(v - R_2 + 1)(R_1 - R_2)$$
$$+ u(v - R_2 + 1)(R_2 - 1) - (u - R_1 + 1)v(R_1 - 1)$$
$$= (u - R_1 + 1)(vR_1 - vR_2$$
$$- (R_2 - 1)(R_1 - R_2) - vR_1 + v)$$
$$+ u(v - R_2 + 1)(R_2 - 1)$$
$$= (u - R_1 + 1)(R_2 - 1)(-v - R_1 + R_2)$$
$$+ u(v - R_2 + 1)(R_2 - 1)$$
$$= (R_2 - 1)(-uv - uR_1 + uR_2 + (R_1 - 1)v$$
$$+ (R_1 - 1)(R_1 - R_2) + uv - uR_2 + u)$$
$$= (R_2 - 1)(R_1 - 1)(v - R_2 - u + R_1) \geq 0$$
$$\Leftrightarrow v - R_2 \geq u - R_1.$$

The last inequality is valid due to the assumption made.

(2) Let us prove that $f(R_1, 1) \leq f(1, 2)$.

Set $R'_1 = R_1 - 1$ and $R'_2 = R_2 - 1$. By virtue of Eq. (5.8), $R'_1 \geq 1$ and $R'_2 \geq 1$. Then,

$$f(R_1, 1) = (u - R'_1)vR'_1 + (u - R'_1)(v - R'_2)R'_2,$$

$$f(1, 2) = u(v - 1) + (u - R'_1)(v - R'_2)(R_1R'_2 - 1).$$

We have

$$f(1, 2) - f(R_1, 1) \geq u(u - R'_1 + R'_2 - 1)$$
$$+ (u - R'_1)^2(R'_1R'_2 - 1) - (u - R'_1)(u - R'_1 + R'_2)R'_1$$
$$= u^2 - uR'_1 + uR'_2 - u + u^2R'_1R'_2 - u^2 - 2uR'^2_1R'_2$$
$$+ 2uR'_1 + R'^3_1R'_2 - R'^2_1 - u^2R'_1 + 2uR'^2_1$$
$$- R'^3_1 - uR'_1R'_2 + R'^2_1R'_2$$
$$= R'_2(u + u^2R'_1 - 2uR'^2_1 + R'^3_1 - uR'_1 + R'^2_1)$$
$$- uR'_1 - u + 2uR'_1 - R'^2_1 - u^2R'_1 + 2uR'^2_1 - R'^3_1.$$

The last expression is a linear function $g(R'_2)$ of the argument R'_2. Let us show that, for all admissible values of the arguments R'_2 and parameters u and R'_1, this function takes nonnegative values. To do that, let us show that, for all admissible u and R'_1, the coefficient $u + u^2R'_1 - 2uR'^2_1 + R'^3_1 - uR'_1 + R'^2_1$ of the argument of linear function is positive and the function's value at the point $R'_2 = 1$ is nonnegative.

We have

$$u + u^2R'_1 - 2uR'^2_1 + R'^3_1 - uR'_1 + R'^2_1$$
$$> u^2R'_1 - 2uR'^2_1 + R'^3_1 - uR'_1 + R'^2_1.$$

Dividing the last expression by $R'_1 > 0$, we obtain $u^2 - u(1 + 2R'_1) + R'^2_1 + R'_1$. The last expression is a quadratic trinomial relative to u; it takes nonnegative values for $u \geq 1 + R'_1 = R_1$, which is valid for any allowable values of u and R'_1.

Here,

$$g(1) = u + u^2R'_1 - 2uR'^2_1 + R'^3_1 - uR'_1 + R'^2_1$$
$$- uR'_1 - u + 2uR'_1 - R'^2_1 - u^2R'_1 + 2uR'^2_1 - R'^3_1 = 0.$$

(3) Let us prove that $f(R_1, 1) \leq f(2, 1)$.

We have

$$f(2, 1) = (u - 1)v + (u - R'_1)(v - R'_2)(R'_1R_2 - 1);$$

$$f(2, 1) - f(R_1, 1) = (u - 1)v + (u - R'_1)(v - R'_2) \tag{5.20}$$
$$\times R_2(R'_1 - 1) - (u - R'_1)vR'_1.$$

Then,

$$(u - R'_1)(v - R'_2)R_2(R'_1 - 1)$$
$$= ((u - R'_1)v - (u - R'_1)R'_2)R_2(R'_1 - 1)$$
$$= (u - R'_1)vR_2(R'_1 - 1) - (u - R'_1)R'_2R_2(R'_1 - 1)$$
$$= (u - R'_1)vR'_1R_2 - (u - R'_1)$$
$$\times R_2(v + R'_2(R'_1 - 1))$$
$$= (u - R'_1)vR'_1 + (u - R'_1)vR'_1R'_2$$
$$-(u - R'_1)R_2(v + R'_2(R'_1 - 1)).$$

By substituting the last expression into the Eq. (5.20) and by reducing similar terms, we obtain

$$f(2, 1) - f(R_1, 1) = (u - 1)v + (u - R'_1)vR'_1R'_2$$
$$-(u - R'_1)(R'_2 + 1)(v + (R'_1 - 1)R'_2)$$
$$= uv - v + uvR'_1R'_2 - vR'^2_1R'_2 - uvR'_2$$
$$-uR'_1R'^2_2 + uR'^2_2$$
$$-uv - \underline{uR'_1R'_2} + uR'_2 + \underline{vR'_1R'_2} + R'^2_1R'^2_2 - R'_1R'^2_2$$
$$+ vR'_1 + R'^2_1R'_2 - R'_1R'_2.$$

Estimating the expression $R'_1R'_2(v - u)$ from below by the expression $R'_1R'_2(R'_2 - R'_1)$ (this estimation is valid by virtue of the supposition we made in the beginning of the proof) and reducing similar terms, we obtain

$$f(2, 1) - f(R_1, 1)$$
$$\geq u(vR'_1R'_2 - vR'_2 - R'_1R'^2_2 + R'^2_2 + R'_2)$$
$$- v - vR'^2_1R'_2 + R'^2_1R'^2_2 + vR'_1 - R'_1R'_2.$$

The right-hand part of the inequality is a linear function g(u). Let us show that g(u) ≥ 0 for all admissible values of the argument u and the parameters v, R'_1, and R'_2. To do that, let us show that for all admissible v, R'_1, and R'_2, the coefficient $vR'_1R'_2 - vR'_2 - R'_1R'^2_2 + R'^2_2 + R'_2$ of the argument of linear function is positive and the function's value at the point $u = R_1 = R'_1 + 1$ is nonnegative.

Indeed,

$$vR'_1R'_2 - vR'_2 - R'_1R'^2_2 + R'^2_2 + R'_2$$
$$= R'_2((v - R'_2)(R'_1 - 1) + 1) > 0.$$

In addition,

$$g(R'_1 + 1) = vR'_1 + R'_2 + R'^2_2 - v(R'_2 + 1)$$
$$= v(R'_1 - R'_2 - 1) + R'_2 + R'^2_2$$
$$\geq (u - R'_1 + R'_2)(R'_1 - R'_2 - 1) + R'_2 + R'^2_2$$
$$= (u - R'_1)(R'_1 - R'_2 - 1) + R'_1R'_2$$
$$\geq R'_1 - R'_2 - 1 + R'_1R'_2 = (R'_1 - 1)(R'_2 + 1) \geq 0.$$

(4) Let us prove that $f(R_1, 1) \leq f(R_1, R_2 - 1)$.

We have

$$f(R_1, R_2 - 1) = (u - R'_1)(v - R'_2 + 1)(R_1R'_2 - 1)$$
$$+ (u - R'_1)(v - R'_2).$$
$$f(R_1, R_2 - 1) - \mathrm{f}(R_1, 1)$$
$$= (u - R'_1)((v - R'_2 + 1)(R_1R'_2 - 1)$$
$$+ (v - R'_2) - vR'_1 - (v - R'_2)R'_2) \vee 0|: \ u - R'_1 \geq 1.$$

By opening the brackets and by reducing similar terms in the last expression, we obtain

$$R_1(vR_2 - 2v - R^2_2 + 3R_2 - 2)$$
$$+ 2v - vR_2 + R^2_2 - 2R_2 \vee 0.$$

The expression compared to zero is a linear function $g(R_1)$ of the argument R_1. Let us show that $g(R_1) \geq 0$ for all admissible values of the argument R_1 and parameters v and R_2. To do that, let us show that for all admissible v and R_2, the coefficient $vR_2 - 2v - R^2_2 + 3R_2 - 2$ of the argument of the linear function and the function's value at the point $R_1 = 2$ are nonnegative.

The expression $v(R_2 - 2) - R^2_2 + 3R_2 - 2$ is itself a linear function $g_1(v)$; in addition, $R_2 - 2 \geq 0$ and $g_1(R_2) = R_2 - 2 \geq 0$. Here, $g(2) \geq g(1) = R_2 - 2 \geq 0$.

(5) Let us prove that $f(R_1, 1) \leq f(R_1 - 1, R_2)$.

We have

$$f(R_1 - 1, R_2) = (u - R'_1 + 1)$$
$$\times (v - R'_2)(R'_1R_2 - 1) + (u - R'_1)(v - R'_2).$$
$$f(R_1 - 1, R_2) - f(R_1, 1)$$
$$= (u - R'_1 + 1)(v - R'_2)(R'_1R_2 - 1)$$
$$+ (u - R'_1)(v - R'_2) - (u - R'_1)vR'_1$$
$$-(u - R'_1)(v - R'_2)R'_2.$$

By opening the brackets, using the inequality $R'_1R'_2(v - u) \geq R'_1R'_2(R'_2 - R'_1)$, and by reducing similar terms in the last expression, we obtain

$$f(R_1 - 1, R_2) - f(R_1, 1)$$

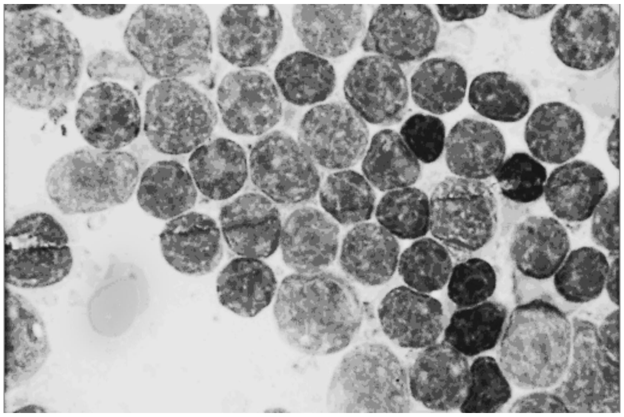

Fig. 6.1. Monochrome photomicrographic image of the preparations of the lymphatic gland.

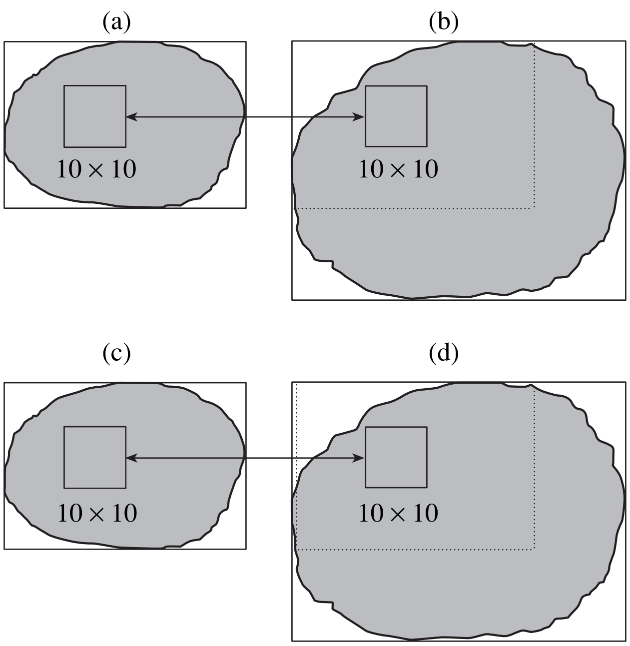

Fig. 6.2. Matching pair of nuclei.

$$\geq uvR'_1R'_2 - uR'_1R'^2_2 - vR'^2_1R'_2 + R'^2_1R'^2_2 + \underline{vR'_1 - v}$$

$$- R'_1R'_2 + R'_2 - uvR'_2 + uR'^2_2 + vR'_1R'_2 - R'_1R'^2_2$$

$$\geq R'_2(uvR'_1 - uR'_1R'_2 - vR'^2_1 + R'^2_1R'_2 - R'_1$$

$$+ 1 - uv + uR'_2 + vR'_1 - R'_1R'_2) \vee 0|: R'_2 \geq 1;$$

$$v(uR'_1 - R'^2_1 - u + R'_1) - uR'_1R'_2$$

$$+ R'^2_1R'_2 - R'_1 + 1 + uR'_2 - R'_1R'_2 \vee 0.$$

The expression compared to zero is a linear function $g(v)$ of the argument v. Let us show that $g(v) \geq 0$ for all admissible values of the argument v and the parameters u, R'_1 and R'_2. To do that, let us show that for all admissible u, R'_1 and R'_2, the coefficient $uR'_1 - R'^2_1 - u + R'_1$ of the argument of the linear function and the function's value at the point $v = R_2 = R'_2 + 1$ are nonnegative:

$$uR'_1 - R'^2_1 - u + R'_1 = (u - R'_1)(R'_1 - 1) \geq 0;$$

$$g(R'_2 + 1) = uR'_1R'_2 + uR'_1 - R'^2_1R'_2 - R'^2_1 - uR'_2 - u$$

$$+ R'_1R'_2 + R'_1 - uR'_1R'_2 + R'^2_1R'_2 - R'_1 + 1 + uR'_2 - R'_1R'_2$$

$$= u(R'_1 - 1) - (R'^2_1 - 1)$$

$$= (R'_1 - 1)(u - R'_1 - 1) = (R'_1 - 1)(u - R_1) \geq 0.$$

The theorem is proven.

The searching procedure may be considered as an implementation of a morphological operation "erosion by the structuring element Φ" and, hence, an efficient procedure of searching for GESS Φ is an efficient implementation of a morphological operation "erosion by the structuring element Φ." In addition, for several types of GESS Φ, inversion of the searching procedure allows one to perform the morphological operation of "dilation by the structuring element Φ."

6. APPLICATION EXAMPLE OF THE INTRODUCED SUBCLASS OF 2D-AEC: CLASSIFICATION OF HEMOBLASTOSES

The proposed subclass of 2D-AEC with rectangular support sets outlined here was used for compiling an efficient algorithm with rectangular support sets. This algorithm was employed for estimating initial data in automated diagnosis of malignant growths in a human hematogenic system (RFBR project no. 00-07-9004 "Knowledge-oriented system of automation of scientific research in the area of morphology of blood-cells and hematogenic organs").

The photomicrographic images of nuclei of lymphocytes in the preparations of lymphatic tissues of the patients were the objects of our analysis. The photomicrographic images were obtained by the digital photo camera with the lens ×100 combined with the microscope; they were represented as 24-bite files in TIFF of 1500 × 1000 pixels in size. Each of the pictures contained 3–40 nuclei of lymphocytes. Figure 6.1 shows the reduced monochrome picture of the slide of lymphatic gland. This picture was additionally processed to enhance contrast. The dark rounded regions against the light background including the light streaks are the nuclei of lymphocytes.

The hematologists indicated the nuclei suitable for analysis (i.e., not smeared, without artifacts, uniformly colored, etc.). The learning sample was formed of 639 of such nuclei taken from 107 preparations (ten patients). Each patient was classified into one of the three classes: malignant growth, nonmalignant growth, and reactive lymphatic gland.

To preliminarily estimate the quality of the source material, first, we estimate the similarity of the cells for different patients and different classes of patients.

The preprocessing of images compensates for different illumination conditions and different colors of stain used for the preparations. After that, the monochrome images with 256 intensity levels were obtained. In these images, each nucleus from the learning sample was matched with the rest of the nuclei from the learning sample over all possible local neighborhoods of the rectangular shape of 10×10 pixel sizes.

Figure 6.2. exemplifies the process of nuclei matching. A nucleus extracted in the monochrome image was placed against some background and treated as a rectangular image with dimensions similar to those of the circumscribed rectangle (see gray schematic images of the nuclei against the white background). The sizes of rectangular images constructed for different nuclei may differ, as is seen in Fig. 6.2. In the case depicted in Fig. 6.2, first, the image of the nucleus S (Fig. 6.2a) is matched with the fragment of the image of the nucleus S' isolated by the dotted line (Fig. 6.2b). In the image (a) and in the separated image fragment (b), it is necessary to search for all possible square neighborhoods of 10×10 pixel sizes (one of these neighborhoods is depicted as a square) and to calculate the value of the proximity function $B_{\hat{\omega}}(S, S', Fr)$ for each of these neighborhoods (it is coded by a matrix $\hat{\omega}$). Here, the proximity function is written with the additional argument Fr, which indicates that it is necessary to take account of the separate fragments of the image (b) of the nucleus S'. Apart from the value of proximity function $B_{\hat{\omega}}(S, S', Fr)$, it is necessary to calculate the number of pixels $EffArea_{\hat{\omega}}(S, S', Fr)$ which have the brightness level different from the level of background both in the first and second images.

In the same manner, at the following steps of processing, the remaining rectangular fragments of the image of the nucleus S' are searched and matched with the image of the nucleus S over all possible neighborhoods of 10×10 pixels. For example, at the second step, one can choose a fragment of the image of nucleus S' bounded by a rectangle shifted by one pixel to the right with respect to the first fragment (see Figs. 6.2c and 6.2d); at the third step, a fragment bounded by a rectangle shifted by one pixel to the right with respect to the second fragment, etc. When the fragment in the upmost right corner of the image of the nucleus S' is considered, the fragment bounded by a rectangle shifted by one pixel down with respect to the first fragment is chosen, etc.

After matching all fragments of the image (b) of the nucleus S' with the image (a) of the nucleus S has been performed, the value of expression

$$\sum_{Fr} \sum_{\hat{\omega} \in \Omega_A} \frac{B_{\hat{\omega}}(S, S', Fr)}{EffArea_{\hat{\omega}}(S, S', Fr)} \tag{6.1}$$

is calculated.

For nuclei S' from the learning set and fixed nucleus S, vector $\Gamma(S) = (\Gamma_1(S), \Gamma_2(S), \ldots, \Gamma_{10}(S))$ of the estimates of the nucleus S for different patients is calculated. Here, the estimate $\Gamma_j(S)$ of the nucleus S for the jth patient is determined by the formula

$$\Gamma_j(S) = \frac{1}{|W_j|} \sum_{\substack{S' \in W_j \\ S' \neq S}} \sum_{Fr} \sum_{\hat{\omega} \in \Omega_A} \frac{B_{\hat{\omega}}(S, S', Fr)}{EffArea_{\hat{\omega}}(S, S', Fr)}, \tag{6.2}$$

where W_j is a set of nuclei of the jth patient; $EffArea_{\hat{\omega}}(S, S', Fr)$ is the number of pixels for matching nuclei S and S'; Ω_A is a system of support sets: a totality of 10×10 squares; $B_{\hat{\omega}}(S, S', Fr)$ is a proximity function (4.1) with $\varepsilon = 0$ and $\varepsilon_{ij} = 8$ for all i and j.

The value of the sum

$$\sum_{\hat{\omega} \in \Omega_A} B_{\hat{\omega}}(S, S', Fr) \tag{6.3}$$

was estimated by the following procedure of searching for the square (further, we will omit the additional argument Fr):

(1) a characteristic proximity matrix $C(S, S')$ was calculated;

(2) a matrix $C_1 = f_1(C)$ was constructed (or $C_1 = f_2(C)$ which depends on the ratio between the image lengths S, S' and the image heights r_1 and r_2);

(3) a matrix $C_2 = f_2(C_1)$ ($C_2 = f_1(C_1)$) was constructed.

The value of the sought sum was equal to the number of unities in the matrix C_2.

The results obtained testify that (i) the initial data were substantially heterogeneous and (ii) a learning set should be enlarged by adding new precedents from the existing three classes and from the new class of “norm.” This means that data presented cannot serve as a basis for a reliable detection of diagnostic features of human hematogenic system. In practice, it demands a substantial extension of the initial data.

CONCLUSIONS

The problem of the construction of efficient algorithm in one subclass of 2D-AEC is equivalent to the problem of the construction of efficient procedure of searching for GESS in the binary raster. We proposed a formalism describing the multistep procedures of searching for solutions to these problems and a criterion of efficiency for searching procedures related to their computational complexity.

A large number of GESS generating the systems of support sets which make sense in pattern recognition can be represented by the union of the rectangles. Therefore, our primary task is a construction of the optimal procedure of searching for a rectangle. Here, we propose an efficient two-step procedure of searching for a rectangle and prove its optimality in a subclass of all two-step procedures of searching for a rectangle (Theorem 5.2).

The constructed two-step procedure of searching for a rectangle served as a basis of the efficient algorithm of the 2D-AEC class which was successfully used for analysis of initial data in the problem of hemoblastoses classification from the images of lymphocyte nuclei in the histological preparations.

The introduced approach to the construction of the efficient algorithm of the 2D-AEC class can be generalized to the case where the three-dimensional matrices are the object descriptions and the support sets are the aggregates of the three-dimensional indices.

In the second part of this work, we plan to investigate the possibility of constructing optimal procedures of searching for a rectangle as well as efficient parallel schemes of searching for arbitrary GESS constructed from rectangles. The obtaining of the upper and lower complexity estimates for the procedures of searching for GESS, in particular, rectangular GESS, is also a challenging problem. In the future research, we suppose to construct efficient procedures for other spatial systems of support sets.

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