Operations of Descriptive Image Algebras with One Ring

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Abstract—The work is devoted to the description of a new class of image algebras—descriptive image algebras (DIA). These algebras are intended for the structural description of possible algorithms for image analysis and understanding. Definitions of DIA and basic DIA are introduced. The choice of the algebra for refinement of the concept of DIA with one ring is discussed. Examples of operations, both resulting and not resulting in construction of DIA with one ring, are presented. Possible interpretations of operations of DIA are considered. By results of investigation of the standard Ritter's image algebra used in construction of DIA with one ring are formulated. An illustrative example of an algorithmic scheme is described with the help of DIA.

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INTRODUCTION

In this paper, a new mathematical object, namely, descriptive image algebras (DIAs), is described. This object is studied in developing a mathematical apparatus for analysis and estimation of information represented in the form of images. For a structural description of possible algorithms for solving these problems, we need a formal instrument that allows us to describe and justify the chosen way of solution. As formalization tools, we chose the algebraic approach, which should provide a unique form of procedures for describing the objects–images and transformations of these objects–images.

The need to develop a mathematical language that ensures that solutions of problems of image processing, analysis, and understanding may be uniformly described by structural algorithmic schemes is justified by the following factors:

(1) there are many algorithms (designed and introduced into practice) for analysis, estimation, and understanding of information represented in the form of images;

(2) the set of algorithms is neither structured nor ordered;

(3) as a rule, methods for image analysis and understanding are designed on the basis of intuitive principles, because the information represented in the form of images is hardly formalized;

(4) the efficiency of these methods is estimated (as is usual in experimental sciences) by the success in solving actual problems—as a rule, the problem of rigorous mathematical justification of an algorithm is not considered.

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1. ALGEBRAIZATION

1.1. Justification of Actuality of Algebraization

"Algebraization" is one of the most topical and promising directions of fundamental research in image analysis and understanding. The main goal of the algebraic approach is the development of a theoretical basis for representations and transformations of images in the form of algebraic structures that enable one to use methods from different areas of mathematics in image analysis and understanding. The idea of constructing a unified theory for different concepts and operations employed in image and signal processing was first put forward by Unger [29], who suggested parallelizing algorithms for image processing and analysis on computers with cellular architecture. The basis for the development of parallel architectures for image processing arose from the idea that a wide class of image transformations may be described by a small set of standard rules.

The general foundation of the algebraic approach to synthesis and analysis of recognition algorithms was laid down in papers by Yu.I. Zhuravlev [5, 6]. The Zhuravlev algebra is a tool for realization of an algebraic approach to pattern recognition and is used for systematization of separated heuristic algorithms, each one designed for solving a classification problem. The Zhuravlev algebra is intended to solve problems with incompletely formalized and partially contradictory data. The idea of the algebraic approach to problems of recognition and classification is the following. There are no rigorous mathematical models for weakly formalized sciences such as geology, biology, medicine, and sociology. However, in many cases, nonstrict methods based on heuristic arguments are very efficient in applications. Therefore, it is sufficient to construct a family of such heuristic algorithms for solving appropriate problems and, then, to construct the algebraic closure of this family, which contains the required solution.

The problem of image understanding is a classical example of a problem with incompletely formalized and partially contradictory information. This allows us to hope that the application of the algebraic approach to image recognition may result in significant achievements.

Image analysis and understanding have a certain peculiarity, due to which the use of the Zhuravlev algebraic approach in the general form is inconvenient. The reasons are the following:

(1) the character of the considered problem is not taken into account if algebraic methods are applied to the information represented in the form of images;

(2) the results of application of the theory cannot always be simply interpreted;

(3) there are many natural transformations of images which are easily interpreted from the user's point of view (for instance, rotation, contraction, stretching, color inversion, etc.) but are hardly representable by standard algebraic operations.

The necessity arises of using algebraic tools to record natural transformations of images. Moreover, the algebraization of image analysis and understanding must include the construction of algebraic descriptions of both the images themselves and algorithms for their processing, analysis, and recognition.

Analyzing the publications related to applications of algebraic methods to image analysis and understanding, we distinguish the following advantages of unified representation of images and algorithms for their processing and analysis:

(1) construction of unified representations for descriptions of images;

(2) efficiency of transition from input data in the form of images to different formal models of the images;

(3) naturalness of uniting the algebraic representation of the information with the developed algebraic tools for pattern recognition, which has been successfully employed;

(4) the possibility of using the methods of mathematical modeling employed in applied domains to which the processed images belong;

(5) the possibility of using the image descriptions in the form of group-theoretic representations;

(6) naturalness of uniting the methods of structural analysis of images with tools of probabilistic analysis;

(7) the possibility of a formalized description for problems of parallelizing with due regard for the specifics of particular computational architectures.

1.2. Survey of Image Algebras

To a great extent, attempts to develop a formal apparatus for uniform and compact representation of procedures for image processing and analysis are stem from applied requirements for effective realization of algorithmic tools for image processing and analysis on computers with specialized architectures, in particular, cellular and parallel.

Mathematical morphology [26], proposed by Minkowski and Hadwiger and developed by Matheron and Serra, seems to be the first attempt to create a theoretical apparatus that allows one to describe many widespread operations of image processing in the composition of a rather small set of standard simple local operations. Such representations allow one to formalize the choice of procedures for image processing and are convenient for implementation on parallel architectures. It might have been the success of mathematical morphology that initiated numerous attempts of algebraization both in the domain of algorithm representations and in closed domains. Mathematical morphology is an efficient tool for uniform representation of local operations of image processing, analysis, and understanding in terms of algebras over sets. It makes it possible to describe algorithms for image transformations in terms of four basic local operations, namely, those of erosion, dilatation, opening, and closing; moreover, any two of these operations form a basis, in terms of which the other two operations may easily be expressed. This is very convenient for the development of software systems, in which the user can quickly design particular algorithms from basic blocks.

On the basis of mathematical morphology, Sternberg [27] introduced the concept of an image algebra. The image algebra made it possible to represent algorithms for image processing in the form of algebraic expressions, where variables are images and operations are geometrical and logical transformations of the images.

It is known that the possibilities of mathematical morphology are very limited. In particular, many important and widely used operations of image processing (feature extraction based on the convolution operation, Fourier transforms, use of the chain code, equalization of a histogram, rotations, recording, and nose elimination), except for the simplest cases, can hardly (if ever) be realized in the class of morphological operations.

The impossibility of constructing a universal algebra for tasks of image processing on the basis of the morphological algebra may be explained by the limitation of the basis consisting of the set-theoretical operations of addition and subtraction in Minkowski's sense. It is known that this basis has the following drawbacks [25]:

(1) complicated realization of widely used operations of image processing;

(2) impossibility of establishing a correspondence between the operations of mathematical morphology and linear algebra;

(3) impossibility of using mathematical morphology for transformations between different algebraic struc-

tures, in particular, sets including real and complex numbers and vector quantities.

These problems have been solved in the standard image algebra (IA) by G. Ritter [25] on the basis of a more general algebraic representation of operations of image processing and analysis. Image algebra generalizes the known local methods for image analysis, in particular, mathematical morphology, and provides the following advantages as compared with mathematical morphology:

(1) it makes it possible to work with both real and complex quantities;

(2) it allows one to include both scalar and vector data into the input information;

(3) it makes image-algebra structures consistent with linear structures;

(4) it provides a more accurate and complete description of its operations and operands;

(5) with the help of a special structure "template," composite operations of image processing are divided into a number of parallel simplest operations.[1]

The bottleneck in applications of methods of image algebra to image recognition is the choice of the sequence of algebraic operations and templates for representation of composite operations of image processing. At present, this choice is based, as a rule, on general representations of the character of images and tasks. Deficiencies of this approach are obvious: first, it is subjective and its success depends to a great extent on the user's experience and, second, it is intended to solve a specific narrow class of problems.

The most general approach to the algebraic description of information for recognition algorithms is Grenander's general pattern theory [2, 13], which unites metric theory with probability theory for certain universal algebras of combinatorial type. The main attention is given to the investigation of the structure of recognizing elements. The idea that underlies Grenander's theory is that knowledge about patterns may be expressed in terms of regular structures.[2] The theory is based on three principles, namely, atomism, combinatority, and observability. By atomism, we mean that the structures are composed of certain basic elements. Combinatority means that explicit rules are formulated for definition of admitted and prohibited structures. The third principle is related to the search for identification rules for determining equivalence classes. It should be noted that Grenander used the notion of image algebra: however, he was dealing with a different algebraic construction.

1.3. Descriptive Approach to Image Analysis and Understanding

Despite the existence of a series of significant publications in the area of algebraization of problems of image processing, understanding, and analysis and image evaluation, we can state that, at present, there is no unique theoretical frame for problems in this enterprise. In the late 1980s and in 1990s, I.B. Gurevich [3, 14, 15] specialized for the first time a general algebraic approach to solving problems of recognition, classification, and prediction [5, 6] to the case of representation of the initial data in the form of images—the descriptive approach to image analysis and understanding.

This approach has been developed for formulation and solution of the following problems:

(1) problems related to obtaining a formal description of a recognition object;

(2) problems related to developing procedures for image analysis and understanding.

These problems belong to the descriptive theory of image analysis, because they are connected with the study of the internal structure of images depending on the operations by which the image may be obtained from other images and objects of a simpler nature. This is why the word "descriptive" appears in the name of this approach. The key problems of the descriptive theory of image analysis are image models and transformations defined on equivalence classes of images. Thus, the realization of the descriptive approach is the special descriptive theory of image analysis.

In the descriptive approach, the following mathematical foundations for solving problems of image analysis and understanding may be distinguished:

(1) specialization of the Zhuravlev algebra to the case of image understanding;

(2) standardization of representation of problems of image analysis and understanding;

(3) standardization of the language for description of procedures for image analysis and understanding;

(4) application of algebraic tools to transformations of algorithms for image analysis and understanding and of image models.

Thus, the algebraic formalism should provide the following possibilities:

(1) construction of algebraic structures such that methods from other areas of mathematics may be used in image processing, analysis, and understanding;

(2) construction of precise and compact descriptions of images, which are convenient both from the point of view of interpreting the actions performed and from the point of view of developing of new methods;

(3) creation of a language for a standard description of image transformations;

(4) description of operations on images in the form of a compact collections of simple transformations.

[1] Let F be a homogeneous algebra and X a topological space. By a template, we mean an image whose set of values is a set of images. In particular, an F-valued template from a set Y into set X is a function $t: Y \longrightarrow F^X$; i.e., t is an F^X-valued image on set Y. An image in Ritter's sense (F-valued image on set X) is a mapping from set X into set F: $I = \{(x, a(x)), x \in X, a(x) \in F\}$ [25].

[2] Regular structures are structures constructed by certain rules.

In the general case, such an algebraic formalism should be a formal system for image representations and transformations, which satisfy the following conditions:

(1) each object of transformations is a hierarchical structure constructed of elementary objects with the help of operations of the image algebra;

(2) as the objects, one may use points, sets, models, operations, and morphisms;

(3) each transformation is a hierarchical structure constructed of a collection of basic transformations with the help of operations of the image algebra.

1.4. Descriptive Image Algebras

Investigations in the area of algebraization and image analysis (see Subsections 2.2, 2.3) of the 1970–1980s represent a source of development of the descriptive image algebra (DIA) [4, 16, 18, 19, 20, 21, 22, 23].

An object that lies most closely to the developed mathematical object is the image algebra proposed and developed by Ritter [24]. Ritter's main goal in developing the image algebra is the design of a standardized language for description of algorithms for image processing intended for parallel execution of operations. A key difference in the new image algebra from the standard Ritter image algebra is that DIA is developed as a descriptive tool, i.e., as a language for description of algorithms and images rather than a language for algorithm parallelizing.

The conceptual difference of the algebra under development from the standard image algebra is that objects of this algebra are (along with algorithms) descriptions of input information. DIA generalizes the standard image algebra and allows one to use (as ring elements) basic models of images and operations on images or the models and operations simultaneously. In the general case, a DIA is the direct sum of rings whose elements may be images, image models, operations on images, and morphisms. As operations, we may use both standard algebraic operations and specialized operations of image processing and transformations represented in an algebraic form. In more detail, the definition of the standard image algebra and that of DIA are considered below (Subsection 3.2 *Image Algebras*).

To use DIA actively, it is necessary to investigate its possibilities and to attempt to unite all possible algebraic approaches, for instance, to use the standard image algebra as a convenient tool for recording certain algorithms for image processing and understanding or to use Grenander's concepts for representation of input information.

In the present paper, the main attention is given to DIAs with one ring, which form the main subclass of basic DIAs. We consider the possibility of using the standard image algebra without the template notion in constructing the DIA. In future, we are going to consider DIAs based on superalgebras (see Definition 7) and investigate other possibilities of application of other algebraic concepts in the theory being developed.

2. PLACE OF DESCRIPTIVE IMAGE ALGEBRAS AMONG OTHER KNOWN IMAGE ALGEBRAS

2.1. Algebras

Recall the definition of an algebra and a multivalued algebra, which are needed below in order to define the image algebras.

Modern algebra, rooted in works of Hilbert, was formed, in general, in the 1920s. Algebra studies sets endowed with specified algebraic operations, more exactly, the operations themselves. Up to the mid-1930s, only a few types of such sets were systematically studied. These sets (groups, rings, and vector spaces) were inherited from algebra of XIX century (see the monograph by van der Waerden [1]).

Definition 1 [1]. *An algebra* is a ring U endowed with a structure of a finite-dimensional vector space over a field.

Definition 2 [1]. *A ring* is a system with a double composition such that the operations on elements of this system satisfy the following rules:

I. Rules of addition:

(1) Associative rule: $a + (b + c) = (a + b) + c$.

(2) Commutative rule: $a + b = b + a$.

(3) Solvability of the equation $a + x = b$ for any a, b.

II. Rules of multiplication:

(1) Associative rule: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

III. Distributive rules:

(1) $a \cdot (b + c) = a \cdot b + a \cdot c$;

(2) $(b + c) \cdot a = b \cdot a + c \cdot a$.

If the multiplication satisfies commutative rule II (b) $a \cdot b = b \cdot a$, then the ring is called *commutative*.

By a system with a double composition, we mean an arbitrary set of elements a, b, …, in which, for any two elements a and b, their sum $a + b$ and product $a \cdot b$ are uniquely defined and belong to the same set.

Definition 3 [1]. A ring is a *skewfield* (a division ring) if:

(1) it contains at least one element different from zero;

(2) the equations

$$\left.\begin{aligned} a \cdot x &= b, \\ y \cdot a &= b \end{aligned}\right\} \quad (1)$$

are solvable for any $a \neq 0$.

If, moreover, the ring is commutative, then it is a *field* or a *rational ring*.

Definition 4 [1]. Suppose that the following are given: (1) a skewfield K whose elements a, b, … are referred to as coefficients or scalars; (2) an additive Abelian group M whose elements x, y, … are referred to as vectors; and (3) a multiplication $x \cdot a$ of vectors by scalars such that

(1) $x \cdot a$ belongs to M;

(2) $(x + y) \cdot a = x \cdot a + y \cdot a$;

(3) $x \cdot (a + b) = x \cdot a + x \cdot b$;

(4) $x \cdot (a \cdot b) = (x \cdot a) \cdot b$;

(5) $x \cdot 1 = x$.

In this case, M is a *vector space over* K; more exactly, a right K-vector space, because coefficients a stands to the right of the vectors. The notion of a left K-vector space is introduced similarly. In the case of a commutative skewfield K, these concepts coincide.

Thus, any algebra possesses the following properties.

Properties of an algebra:

Properties of a field $P(\alpha, \beta, \gamma \in P)$

(1) $\forall\alpha, \beta \in P, \exists!(\alpha + \beta) \in P$

(a) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$;

(b) $\alpha + \beta = \beta + \alpha$;

(c) $\exists 0 \in P, \forall\alpha \in P, \alpha + 0 = \alpha$;

(d) $\forall\alpha \in P, \exists(-\alpha), \alpha + (-\alpha) = 0$;

(2) $\forall\alpha, \beta \in P, \exists!(\alpha \cdot \beta) \in P$;

(a) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$;

(b) $\alpha \cdot \beta = \beta \cdot \alpha$;

(c) $\exists 1 \in P, \forall\alpha \in P, 1 \cdot \alpha = \alpha$;

(d) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$

(this is a property of a ring being a field of vectors)

Properties of a ring $U(a, b, c \in U)$

(1) $\forall a, b \in U, \exists!(a + b) \in U$;

(a) $a + (b + c) = (a + b) + c$;

(b) $a + b = b + a$;

(c) $\exists 0 \in U, \forall a \in U, a + 0 = a$;

(d) $\forall a \in U, \exists(-a), a + (-a) = 0$;

(2) $\forall a, b \in U, \exists!(a \cdot b) \in U$;

(a) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;

(b) $(\alpha \cdot a + \beta \cdot b) \cdot c = \alpha \cdot a \cdot c + \beta \cdot b \cdot c$;

Properties of a vector space $(\forall\alpha \in P, a \in U: \alpha \cdot a \in U)$

(1) $\alpha \cdot (\beta \cdot a) = (\alpha \cdot \beta) \cdot a$;

(2) $(\alpha + \beta) \cdot a = \alpha \cdot a + \beta \cdot a$;

(3) $\alpha \cdot (a + b) = \alpha \cdot a + \alpha \cdot b$;

The next stage in algebra began in the mid-1930s, when Birkhoff [12] began a study of arbitrary universal algebras.

Definition 5 [12]. *An algebra* (in the sense of Lipson and Birkhoff) is a system $A = [\Phi, F]$ such that

(1) $\Phi = \{S_i\}$ is a nonempty set of different types of elements. Each type is referred to as a type of algebra A. The sets S_i are indexed by a set I, i.e., $S_i \in \Phi$ for $i \in I$.

(2) $F = \{f_\alpha\}$ is a set of finitary operations; each operation f_α is a mapping f_α:

$$S_{i(1,\alpha)} \times S_{i(2,\alpha)} \times \ldots \times S_{i(n(\alpha),\alpha)} \longrightarrow S_{r(\alpha)} \qquad (2)$$

for a nonnegative integer-valued function $n(\alpha)$, where i_α: $k \longrightarrow i(k, \alpha)$ are functions from $n(\alpha) = \{1, 2, \ldots, n(\alpha)\}$ into the set I and $r(\alpha) \in I$. Operations f_α are indexed by a set Ω, i.e., $f_\alpha \in F$ for $\alpha \in \Omega$.

Another complex generalization of algebras is represented by graded algebras, in particular, Z_2-graded algebras (superalgebras).

Definition 6 [7]. *A graded algebra* is an algebra A whose additive group is represented in the form of a (weak) direct sum of groups A_i, $i = 0, 1, 2, \ldots$, such that $A_iA_j \subseteq A_{i+j}$ for any i, j. The additive group of a graded algebra (considered as a module over the ring of integers) is a positively graded module.

The following is an example of a graded algebra: the algebra $A = F[x]$ of polynomials over a field F, where A_i is the subspace generated by monomials of degree i $(A_0 = F)$. $A = \sum_{i \geq 0} A_i$.

Definition 7 [8]. *A superalgebra* is a Z_2-graded algebra, i.e., a superspace A over K endowed with an even linear mapping $A \otimes A \longrightarrow A$. A *superspace* is a k-vector space endowed with a Z_2-grading $V = V_{\bar{0}} \oplus V_{\bar{1}}$.

A.I. Mal'tsev [9] laid the foundation of another general theory that borders on mathematical logics, namely, the theory of algebraic systems.

2.2. Image Algebra

At the present time, by image algebra, we mean a mathematical theory describing image transformations and analysis in continuous and discrete domains. Initially, image algebra was understood as a tool for description of image transformations created for the convenience of parallelizing computations on computers. In the 1980s, Sternberg formalized this notion and introduced the following definition.

Definition 8 [27]. *Image algebra* is the representation of algorithms for image processing on a cellular computer in the form of algebraic expressions whose variables are images and whose operations are procedures for constructing logical and geometrical combinations of images.

This image algebra is described on the basis of mathematical morphology and is identified by the author with mathematical morphology. In 1985, Sternberg [28] noted that the languages for image processing were being developed for each processor architecture and none of them has been created for one computer and run on another. However, there are explicit language structures that satisfy the same principles. It is for description of these structures that image algebra (or mathematical morphology) appeared.

Ritter's image algebra generalizes mathematical morphology, unites the apparatus of local methods for image analysis with linear algebra, and generates more complex structures. Examples of such structures are templates and morphological algorithms. In [25], various operations and operands of standard image algebra are described, as well as applications of these structures to actual problems. Since the standard image algebra does not just generalize mathematical morphology, but is a wider and more convenient structure, the language of image algebra admits both implementation of known algorithms and design of new algorithms. The structure of the standard image algebra may be extended by introducing new operations. Hence, it may be successfully applied in the cases where a satisfactory result cannot be obtained with the help of morphology and linear algebra.

Definition 9 [25]. *A standard image algebra* is a heterogeneous (or multivalued) algebra (see Definition 5 in this section) with a complex structure of operands and operations if the basic operands are images (sets of points) and values and characteristics related to these images (sets of values related to these points).

Analyzing the existing algebraic apparatus, we came to the statement of the following requirements on the language designed for recording algorithms for solving problems of image processing and understanding:

(1) the new algebra must make possible processing of images as objects of analysis and recognition;

(2) the new algebra must make possible operations on image models, i.e., arbitrary formal representations of images, which are objects and, sometimes, a result of analysis and recognition; introduction of image models is a step in the formalization of the initial data of the algorithms;

(3) the new algebra must make possible operations on main models of procedures for image transformations; it is convenient to use the procedures for image modifications both as operations of the new algebra and as its operands for construction of compositions of basic models of procedures.

Definition 10 [4, 20]. An algebra is called a *descriptive image algebra* if its operands are either image models (for instance, as a model, we may take the image itself or a collection of values and characteristics related to the image) or operations on images, or models and operations simultaneously.

Definition 11 [4, 20]. A descriptive image algebra is a *basic descriptive image algebra* if its operands are either only image models or only operations on images.

It should be noted that, due to the variety of "algebras" (several versions of definitions are given in Subsection 3.1 *Algebras*), we should indicate which algebra is meant in Definition 10.

For the generality of the results and extension of the domain of applications of the new algebra, to define DIA with one ring, we use the definition of the classical algebra of Van der Waerden (Definition 1 in Subsection 3.1).

Definition 12. A ring U, which is a finite-dimensional vector space over a field P, is a *descriptive image algebra with one ring* if its operands are either image models or operations on images.

Thus, a DIA with one ring must satisfy the properties of classical algebras. A DIA with one ring is a basic DIA, because it contains a ring of elements of the same nature, i.e., either a ring of image models or a ring of operations on images.

2.3. The Place of Descriptive Algebra of Images in Algebraic Structure

In Subsection 2.1, we presented definitions of different algebras, which are far from their entirety. The goals of this subsection are, first, to indicate the place chosen for DIA in the structure of an algebra and, second, the variety of algebras. Figure 1 presents a classification reflecting the authors' point of view on the contemporary hierarchy of algebras and the place of DIA in this hierarchy.

3. DESCRIPTIVE IMAGE ALGEBRAS WITH ONE RING

3.1. Operands and Operations of DIA with One Ring

To design efficient algorithmic schemes for image analysis and understanding, it is necessary to investigate different types of operands and different types of operations applicable to the chosen operands, which generate the DIA. In this subsection, we present three types of operands of a DIA with one ring and five combinations of operations over these operands which either generate or do not generate a DIA.

By virtue of constraints imposed by the chosen definition of an algebra (Definition 1), it is not every operation of image processing and analysis that generates a DIA with one ring. The main goal of these examples is to demonstrate a method for attesting that operations and operands belong to the class of DIA with one ring.

Without loss of generality, in constructing the examples, we use the formal definition of an image in Ritter's sense [24, 25] and operations of standard image algebra as an element of formalization of the notion of an image.

Let F be a set of values and X a set of points. Recall that an image according to Ritter whose values on set X belong to F is a mapping from the set X

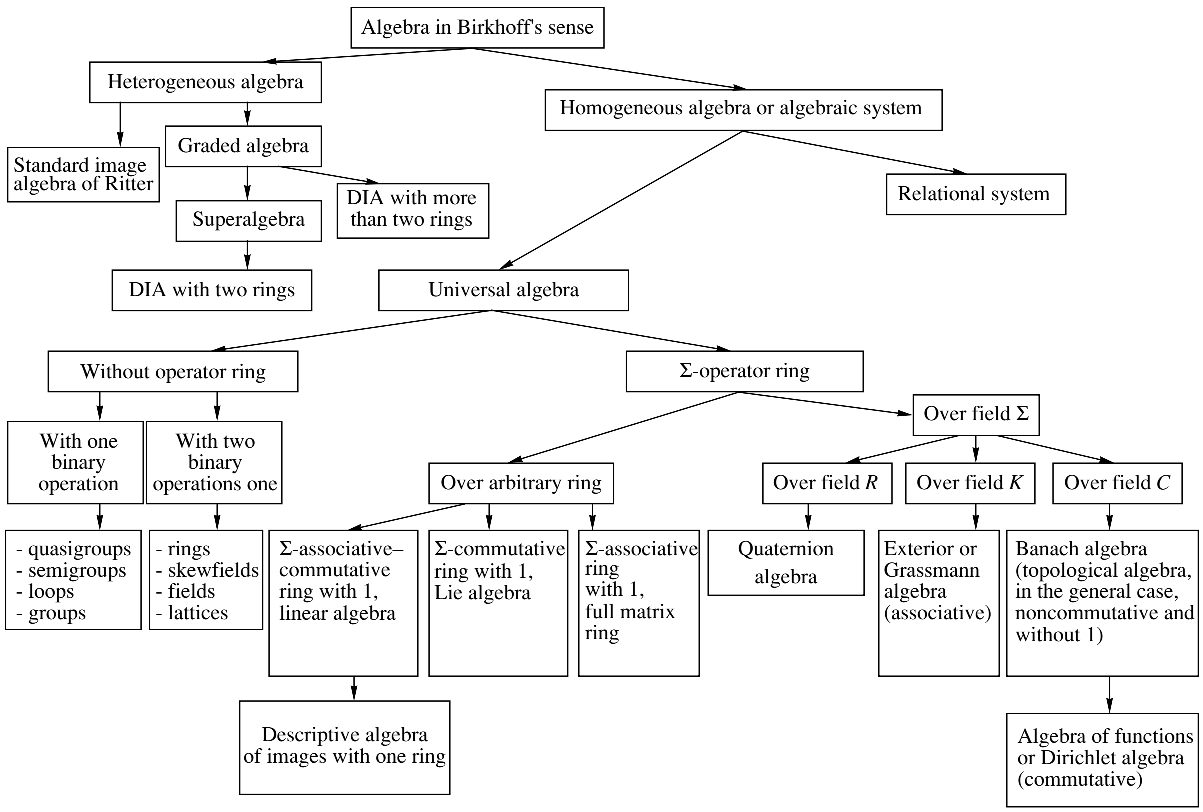

Fig. 1. Algebraic scheme.

into the set F (an element of the set F^X): $I = \{(x, a(x)), x \in X, a(x) \in F\}$.

In examples of DIA with one ring described below, as operands, we use the following:

(1) arbitrary images $I_i = \{(x, f_i(x)), x \in X, f_i(x) \in F_i\}$, $i = 1, 2, \ldots$, of the same shape and size (constructed on the same set X);

(2) images $I_j = \{(x, f_j(x)), x \in X, f_j(x) \in X\}$, $j = 1, 2, \ldots$, constructed on the same initial set X with the set of values X coinciding with the initial set;

(3) operations of the standard image algebra [25].

As operations over these operands, we take both the standard algebraic operations (addition, subtraction, multiplication, and division) and superpositions of these operations with other operations whose particular form is considered below in detail. The choice of precisely these operations is justified by the operands being used and the main goal of the examples of DIA, namely, to demonstrate the method for testing that the sets with these operations belong to the class of DIAs with one ring.

3.2. Interpretability of Operations of DIA

Note that, since the main goal of investigation of DIAs is the application of algebraic methods for designing algorithmic schemes of solving applied problems of image analysis and understanding in practice, the question as to the real meaning of the proposed constructions arises. In the formal construction of DIA, algebraic structures may appear that cannot be interpreted in the context of image analysis and understanding. The question about the interpretability of the obtained formal constructions arises.

For the first time, the problem of formal construction of examples of SIA appeared in papers [4, 19, 20], where examples of operations were considered, which have a certain physical sense; however, the question on the physical interpretation of these operation was not uniquely answered. Further investigation of this problem results in the following formalization.

Definition 13. By the *physical sense of an operation*, we mean a semantic description of the process of transforming an initial image (initial images) into a final image (final images) or a description of endowing an initial image with a collection of characteristics.

For the further formalization, we introduce definitions of an image representation and an image model.

Definition 14. By an *image representation*, we mean any description of the image obtained by acting on this image either with formal (mathematical) tools or with transformations admissible for this image. An *image model* is any formal image representation.

Definition 15. An operation on an image (images), its fragments, or image representation (representations) is called a *physically interpretable operation in the context of image analysis and understanding* if

(1) the result of its application is an image or its fragments;

(2) the result of its application is an image representation such that the semantically substantial geometrical objects, brightness characteristics and configurations obtained due to regular repetitions of geometrical objects and brightness characteristics of the initial image can be reconstructed starting from this representation; or

(3) the result of its application is an image characteristic (characteristics), which may be put into unique correspondence with geometrical objects, brightness characteristics or configurations obtained due to regular repetitions of geometrical objects and brightness characteristics of the initial image.

Definition 16. An operation over certain objects is called *visually interpretable in the context of image analysis and understanding* if, as a result of the operation, we obtain an image (images) such that, starting from this image, we can reconstruct a bijective correspondence between semantically substantial geometrical objects, brightness characteristics, and configurations obtained due to regular repetitions of geometrical objects and brightness characteristics on the obtained image (images) and on initial objects.

Proposition 1. Any visually interpretable operation is physically interpretable.

Corollary. If an operation is not physically interpretable, then this operation is not visually interpretable.

One can distinguish the physical interpretability in the strong and weak sense.

Definition 17. A operation is *physically interpretable in the strong sense* if it is also visually interpretable.

Definition 18. An operation is *physically interpretable in the weak sense* if it is physically interpretable but not visually interpretable.

For example, visually interpretable operations are the rotation of the image, shift of the image, increase in image contrast ratio, increase in image brightness, noise elimination in the image, image smoothing, contour extraction in the image, and other operations of image processing. As examples of visually interpretable operations, we can also mention the operations of image construction by a certain rule from a set of initial objects, for instance, image reconstruction from equations specifying the image shape.

Physically interpretable operations are also certain operations of constructing of image models and operations on images such as calculation of the histogram of the image and calculation of statistical features of the image.

Proposition 2. An operation is not physically interpretable in the context of image analysis and understanding if

(1) its operands are different from images, image representations, and fragments of images;

(2) applying the operation to an image (images), we obtain an image model such that, on its basis, it is impossible to reconstruct semantically substantial geometrical objects, brightness characteristics, or configurations, which appear due to a regular repetition of geometrical objects and brightness characteristics, of the initial image;

(3) applying the operation to the image, we calculate characteristics such that we cannot uniquely associate of geometrical objects, brightness characteristics, or configurations, which appear due to a regular repetition of geometrical objects and brightness characteristics, of the initial image with these characteristics of properties;

(4) the operation cannot be applied to images, image representations, and fragments of images.

In examples below, a special attention is given to the interpretability (in the sense of Definitions 15 and 16) of the operations introduced.

3.3. Description Scheme of Examples

All examples presented in this section are described by the same scheme: we introduce operands (I) and operations (II) of the proposed algebra, present the results of investigation of the physical and visual interpretability of the operation introduced (III), formulate certain conditions (IV), and, on the basis of items I–IV, prove a theorem (V).

I. Elements of a set U whose nature corresponds to the nature of objects described with the help of DIA. Recall that operands of a DIA are either image models (as a model, for instance, we can take both the image itself and a collection of quantities and characteristics related to the image) or an operation on images, or models and operations simultaneously. The set U, together with introduced operations, is treated as a candidate for the basic descriptive image algebra.

II. Operations introduced on the set U.

III. Interpretation of the introduced operations.

IV. Conditions necessary for statement and proof of the theorem in item V.

V. Statement and proof of the theorem on validity/invalidity of properties of DIA for the construction considered in the example.

3.4. Examples of Operations Generating Descriptive Image Algebras

3.4.1. Description of examples

This subsection contains three examples.

In the first example, as operands of the set U, we consider images defined on a fixed set X with different fixed range domains F_i, $i = 1, 2, \ldots$: $I_i = \{(x, f_i(x)), x \in X, f_i(x) \in F_i\}$. As the operation of addition of two images I_1 and I_2, we use the operation producing the function, which determines the resulting image, by adding the functions $f_1(x)$ and $f_2(x)$ describing the images I_1 and I_2, respectively, at each point of the set X. As the operation of multiplication of two images I_1 and I_2, we use the operation producing the function, which determines the resulting image, by multiplying the functions $f_1(x)$ and $f_2(x)$ describing the images I_1 and I_2, respectively, at each point of the set X. As the operation of multiplication of an image I by an element of the field, we use the operation producing the function, which determines the resulting image, by multiplying the function f describing the image I by the element of the field at each point of the set X.

In the second example, as operands of the set U, we consider images defined on a fixed set X with the range domain X: $I_i = \{(x, f_i(x)), x \in X, f_i(x) \in X\}$, $i = 1, 2, \ldots$. As the operation of addition of two images I_1 and I_2, we use the operation producing the function, which determines the resulting image, by adding functions $f_1(x)$ and $f_2(x)$ describing the images at each point of the set X. As the operation of multiplication of two images I_1 and I_2, we use the operation producing the function, which determines the resulting image, by superposition of functions $f_1(x)$ and $f_2(x)$ describing the images at each point of the set X. As the operation of multiplication of an image I by an element of the field, we use the operation producing the function, which determines the resulting image, by multiplying function f describing the image by the element of the field at each point of the set X.

In the third example, as operands of the set U, we consider standard binary operations on images. On these operands, specialized operations of addition of two operands and multiplication of two operands, as well as an operation of multiplication by an element of the field, are introduced.

All three examples demonstrate operands and operations generating a DIA with one ring.

3.4.2. Example 1

I. Elements of the set U

Images defined on a fixed set X with different fixed range domains F_i, $i = 1, 2, \ldots$: $I_i = \{(x, f_i(x)), x \in X, f_i(x) \in F_i\}, \ldots i = 1, 2,$.

As examples of functions $f_i(x)$ and sets F_i, X specifying the image, we can take:

* $X = [0\ldots N; 0\ldots N]$ is a square in the two-dimensional plane;

* $f_i(x) = (r_i(x), g_i(x), b_i(x))$, here $r_i(x)$, $g_i(x)$, and $b_i(x)$ are three components of colors, respectively, red, green, and blue;

* sets $F_i \subset R^3$ are some sets bounded the color gamma of images, for instance, corresponding to the gamma of black–white images, red images, green images, yellow images, etc.

II. Operations on set U.

Let $I_1 = \{(x, a(x)), x \in X, a(x) \in F_1\}$, $I_2 = \{(x, b(x)), x \in X, b(x) \in F_2\}$.

* Operation of addition of two images I_1 and I_2:

$$I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\}; \tag{3}$$

* Operation of multiplication of two images I_1 and I_2:

$$I_1 \cdot I_2 = \{(x, a(x) \cdot b(x)), x \in X\}; \tag{4}$$

* Operation of multiplication of image I by an element of the field of real numbers $\alpha \in R$:

$$\alpha \cdot I = \{(x, \alpha \cdot f(x)), x \in X\}. \tag{5}$$

III. Interpretation of these operations

III.1 Operation of addition of two elements of the set

The physical sense of the operation corresponds to the pointwise addition of two images.

Proposition 3. The operation of addition of two elements of the set introduced in (3) is *physically interpretable* but, in the general case *not visually interpretable*; i.e., it is a physically interpretable operation in the weak sense.

Proof.

1. Physical interpretability of the operation: by Definition 15, the operation is physically interpretable if, applying it, we obtain an image.

2. Visual interpretability of the operation: this operation is not, in the general case, visually interpretable, because, applying it to two arbitrary images, we obtain a semantically meaningless image (Definition 16).

The proposition is proved.

An example of the use of this operation is that it can be used in the problem of overlapping of certain images (construction of a face from objects such as eyes, nose, mouth).

III.2 Operation of multiplication of two elements of the set

The physical sense of the operation corresponds to the pointwise multiplication of two images.

Proposition 4. The operation of multiplication of two elements of the set introduced in (4) is *physically interpretable* but, in the general case *not visually interpretable*, i.e., it is a physically interpretable operation in the weak sense.

Proof.

1. Physical interpretability of the operation: by Definition 15, the operation is physically interpretable if, applying it, we obtain an image.

2. Visual interpretability of the operation: this operation is not, in the general case, visually interpretable, because, applying it to two arbitrary images, we obtain a semantically meaningless image (Definition 16).

The proposition is proved.

An example of the use of this operation is that it can be used in working with binary images, namely, in multiplication of an image by a mask represented in the form of a binary image.

III.3 Operation of multiplication of an element of set U by an element of the field of real numbers

The physical sense of the operation: the operation corresponds to the pointwise multiplication of an image by a real number.

Proposition 5. The operation of multiplication of an element of set U by an element of the field of real numbers introduced in (5) is *physically interpretable* and *visually interpretable*; i.e., it is a physically interpretable operation in the strong sense.

Proof.

1. Physical interpretability of the operation: by Definition 15, the operation is physically interpretable if, in applying it, we obtain an image.

2. Visual interpretability of the operation: this operation results in a proportional increase or decrease in the brightness of the image (if the image is represented as a function of pixel brightness) or just a change in the color. Therefore, by Definition 16, this operation is visually interpretable.

The proposition is proved.

An example of the use of this operation is that it may be used in tasks where the brightness of the image must be proportionally increased or decreased.

IV. Necessary conditions for stating and proving the theorem in item V.

Let a set of sets $\{F_i\}_1^\infty$ be given; i.e.,

* an operation of the addition of two elements from $F_i, F_j \subset R^n$: $i, j = 1, 2, \ldots, \forall a \in F_i, b \in F_j$: $\exists!\ a + b \in F_k$, $k = 1, 2, \ldots, F_k \subset R^n$, be given with the following properties ($\forall a \in F_i, b \in F_j, c \in F_y, i, j, y = 1, 2, \ldots$):

$$1.1.\ a + (b + c) = (a + b) + c; \tag{6}$$

$$1.2.\ a + b = b + a; \tag{7}$$

$$1.3.\ \forall i\colon \forall a \in F_i, \quad \exists 0 \in F_i\colon a + 0 = a; \tag{8}$$

$$1.4.\ \forall i\colon \forall a \in F_i, \quad \exists(-a) \in F_i\colon a + (-a) = a; \tag{9}$$

* an operation of multiplication of two elements from F_i, F_j: $i, j = 1, 2, \ldots, a \in F_i, b \in F_j$: $\exists!\ a \cdot b \in F_k$, $k = 1, 2, \ldots, F_k \subset R^n$, be given with the following properties ($\forall a \in F_i, b \in F_j, c \in F_y, i, j, y = 1, 2, \ldots$):

$$1.5.\ a \cdot (b \cdot c) = (a \cdot b) \cdot c; \tag{10}$$

* on the set F_i ($i = 1, 2, \ldots$), an operation of multiplication by elements of the field of real numbers R: $\forall \alpha \in R, a \in F_i$: $\exists! \alpha \cdot a \in F_i$ ($i = 1, 2, \ldots$), be given with the following properties ($\forall a \in F_i, b \in F_j, c \in F_y, i, j, y = 1, 2, \ldots, \forall \alpha, \beta \in R$):

$$1.6.\ (\alpha \cdot a + \beta \cdot b) \cdot c = \alpha \cdot a \cdot c + \beta \cdot b \cdot c; \tag{11}$$

$$1.7.\ \alpha \cdot (\beta \cdot a) = (\alpha \cdot \beta) \cdot a; \tag{12}$$

$$1.8.\ (\alpha + \beta) \cdot a = \alpha \cdot a + \beta \cdot a; \tag{13}$$

$$1.9.\ \alpha \cdot (a + b) = \alpha \cdot a + \alpha \cdot b. \tag{14}$$

If all $F_i \equiv F \subset R^n$, then the set F with these operations is an algebra over the field of real numbers. The sets considered above in item I as examples of sets F_i satisfy conditions 1.1–1.9.

V.Statement and proof of the theorem on validity/invalidity of properties of DIA for the construction considered in the example.

Theorem 1.

Let

* R be the field of real numbers;

* $I = \{(x, f(x)), x \in X, f(x) \in F\}$ ($F \in \{F_i\}_1^\infty$, where F is a set of values of image I on set X) be elements of set U;

* $I_1 = \{(x, a(x)), x \in X, a(x) \in F_1\}$, $I_2 = \{(x, b(x)), x \in X, b(x) \in F_2\}$, ($F_1, F_2 \in \{F_i\}_1^\infty$).

We introduce

* an operation of addition of two images I_1 and I_2:

$$I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\};$$

* an operation of multiplication of two images I_1 and I_2:

$$I_1 \cdot I_2 = \{(x, a(x) \cdot b(x)), x \in X\};$$

* an operation of multiplication of image I by an element of the field of a real number $\alpha \in R$:

$$\alpha \cdot I = \{(x, \alpha \cdot f(x)), x \in X\}.$$

Then, the set U with these operations of addition, multiplication, and multiplication by a real number is a *basic descriptive image algebra over the field of real numbers.*

Proof.

The proof is based on verification of the properties of algebras (see Section 3, Definition 1) for the set U with the above operations of addition, multiplication, and multiplication by a real number.

I. By condition of the theorem, R is the field of real numbers.

II. Verification of the properties of ring U ($I_1, I_2, I_3 \in U$, $I_1 = \{(x, a(x)), x \in X, a(x) \in F_1\}$, $I_2 = \{(x, b(x)), x \in X, b(x) \in F_2\}$, $I_3 = \{(x, c(x)), x \in X, c(x) \in F_3\}$).

1. Verification of the properties of the operation of addition.

$$\forall I_1, I_2 \in U, \quad \exists!(I_1 + I_2) \in U.$$

(a) Verification of the associative property:

$$I_1 + (I_2 + I_3) = I_1 + \{(x, b(x) + c(x)), x \in X\}$$
$$= \{(x, a(x) + (b(x) + c(x))), x \in X\};$$
$$(I_1 + I_2) + I_3 = \{(x, a(x) + b(x)), x \in X\} + I_3$$
$$= \{(x, (a(x) + b(x)) + c(x)), x \in X\}$$
$$= \{\text{by property 1.1 of item IV in this example}\}$$
$$= \{(x, a(x) + (b(x) + c(x))), x \in X\};$$

hence, $I_1 + (I_2 + I_3) = (I_1 + I_2) + I_3$.

(b) Verification of the commutative property:

$$I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\};$$
$$I_2 + I_1 = \{(x, b(x) + a(x)), x \in X\}$$
$$= \{\text{by property 1.2 of item IV in this example}\}$$
$$= \{(x, a(x) + b(x)), x \in X\};$$

hence, $I_1 + I_2 = I_2 + I_1$.

(c) Verification of the existence of zero:

Zero of the set U has the form $O = \{(x, 0), x \in X\}$. Let us prove this.

$$I_1 + O = \{(x, a(x) + 0), x \in X\}$$
$$= \{\text{by property 1.3 of item IV in this example}\}$$
$$= \{(x, a(x)), x \in X\} = I_1;$$

hence, $\exists O \in U, \forall I_1 \in U, I_1 + O = I_1$.

(d) Verification of the existence of an opposite element.

For an image I_1, the opposite element is $(-I_1) = \{(x, -a(x)), x \in X\}$. Let us prove this.

$$I_1 + (-I_1) = \{(x, a(x) - a(x)), x \in X\}$$
$$= \{\text{by property 1.4 of item IV in this example}\}$$
$$= \{(x, 0), x \in X\} = O;$$

hence, $\forall I_1 \in U, \exists(-I_1), I_1 + (-I_1) = 0$.

2. Verification of the properties of the operation of multiplication.

$$\forall I_1, I_2 \in U, \exists!(I_1 \cdot I_2) \in U.$$

(a) Verification of the associative property:

$$I_1 \cdot (I_2 \cdot I_3) = I_1 \cdot (\{(x, b(x) \cdot c(x)), x \in X\})$$
$$= \{(x, a(x) \cdot (b(x) \cdot c(x))), x \in X\};$$
$$(I_1 \cdot I_2) \cdot I_3 = (\{(x, a(x) \cdot b(x)), x \in X\}) \cdot I_3$$
$$= \{(x, (a, (x) \cdot b(x)) \cdot c(x)), x \in X\}$$
$$= \{\text{by property 1.5 of item IV in this example}\}$$
$$= \{(x, a(x) \cdot (b(x) \cdot c(x))), x \in X\};$$

hence, $I_1 \cdot (I_2 \cdot I_3) = (I_1 \cdot I_2) \cdot I_3$.

3. Verification of the distributive property:

For any $\alpha, \beta \in R$, $(\alpha \cdot I_1 + \beta \cdot I_2) \cdot I_3 = (\{(x, \alpha \cdot a(x) + \beta \cdot b(x)), x \in X\}) \cdot I_3 = \{(x, (\alpha \cdot a(x) + \beta \cdot b(x)) \cdot c(x)), x \in X\}$ = {by property 1.6 of item IV in this example} = $\{(x, \alpha \cdot a(x) \cdot c(x) + \beta \cdot b(x) \cdot c(x)), x \in X\}$;

$$\alpha \cdot I_1 \cdot I_3 + \beta I_2 \cdot I_3 = \{(x, \alpha \cdot a(x) \cdot c(x)), x \in X\}$$
$$+ \{(x, \beta \cdot b(x) \cdot c(x)), x \in X\}$$
$$= \{(x, \alpha \cdot a(x) \cdot c(x) + \beta \cdot b(x) \cdot c(x)), x \in X\};$$

hence, $(\alpha \cdot I_1 + \beta \cdot I_2) \cdot I_3 = \alpha \cdot I_1 \cdot I_3 + \beta \cdot I_2 \cdot I_3$.

Thus, the operations of addition and multiplication satisfy the properties of a ring.

III. Verification of the properties of a ring being a vector space over field R ($\forall \alpha \in P, I \in U, I = \{(x, f(x)), x \in X\}$: $\alpha \cdot I \in U$).

1. Verification of property 1 of a ring being a vector space over field R:

$$\alpha \cdot (\beta \cdot I) = \alpha \cdot \{(x, \beta \cdot f(x)), x \in X\}$$
$$= \{(x, \alpha \cdot (\beta \cdot f(x))), x \in X\}$$
$$= \{\text{by property 1.7 of item IV in this example}\}$$
$$= \{(x, \alpha \cdot \beta \cdot f(x)), x \in X\};$$
$$(\alpha, \beta) \cdot I = \{(x, \alpha \cdot \beta \cdot f(x)), x \in X\};$$

hence, $\alpha \cdot (\beta \cdot I) = (\alpha \cdot \beta) \cdot I$.

2. Verification of property 2 of a ring being a vector space over field R:

$$(\alpha + \beta) \cdot I = \{(x, (\alpha + \beta) \cdot f(x)), x \in X\}$$
$$= \{\text{by property 1.8 of item IV in this example}\}$$
$$= \{(x, \alpha \cdot f(x) + \beta \cdot f(x)), x \in X\};$$
$$\alpha \cdot I + \beta \cdot I = \{(x, \alpha \cdot f(x)), x \in X\} + \{(x, \beta \cdot f(x)), x \in X\}$$
$$= \{(x, \alpha \cdot f(x) + \beta \cdot f(x)), x \in X\};$$

hence, $(\alpha + \beta) \cdot I = \alpha \cdot I + \beta \cdot I$.

3. Verification of property 3 of a ring being a vector space over field R:

$$\alpha \cdot (I_1 + I_2) = \alpha \cdot \{(x, a(x) + b(x)), x \in X\}$$
$$= \{(x, \alpha \cdot (a(x) + b(x))), x \in X\}$$
$$= \{\text{by property 1.9 of item IV in this example}\}$$
$$= \{(x, \alpha \cdot a(x) + \alpha \cdot b(x)), x \in X\};$$
$$\alpha \cdot I_1 + \alpha \cdot I_2 = \{(x, \alpha \cdot a(x)), x \in X\}$$
$$+ \{(x, \alpha \cdot b(x)), x \in X\} = \{(x, \alpha \cdot a(x) + \alpha \cdot b(x)), x \in X\};$$

hence, $\alpha \cdot (I_1 + I_2) = \alpha \cdot I_1 + \alpha \cdot I_2$.

Thus, all the properties of a ring being a vector space are fulfilled.

Hence, the set U with the introduced operations of addition, multiplication, and multiplication by real numbers is an algebra over the field of real numbers. By virtue of the nature of elements of the set U, this algebra is a *basic descriptive image algebra*.

The theorem is proved.

3.4.3. Example 2

I. Elements of set U

Images defined on a fixed set X with the domain of values X: $I = \{(x, f(x)), x \in X, f(x) \in X\}$.

As examples of such functions and sets specifying images, we can consider the following:

* set $X = R^3$;

* function $f(x)$ of image I of a three-dimensional scene is defined at a point (x, y) of the space and takes into account the distance from a source of light l, i.e., $x = (x, y, l)$;

* function $f(x)$ is equal to $(r(x), g(x), b(x))$, where $r(x)$, $g(x)$, and $b(x)$ are three components of the color, respectively, red, green, and blue.

II. Operations introduced on the set U

Let $I_1 = \{(x, a(x)), x \in X, a(x) \in X\}$, $I_2 = \{(x, b(x)), x \in X, b(x) \in X\}$.

* Operation of addition of two images I_1 and I_2:

$$I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\}; \quad (15)$$

* Operation of multiplication of two images I_1 and I_2:

$$I_1 \cdot I_2 = \{(x, a(b(x))), x \in X\}; \quad (16)$$

* Operation of multiplication of image I by a real number $\alpha \in R$:

$$\alpha \cdot I = \{(x, \alpha \cdot f(x)), x \in X\}. \quad (17)$$

III. Interpretation of the introduced operations

III.1. Operation of addition of two elements of the set

The physical sense of the operation corresponds to the pointwise addition of two images.

The representation of images in Example 2 is a particular case of the image representation in Example 1. Therefore, we claim that, by Proposition 3, this operation of addition of two elements of the set is physically interpretable but, in the general case, not visually interpretable.

An example of this operation is that it may be used in the problem of overlapping of certain images (construction of a face from objects such as eyes, nose, mouth). If an image is specified by the function described in item I of the example, then this operation may be used for addition of two images of the same scene with different illumination intensities of objects in the scene. Addition of such images allows one to obtain an image with all objects of the scene or most of them being illuminated.

III.2. Operation of multiplication of two elements of the set

The physical sense of the operation: definition of an image on the set specified by another image.

Proposition 6. The operation, introduced in (15), of multiplication of two elements of the set is *physically interpretable* but, in the general case, *not visually interpretable*; i.e., this operation is physically interpretable in the weak sense.

Proof.

1. Physical interpretability of the operation: by Definition 15, the operation is physically interpretable if, applying it, we obtain an image.

2. Visual interpretability of the operation: in the general case, this operation is not visually interpretable, because, applying it to two arbitrary images, we obtain a semantically meaningless image (Definition 16).

The proposition is proved.

Example of the use of this operation. If an image is specified by the function described in item I of the example, then this operation may be applied in the following case. Suppose that we know the value $f(x, y, \infty) = \{r(x, y, \infty), g(x, y, \infty), b(x, y, \infty)\}$; i.e., we know the colors of the scene with a light source at infinity. Then, if the position of the light source relative to the same three-dimensional scene is known, then the image of this scene under illumination may be specified as the function $g(x, y, l) = g(f(x, y, \infty), l)$.

* Operation of multiplication of an element of set U by a real number.

The physical sense of the operation: the operation corresponds to pointwise multiplication by a real number.

The image representation in Example 2 is a particular case of the image representation in Example 1. Therefore, we may say that, by Proposition 5, the introduced operation of multiplication of an element of the set U by a real number is physically and visually interpretable.

IV. Necessary conditions for stating and proving the theorem in item V

Suppose that a set X and a set of functions Φ on this set are given such that $\forall f \in \Phi, f: X \longrightarrow X$:

* an operation of addition of two functions from set Φ is defined: $\forall a(x), b(x) \in \Phi$: $\exists!(a + b)(x) \equiv a(x) + b(x) \in \Phi$, with the following properties ($\forall a(x), b(x), c(x) \in \Phi$):

$$2.1.\ a(x) + (b(x) + c(x)) = (a(x) + b(x)) + c(x); \quad (18)$$

$$2.2.\ a(x) + b(x) = b(x) + a(x); \quad (19)$$

$$2.3.\ \forall a(x) \in \Phi, \exists(-a(x)) \in \Phi: a(x) + (-a(x)) = 0; \quad (20)$$

$$2.4.\ \forall a(x) \in \Phi, \exists 0 \in \Phi: a(x) + 0 = a(x); \quad (21)$$

* an operation of superposition of two functions from Φ is defined: $\forall a(x), b(x) \in \Phi$: $\exists!(ab)(x) \equiv a(b(x)) \in \Phi$;

* on the set Φ, an operation of multiplication by elements of the field of real numbers R is defined: $\forall \alpha \in R, a(x) \in \Phi$: $\exists! \alpha \cdot a(x) \in \Phi$, with the following properties ($\forall a(x), b(x), c(x) \in \Phi, \forall \alpha, \beta \in R$):

$$2.5.\ (\alpha \cdot a(x) + \beta \cdot b(x)) \cdot c(x) = \alpha \cdot a(c(x)) + \beta \cdot b(c(x)); \quad (22)$$

$$2.6.\ \alpha \cdot (\beta \cdot a(x)) = \alpha \cdot \beta \cdot a(x); \quad (23)$$

$$2.7.\ (\alpha + \beta) \cdot a(x) = \alpha \cdot a(x) + \beta \cdot a(x); \quad (24)$$

$$2.8.\ \alpha \cdot (a(x) + b(x)) = \alpha \cdot a(x) + \alpha \cdot b(x). \quad (25)$$

For example, in the case $X \equiv R$, as elements of the set X, we may take polynomials of all degrees.

V. Statement and proof of the theorem on validity/invalidity of the properties of DIA for the construction considered in the example.

Theorem 2.

Let

* R be the field of real numbers;

* $I = \{(x, f(x)), x \in X, f(x) \in X\}$ $(f(x) \in \Phi)$ be elements of a set U;

* $I_1 = \{(x, a(x)), x \in X, a(x) \in X\}$, $I_2 = \{(x, b(x)), x \in X, b(x) \in X\}$ $(a(x), b(x) \in \Phi)$;

We introduce

* an operation of addition of two images I_1 and I_2:

$$I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\};$$

* an operation of multiplication of two images I_1 and I_2:

$$I_1 \cdot I_2 = \{(x, a(b(x))), x \in X\};$$

(when the multiplication is introduced in this way, the result of application of this operation does not go beyond the set U of images considered);

* an operation of multiplication of image I by real numbers $\alpha \in R$:

$$\alpha \cdot I = \{(x, \alpha \cdot f(x)), x \in X\}.$$

Then, the set U with these operations of addition, multiplication, and multiplication by real numbers is a *descriptive image algebra over the field of real numbers.*

Proof.

The proof is based on verification of the properties of an algebra (see Section 3, Definition 1) for the set U with the introduced operations of addition, multiplication, and multiplication by real numbers.

I. By condition of the theorem, R is a field of real numbers.

II. Verification of the properties of ring U $(I_1, I_2, I_3 \in U$, $I_1 = \{(x, a(x)), x \in X, a(x) \in X\}$, $I_2 = \{(x, b(x)), x \in X, b(x) \in X\}$, $I_3 = \{(x, c(x)), x \in X, c(x) \in X\})$.

1. Verification of the properties of the operation of addition.

$$\forall I_1, I_2 \in U, \quad \exists!(I_1 + I_2) \in U.$$

(a) Verification of the associative property:

$$I_1 + (I_2 + I_3) = I_1 + \{(x, b(x) + c(x)), x \in X\}$$
$$= \{(x, a(x) + (b(x) + c(x))), x \in X\};$$
$$(I_1 + I_2) + I_3 = \{(x, a(x) + b(x)), x \in X\} + I_3$$
$$= \{\text{by property 2.1 of item IV in this example}\}$$
$$= \{(x, a(x) + (b(x) + c(x))), x \in X\};$$

hence, $I_1 + (I_2 + I_3) = (I_1 + I_2) + I_3$.

(b) Verification of the commutative property:

$$I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\};$$
$$I_2 + I_1 = \{(x, b(x) + a(x)), x \in X\}$$
$$= \{\text{by property 2.2 of item IV in this example}\}$$
$$= \{(x, a(x) + b(x)), x \in X\};$$

hence, $I_1 + I_2 = I_2 + I_1$.

(c) Verification of the existence of zero.

Zero of the set U has the form $O = \{(x, 0), x \in X\}$. Let us prove this.

$$I_1 + O = \{(x, a(x) + 0), x \in X\}$$
$$= \{\text{by property 2.3 of item IV in this example}\}$$
$$= \{(x, a(x)), x \in X\} = I_1;$$

hence, $\exists O \in U, \forall I_1 \in U, I_1 + O = I_1$.

(d) Verification of the existence of an opposite element.

For an image I_1 the opposite element is $(-I_1) = \{(x, -a(x)), x \in X\}$. Let us prove this fact.

$$I_1 + (-I_1) = \{(x, a(x) - a(x)), x \in X\}$$
$$= \{\text{by property 2.4 of item IV in this example}\}$$
$$= \{(x, 0), x \in X\} = O;$$

hence, $\forall I_1 \in U, \exists(-I_1), I_1 + (-I_1) = 0$.

2. Verification of the property of the operation of multiplication.

$$\forall I_1, I_2 \in U, \quad \exists!(I_1 \cdot I_2) \in U.$$

(a) Verification of the associative property:

$$I_1 \cdot (I_2 \cdot I_3) = I_1 \cdot (\{(x, b(c(x))), x \in X\})$$
$$= \{(x, a(b(c(x)))), x \in X\};$$
$$(I_1 \cdot I_2) \cdot I_3 = (\{(x, a(b(x))), x \in X\}) \cdot I_3$$
$$= \{(x, a(b(c(x)))), x \in X\};$$

hence, $I_1 \cdot (I_2 \cdot I_3) = (I_1 \cdot I_2) \cdot I_3$.

3. Verification of the distributive property:

For any $\alpha, \beta \in R$, $(\alpha I_1 + \beta I_2) \cdot I_3 = (\{(x, \alpha a(x) + \beta b(x)), x \in X\}) \cdot I_3 = \{(x, (\alpha a(c(x)) + \beta b(c(x)))), x \in X\}$;

$$\alpha I_1 \cdot I_3 + \beta I_2 \cdot I_3 = \{(x, \alpha a(c(x))), x \in X\}$$
$$+ \{(x, \beta b(c(x))), x \in X\}$$
$$= \{(x, (\alpha a(c(x)) + \beta b(c(x)))), x \in X\};$$

hence, $(\alpha I_1 + \beta I_2) \cdot I_3 = \alpha I_1 \cdot I_3 + \beta I_2 \cdot I_3$.

Thus, the operation of addition and multiplication satisfy the properties of rings.

III. Verification of the properties of a ring being a vector space over the field R $(\forall \alpha \in P, I \in U, I = \{(x, f(x)), x \in X\}: \alpha I \in U)$.

1. Verification of property 1 of a ring being an R-vector space:

$$\alpha(\beta I) = \alpha\{(x, \beta f(x)), x \in X\} = \{(x, \alpha(\beta f(x))), x \in X\}$$
$$= \{\text{by property 2.6 of item IV in this example}\}$$

$$= \{(x, \alpha\beta f(x)), x \in X\};$$

$$(\alpha\beta)I = \{(x, \alpha\beta f(x)), x \in X\};$$

hence, $\alpha(\beta I) = (\alpha\beta)I$.

2. Verification of property 2 of a ring being an R-vector space:

$$(\alpha + \beta)I = \{(x, (\alpha + \beta)f(x)), x \in X\}$$

$$= \{\text{by property 2.7 of item IV in this example}\}$$

$$= \{(x, \alpha f(x) + \beta f(x)), x \in X\};$$

$$\alpha I + \beta I = \{(x, \alpha f(x)), x \in X\}$$

$$+ \{(x, \beta f(x)), x \in X\} = \{(x, \alpha f(x) + \beta f(x)), x \in X\};$$

hence, $(\alpha + \beta)I = \alpha I + \beta I$.

3. Verification of property 3 of a ring being an R-vector space:

$$\alpha(I_1 + I_2) = \alpha\{(x, a(x) + b(x)), x \in X\}$$

$$= \{(x, \alpha(a(x) + b(x))), x \in X\}$$

$$= \{\text{by property 2.8 of item IV in this example}\}$$

$$= \{(x, \alpha a(x) + \alpha b(x)), x \in X\};$$

$$\alpha I_1 + \alpha I_2 = \{(x, \alpha a(x)), x \in X\}$$

$$+ \{(x, ab(x)), x \in X\} = \{(x, \alpha a(x) + \alpha b(x)), x \in X\};$$

hence, $\alpha(I_1 + I_2) = \alpha I_1 + \alpha I_2$.

Thus, all the properties of a ring being an R-vector space hold.

Hence, the set U with the introduced operations of addition, multiplication, and multiplication by real numbers is an algebra over the field of real numbers. By virtue of nature of the elements of the set U, this algebra is a *basic descriptive image algebra*.

The theorem is proved.

3.4.4. Example 3

I. Elements of set U

Standard binary operations on images.

* If A, B, and C… are images defined on a fixed set X with the range domain X, then the following operations on images can be introduced [25]:

$$A + B = \{(x, c(x)): c(x) = a(x) + b(x), x \in X\}; \quad (26)$$

$$A \cdot B = \{(x, c(x)): c(x) = a(x) \cdot b(x), x \in X\}; \quad (27)$$

$$A \vee B = \{(x, c(x)): c(x) = a(x) \vee b(x), x \in X\}; \quad (28)$$

$$A \wedge B = \{(x, c(x)): c(x) = a(x) \wedge b(x), x \in X\}; \quad (29)$$

$$\frac{A}{B} = \left\{(x, c(x)): c(x) = \frac{a(x)}{b(x)}, \right. \quad (30)$$
$$\left. \text{if } b(x) \neq 0, \text{ otherwise } c(x) = 0; x \in X\right\};$$

$$A^B = \{(x, c(x)): c(x) = a(x)^{b(x)}, \quad (31)$$
$$\text{if } a(x) > 0, \text{ otherwise } c(x) = 0, x \in X\};$$

$$A - B = \{(x, c(x)): c(x) = a(x) - b(x), x \in X\}. \quad (32)$$

* We may say that $r_1, r_2, \ldots \in \{+, *, \vee, \wedge, -, \backslash, A^B\ldots\}$ are standard binary operations on two images;

* $r(A, B)$ is the image obtained by applying an operation r to images A and B.

II. Operations introduced on the set U

Let $r_1, r_2, \ldots \in \{+, *, \vee, \wedge, -, \backslash, A^B\}$; i.e., $r_1, r_2, \ldots$ are operations on two images.

* Operation of addition of two operations r_1 and r_2:

$$(r_1 \oplus r_2)(A, B) = r_1(A, B) + r_2(A, B); \quad (33)$$

* Operation of multiplication of two operations r_1 and r_2:

$$(r_1 \otimes r_2)(A, B) = r_1(r_2(A, B), r_2(A, B)); \quad (34)$$

* Operation of multiplication of operation r by a real number $\alpha \in R$:

$$(\alpha r)(A, B) = \alpha r(A, B) \quad (35)$$

(the right-hand side denotes multiplication of an image by an element of the field).

III. Interpretation of the introduced operations

III.1. Operation of addition of two elements of the set.

The physical sense of the operation is as follows: the operation corresponds to successive application of operations r_1 and r_2 accompanied by addition of the images obtained by applying operations r_1 and r_2 to the image.

Proposition 7. The operation introduced in (33) of addition of two elements of the set is neither physically nor visually interpretable.

Proof.

1. Physical interpretability of the operation: by Proposition 2, the operation is not physically interpretable in the context of image analysis and understanding if its operands are not images, image representations, or fragments of images.

2. Visual interpretability of the operation: by the corollary of Proposition 1, if an operation is not physically interpretable, it is not visually interpretable.

The proposition is proved.

An example of this operation is that it may be used for combining operations on images.

III.2. Operation of multiplication of two elements of the set.

The physical sense of the operation is that it corresponds to the following sequence of actions. The second operation r_2 is applied to both images; as the first and second operands of the first operation r_1, the results of applying the second operation r_2 are taken.

Similarly to the operation of addition of two elements of the set, the operation of multiplication of two elements of the set is also neither physically nor visually interpretable.

An example of this operation is that it may be used for combining operations on images.

III.3. Operation of multiplication of an element of set U by a real number

The physical sense of the operation is this: the operation corresponds to the pointwise multiplication of the image obtained as a result of operation r by a real number.

Similarly to the operation of addition of two elements of the set, the operation of multiplication of an element of the set U by a real number is also neither physically not virtually interpretable.

An example of this operation is as follows: it may be applied for correcting the brightness and color of the image obtained as a result of operation r.

IV. Necessary conditions for formulating and proving the theorem in item V

Set X satisfies the conditions described in Example 2.

V. Statement and proof of the theorem on validity/invalidity of properties of DIA for the construction considered in the example

Theorem 3.

Let

* R be a field of real numbers;

* elements of set U: $r_1, r_2, \ldots \in \{+, *, \vee, \wedge, -, \backslash, A^B, \ldots\}$, i.e., $r_1, r_2, \ldots$ – are operations on two images defined on a fixed set X with the range domain X.

We introduce

* an operation of addition of two operations r_1 and r_2:

$$(r_1 \oplus r_2)(A, B) = r_1(A, B) + r_2(A, B);$$

* an operation of multiplication of two operations r_1 and r_2:

$$(r_1 \otimes r_2)(A, B) = r_1(r_2(A, B), r_2(A, B));$$

* an operation of multiplication of operation r by a real number $\alpha \in R$:

$$(\alpha r)(A, B) = \alpha r(A, B)$$

(the right-hand side denotes multiplication of an image by an element of the field).

Then, the *set U* with these operations of addition, multiplication, and multiplication by real numbers is a *basic descriptive image algebra over the field of real numbers.*

Proof.

The proof is based on verification of the properties of algebras (see Section 3, Definition 1) for the set U with the introduced operations of addition, multiplication, and multiplication by real numbers.

I. By assumption of the theorem, R is the field of real numbers.

II. Verification of the properties of ring U ($r_1, r_2, r_3 \in U$, $r_1 \longrightarrow r_1(A, B)$, $r_2 \longrightarrow r_2(A, B)$, $r_3 \longrightarrow r_3(A, B)$).

1. Verification of the properties of the operation of addition.

$$\forall r_1, r_2 \in U, \ \exists!(r_1 + r_2) \in U.$$

(a) Verification of the associative property:

$$(r_1 \oplus (r_2 \oplus r_3))(A, B) = r_1(A, B) + (r_2(A, B) + r_3(A, B))$$
$$= \{\text{the associative property for addition of images (Theorem 2)}\} = (r_1(A, B) + r_2(A, B)) + r_3(A, B);$$

$$((r_1 \oplus r_2) \oplus r_3)(A, B)$$
$$= (r_1(A, B) + r_2(A, B)) + r_3(A, B);$$

hence, $(r_1 \oplus (r_2 \oplus r_3))(A, B) = ((r_1 \oplus r_2) \oplus r_3)(A, B)$.

(b) Verification of the commutative property:

$$(r_1 \oplus r_2)(A, B) = r_1(A, B) + r_2(A, B);$$

$$(r_2 \oplus r_1)(A, B) = r_2(A, B) + r_1(A, B)$$
$$= \{\text{ring property 1b in Theorem 2}\} = r_1(A, B) + r_2(A, B);$$

hence, $(r_1 \oplus r_2)(A, B) = (r_2 \oplus r_1)(A, B)$.

(c) Verification of the existence of zero.

Zero of the set U has the form $O(A, B) = I_0$, where I_0 is the zero element of the ring of images. By the ring property 1c in Theorem 2, $I_0 = O(A, B)$. Let us prove this:

$$(r_1 \oplus O)(A, B) = r_1(A, B) + O(A, B) = r_1(A, B);$$

hence, $\exists O \in U, \forall r_1 \in U, (r_1 \oplus O)(A, B) = r_1(A, B)$.

(d) Verification of the existence of the opposite element.

For element r_1 of the set U, the opposite element is $(-r_1)(A, B) = -r_1(A, B)$, where $-r_1(A, B)$ is the opposite element for $r_1(A, B)$ in the ring of images. Let us prove this:

$$(r_1 \oplus (-r_1))(A, B) = r_1(A, B) + (-r_1(A, B))$$
$$=\{\text{ring property 1d in Theorem 2}\} = O(A, B);$$

hence, $\forall r_2 \in U \exists(-r_1), (r_1 \oplus (-r_1))(A, B) = O(A, B)$.

2. Verification of the properties of the operation of multiplication.

$$\forall r_1, r_2 \in U, \ \exists!(r_1 \otimes r_2) \in U.$$

(a) Verification of the associative property:

$$(r_1 \otimes (r_2 \otimes r_3))(A, B)$$
$$= (r_1 \otimes (r_2(r_3(A, B), r_3(A, B))))(A, B)$$
$$= r_1(r_2((r_3(A, B), r_3(A, B)), r_2(r_3(A, B), r_3(A, B))));$$

$$((r_1 \otimes r_2) \otimes r_3)(A, B)$$
$$= ((r_1(r_2(A, B), r_2(A, B))) \otimes r_3)(A, B)$$
$$= r_1(r_2(r_3(A, B), r_3(A, B)), r_2(r_3(A, B), r_3(A, B)));$$

hence, $(r_1 \otimes (r_2 \otimes r_3))(A, B) = ((r_1 \otimes r_2) \otimes r_3)(A, B)$.

3. Verification of the distributive property.

For any $\alpha, \beta \in R$, we have

$$((\alpha r_1 \oplus \beta r_2) \otimes r_3)(A, B)$$
$$= ((\alpha r_1(A, B) + \beta r_2(A, B)) \otimes r_3)(A, B)$$
$$= \alpha r_1(r_3(A, B), r_3(A, B)) + \beta r_2(r_3(A, B), r_3(A, B));$$

$$((\alpha r_1 \otimes r_3) \oplus (\beta r_2 \otimes r_3))(A, B)$$
$$= ((\alpha r_1(r_3(A, B), r_3(A, B)))$$
$$\oplus (\beta r_2(r_3(A, B), r_3(A, B))))(A, B)$$
$$= \alpha r_1(r_3(A, B), r_3(A, B)) + \beta r_2(r_3(A, B), r_3(A, B));$$

hence, $((\alpha r_1 \oplus \beta r_2) \otimes r_3) = ((\alpha r_1 \otimes r_3) \oplus (\beta r_2 \otimes r_3))(A, B)$.

Thus, the operations of addition and multiplication satisfy the properties of a ring.

III. Verification of the properties of a ring being an R-vector space ($\forall \alpha \in P, r \in U, r \longrightarrow r(A, B)$: $\alpha r \in U$).

1. Verification of property 1 of a ring being an R-vector space:

$$(\alpha(\beta r))(A, B) = (\alpha(\beta r(A, B)))(A, B) = \alpha\beta r(A, B);$$

$$((\alpha\beta)r)(A, B) = (\alpha\beta)r(A, B) = \alpha\beta r(A, B);$$

hence, $(\alpha(\beta r))(A, B) = ((\alpha\beta)r)(A, B)$.

2. Verification of property 2 of a ring being an R-vector space:

$$((\alpha + \beta)r)(A, B) = (\alpha + \beta)r(A, B)$$
$$= \alpha r(A, B) + \beta r(A, B);$$

$$(\alpha r \oplus \beta r)(A, B) = \alpha r(A, B) + \beta r(A, B);$$

hence, $((\alpha + \beta)r)(A, B) = (\alpha r \oplus \beta r)(A, B)$.

3. Verification of property 3 of a ring being an R-vector space:

$$(\alpha(r_1 \oplus r_2))(A, B) = \alpha(r_1(A, B) + r_2(A, B))$$
$$= \alpha r_1(A, B) + \alpha r_2(A, B);$$

$$(\alpha r_1 \oplus \alpha r_2)(A, B) = \alpha r_1(A, B) + \alpha r_2(A, B);$$

hence, $(\alpha(r_1 \oplus r_2))(A, B) = (\alpha r_1 \oplus \alpha r_2)(A, B)$.

Thus, all the properties of a ring being an R-vector space hold.

Hence, the set U with the introduced operations of addition, multiplication, and multiplication by real numbers is an algebra over the field of real numbers. By virtue of nature of elements of the set U, this algebra is a *basic descriptive image algebra.*

The theorem is proved.

3.5. Examples of Operations That Do not Generate Descriptive Image Algebras

3.5.1. Description of examples

This subsection contains two examples (Example 4 and Example 5).

As operands of set U in Example 4, we consider images defined on a fixed set X with different fixed range domains F_i, $i = 1, 2, \ldots$: $I_i = \{(x, f_i(x)), x \in X, f_i(x) \in F_i\}$. As the operation of addition of two images I_1 and I_2, we use the operation that produces the function determining the resulting image, which is the pointwise sum (at each point of set X) of functions $f_1(x)$ and $f_2(x)$ describing the images. As the operation of multiplication of two images I_1 and I_2, we use the operation producing the function determining the resulting image, which is the pointwise superposition (at each point) of functions $f_1(x)$ and $f_2(x)$ describing the images. At the points where the value of function $f_2(x) \notin X \subset F_2$, instead of the superposition, we take the value of function $f_2(x)$. As the operation of multiplication of image I by an element of the field, we use the operation producing the function determining the resulting image by multiplying function f describing the image by the element of the field at each point of set X.

As operands of set U in Example 5, we consider images defined on fixed sets X_i with different fixed range domains F_i, $i = 1, 2, \ldots$: $I_i = \{(x, f_i(x)), x \in X_i, f_i(x) \in F_i\}$. As the operation of addition of two images I_1 and I_2, we use the operation producing the function determining the resulting image, which is the pointwise sum (at each point of the intersection of sets X_1 and X_2) of functions $f_1(x)$ and $f_2(x)$ determining the images; at the points of sets X_1 and X_2 where only one image is defined, this image is treated as the result of the operation. As the operation of multiplication of two images I_1 and I_2, we use the operation producing the function determining the resulting image, which is the pointwise product (at each point of the intersection of sets X_1 and X_2) of functions $f_1(x)$ and $f_2(x)$ describing the images; at the points of sets X_1 and X_2 where only one image is defined, this image is treated as the result of the operation. As the operation of multiplication of image I by an element of the field, we use the operation producing the function determining the resulting image, which is the pointwise product (at each point of set X) of function f determining the image by the element of the field.

These examples demonstrate operands and operations, which do not generate a DIA with one ring.

3.5.2. Example 4

I. Elements of set U

Images defined on a fixed set X with different fixed range domains F_i, $i = 1, 2, \ldots$: $I = \{(x, f(x)), x \in X, f_i(x) \in F_i\}$.

II. Operations on the set U

Let $I_1 = \{(x, a(x)), x \in X, a(x) \in F_1\}$ and $I_2 = (x, b(x)), x \in X, b(x) \in F_2\}$.

* Operation of addition of two images I_1 and I_2:

$$I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\}; \quad (36)$$

* Operation of multiplication of two images I_1 and I_2:

$$I_1 \cdot I_2 = \begin{cases} (x, a(b(x))), & b(x) \in X \\ (x, b(x)), & b(x) \notin X \end{cases}; \quad (37)$$

* Operation of multiplication of image I by a real number $\alpha \in R$:

$$\alpha \cdot I = \{(x, \alpha \cdot f(x)), x \in X\}. \quad (38)$$

III. Interpretation of the introduced operations

III.1. Operation of addition of two elements of the set (see Example 1).

III.2. Operation of multiplication of two elements of the set.

The physical sense of the operation is as follows: definition of an image on a set specified by another image (if this operation is undefined ($F \not\subset X$), as the result of the operation of multiplication, we take the value of the second operand).

Proposition 8. The operation introduced in (37) of multiplication of two elements of the set is *physically interpretable but not visually interpretable*, i.e., a physically interpretable operation in the weak sense.

Proof.

1. Physical interpretability of the operation: by Definition 15, an operation is physically interpretable if, applying this operation, we obtain an image.

2. Visual interpretability of the operation: in the general case, this operation is not visually interpretable, because, applying it to two arbitrary images, we obtain a semantically meaningless image (Definition 16).

The proposition is proved.

This operation is similar to the notion of Ritter's template [24, 25].

III.3. Operation of multiplication of an element of set U by a real number (see Example 1).

IV. Necessary conditions for formulating and proving the theorem in item V

Let a set of sets $\{F_i\}_1^\infty$ be given. Suppose that, on each set F_i, a set of function Φ_i is specified such that $\forall i = 1, 2, \ldots, \forall f \in \Phi_i$: $f: X \longrightarrow F_i$:

* an operation of addition of two functions from Φ_i, $\Phi_j \subset R^n$: $i, j = 1, 2, \ldots, \forall a(x) \in \Phi_i, b(x) \in \Phi_j$: $\exists!(a + b)(x) \equiv b(x) \in \Phi_k$, k = 1, 2, …, is introduced with the following properties ($\forall a(x) \in \Phi_i, b(x) \in \Phi_j, c(x) \in \Phi_y$, $i, j, y = 1, 2, \ldots$):

$$4.1.\ a(x) + (b(x) + c(x)) = (a(x) + b(x)) + c(x); \quad (39)$$

$$4.2.\ a(x) + b(x) = b(x) + a(x); \quad (40)$$

$$4.3.\ \forall i: \forall a(x) \in \Phi_i,\ \exists 0 \in \Phi_i: a(x) + 0 = a(x); \quad (41)$$

$$4.4.\ \forall i: \forall a(x) \in \Phi_i,\ \exists(-a(x)) \in \Phi_i: a(x) + (-a(x)) = 0; \quad (42)$$

* an operation of superposition of two elements from Φ_i, Φ_j: $i, j = 1, 2, \ldots, \forall a(x) \in \Phi_i, b(x) \in \Phi_j$ is introduced; at the points where $b(x) \in X$: $\exists!(ab)(x) \equiv a(b(x)) \in \Phi_k$, $k = 1, 2, \ldots$;

* on the set of functions Φ_i, $i = 1, 2, \ldots$, an operation of multiplication by elements of field R is introduced: $\forall \alpha \in R$, $a(x) \in \Phi_i$, $i = 1, 2, \ldots$: $\exists! \alpha a(x) \in \Phi_i$, $i = 1, 2, \ldots$, with the following properties: ($\forall a(x) \in \Phi_i$, $b(x) \in \Phi_j$, $c(x) \in \Phi_y$, $i, j, y = 1, 2, \ldots, \forall \alpha, \beta \in R$):

$$4.5.\ (\alpha a(x) + \beta b(x))c(x) = \alpha a(x)c(x) + \beta b(x)c(x); \quad (43)$$

$$4.6.\ \alpha(\beta a(x)) = \alpha\beta a(x); \quad (44)$$

$$4.7.\ (\alpha + \beta)a(x) = \alpha a(x) + \beta a(x); \quad (45)$$

$$4.8.\ \alpha(a(x) + b(x)) = \alpha a(x) + \alpha b(x). \quad (46)$$

V. Statement and proof of the theorem on validity/invalidity of the properties of DIA for the construction considered in the example

Theorem 4.

Let

* R be the field of real numbers;

* $I = \{(x, f(x)), x \in X, f(x) \in F\}$ ($F \in \{F_i\}_1^\infty$, where F is the set of values of image I on set X, $f(x) \in \{\Phi_i\}_1^\infty$) are elements of set U;

* $I_1 = \{(x, a(x)), x \in X, a(x) \in F_1\}$, $I_2 = \{(x, b(x)), x \in X, b(x) \in F_2\}$, ($F_1, F_2 \in \{F_i\}_1^\infty$, $a(x), b(x) \in \{\Phi_i\}_1^\infty$).

We introduce

* an operation of addition of two images I_1 and I_2:

$$I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\};$$

* an operation of multiplication of two images I_1 and I_2:

$$I_1 \cdot I_2 = \begin{cases} (x, a(b(x))), & b(x) \in X \\ (x, b(x)), & b(x) \notin X \end{cases};$$

* an operation of multiplication of image I by real number $\alpha \in R$:

$$\alpha \cdot I = \{(x, \alpha \cdot f(x)), x \in X\}.$$

Then, the resulting construction is not an algebra but *is an additive group.*

Proof.

The proof is based on testing whether the set U with the introduced operations of addition, multiplication, and multiplication by real numbers satisfies the properties of an algebra (see Section 3, Definition 1) and whether the set U with the operation of addition satisfies the properties of a group (see Section 3, Definition 2).

I. By assumption of the theorem, R is the field of real numbers.

II. Verification of the properties of ring U ($I_1, I_2, I_3 \in U$, $I_1 = \{(x, a(x)), x \in X, a(x) \in F_1\}$, $I_2 = \{(x, b(x)), x \in X, b(x) \in F_2\}$, $I_3 = \{(x, c(x)), x \in X, c(x) \in F_3\}$).

1. Verification of the properties of the operation of addition.

$$\forall I_1, I_2 \in U,\ \exists!(I_1 + I_2) \in U.$$

(a) Verification of the associative property.

$$I_1 + (I_2 + I_3) = I_1 + \{(x, b(x) + c(x)), x \in X\}$$
$$= \{(x, a(x) + (b(x) + c(x))), x \in X\};$$
$$(I_1 + I_2) + I_3 = \{(x, a(x) + b(x)), x \in X\} + I_3$$
$$= \{(x, (a(x) + b(x)) + c(x)), x \in X\}$$
$$= \{\text{by property 4.1 of item IV in this example}\}$$
$$= \{(x, a(x) + (b(x) + c(x))), x \in X\};$$

hence, $I_1 + (I_2 + I_3) = (I_1 + I_2) + I_3$.

(b) Verification of the commutative property.

$$I_1 + I_2 = \{(x, a(x) + b(x)), x \in X\};$$
$$I_2 + I_1 = \{(x, b(x) + a(x)), x \in X\}$$
$$= \{\text{by property 4.2 of item IV in this example}\}$$
$$= \{(x, a(x) + b(x)), x \in X\};$$

hence, $I_1 + I_2 = I_2 + I_1$.

(c) Verification of the existence of zero.

Zero of the set U has the form $O = \{(x, 0), x \in X\}$. Let us prove this.

$$I_1 + O = \{(x, a(x) + 0), x \in X\}$$
$$= \{\text{by property 4.3 of item IV in this example}\}$$
$$= \{(x, \mathrm{a}(x)), x \in X\} = I_1;$$

hence, $\exists O \in U, \forall I_1 \in U, I_1 + O = I_1$.

(d) Verification of the existence of an opposite element.

For image I_1, the opposite element is $(–I_1) = \{(x, –a(x)), x \in X\}$. Let us prove this.

$$I_1 + (-I_1) = \{(x, a(x) - a(x)), x \in X\}$$
$$= \{\text{by property 4.4 of item IV in this example}\}$$
$$= \{(x, 0), x \in X\} = O;$$

hence, $\forall I_1 \in U, \exists(-I_1), I_1 + (-I_1) = 0$.

2. Verification of the properties of the operation of multiplication.

$$\forall I_1, I_2 \in U,\ \exists!(I_1 \cdot I_2) \in U.$$

(a) Verification of the associative property.

$$I_1 \cdot (I_2 \cdot I_3) = I_1 \cdot \left(\begin{cases}(x, b(c, (x))), & c(x) \in X \\ (x, c(x)), & c(x) \notin X\end{cases}\right) = \begin{cases}\begin{cases}(x, a(b(c(x)))), & b(c(x)) \in X \\ (x, b(c(x))), & b(c(x)) \notin X\end{cases} & c(x) \in X \\ (x, c(x)), \quad c(x) \notin X\end{cases}$$

$$= \begin{cases}(x, a(b(c(x)))), & b(c(x)) \in X,\ c(x) \in X \\ (x, b(c(x))), & b(c(x)) \notin X,\ c(x) \in X \\ (x, c(x)), & c(x) \notin X\end{cases};$$

$$(I_1 \cdot I_2) \cdot I_3 = \left(\begin{cases}(x, a(b(x))), & b(x) \in X \\ (x, b(x)), & b(x) \notin X\end{cases}\right) \cdot I_3$$

$$= \begin{cases}\begin{cases}(x, a(b(c(x)))), & b(c(x)) \in X,\ c(x) \in X \\ (x, b(c(x))), & b(c(x)) \notin X,\ c(x) \in X \\ (x, c(x)), & c(x) \notin X\end{cases} & (b(x) \in X) \\ \begin{cases}(x, b(c(x))), & c(x) \in X \\ (x, c(x)), & c(x) \notin X\end{cases} & b(x) \notin X\end{cases}$$

$$= \begin{cases} (x, a(b(c(x)))), \ b(x) \in X, \ b(c(x)) \in X, \ c(x) \in X \\ (x, b(c(x))), \ b(c(x)) \notin X, \ c(x) \in X, \ b(x) \in X \quad \text{or } c(x) \in X, \ b(x) \notin X. \\ (x, c(x)), \ c(x) \notin X \end{cases}$$

Consider a point $x \in X$ such that $b(x) \notin X$, $b(c(x)) \in X$, $c(x) \in X$:

$$I_1 \cdot (I_2 \cdot I_3) = \{(x, a(b(c(x))))\};$$

$$(I_1 \cdot I_2) \cdot I_3 = \{(x, b(c(x)))\};$$

hence, $I_1 \cdot (I_2 \cdot I_3) \neq (I_1 \cdot I_2) \cdot I_3$.

The operation of multiplication is not associative; i.e., not all properties of rings hold for elements of the set U. The properties of a group hold for the operation of addition. Without additional constraints, with these definitions of the operations of addition and multiplication, we have constructed only a *group with respect to the operation of addition.*

The theorem is proved.

3.5.3. Example 5

I. Elements of set U

Images defined on fixed sets X_i with different fixed range domains F_i, $i = 1, 2, \ldots$: $I = \{(x, f(x)), x \in X_i, f(x) \in F_i\}$.

II. Operations introduced on the set U

Let $I_1 = \{(x, a(x)), x \in X_1, a(x) \in F_1\}$ and $I_2 = \{(x, b(x)), x \in X_2, b(x) \in F_2\}$.

* Operation of addition of two images I_1 and I_2:

$$I_1 + I_2 = \begin{cases} (x, a(x) + b(x)), \ x \in X_1 \cap X_2 \\ (x, a(x)), \ x \in X_1 \backslash X_2 \\ (x, b(x)), \ x \in X_2 \backslash X_1 \end{cases}; \quad (47)$$

* Operation of multiplication of two images I_1 and I_2:

$$I_1 \cdot I_2 = \begin{cases} (x, a(x) \cdot b(x)), \ x \in X_1 \cap X_2 \\ (x, a(x)), \ x \in X_1 \backslash X_2 \\ (x, b(x)), \ x \in X_2 \backslash X_1 \end{cases}; \quad (48)$$

* Operation of multiplication of images I by real number $\alpha \in R$:

$$\alpha \cdot I = \{(x, \alpha \cdot f(x)), x \in X\}. \quad (49)$$

III. Interpretation of the introduced operations

III.1. Operation of addition of two elements of the set.

The physical sense of the operation: the picture of total brightness of two images on the intersection of sets these images are specified on; at the points of sets X_1 and X_2 where only one image is specified, this image is treated as the result of the operation.

Proposition 13. The operation introduced in (47) of addition of two elements of the set is *physically interpretable but not visually interpretable*; i.e., this operation is physically interpretable in the weak sense.

Proof.

1. Physical interpretability of the operation: by Definition 15, an operation is physically interpretable if, applying this operation, we obtain an image.

2. Visual interpretability of the operation: in the general case, this operation is not visually interpretable, because, applying it to two arbitrary images, we obtain a semantically meaningless image (Definition 16).

The proposition is proved.

An example of this operation is as follows: it can be used in the problem of overlapping of certain images (composition of a face of the objects such as eyes, nose, mouth).

III.2. Operation of multiplication of two elements of the set.

The physical sense of the operation is this: pointwise multiplication of two images under the condition that, at the points of the set X where only one image is defined (the first or second operand); this image (the first or second operand, respectively) is the result of the operation of multiplication.

Proposition 9. The operation introduced in (48) of multiplication of two elements of the set is *physically interpretable but not visually interpretable*; i.e., it is a physically interpretable operation in the weak sense.

Proof.

1. Physical interpretability of the operation: by Definition 15, an operation is physically interpretable if, applying this operation, we obtain an image.

2. Visual interpretability of the operation: in the general case, this operation is not visually interpretable, because, applying it to two arbitrary images, we obtain a semantically meaningless image (Definition 16).

The proposition is proved.

An example of this operation is as follows: it may be used in processing of binary images, namely, in multiplying an image by a mask represented as an image.

III.3. Operation of multiplication of an element of set U by a real number.

See Example 1.

IV. Necessary conditions for formulating and proving the proposition in item V

Suppose that a set of sets $\{X_i\}_1^\infty$ and a set of sets $\{F_i\}_1^\infty$ are given and, on each set F_i, a set of functions Φ_i is given such that $\forall i = 1, 2, \ldots, \forall f \in \Phi_i$: f: $X_i \longrightarrow F_i$, the sets $\{X_i\}_1^\infty$ are not necessarily disjoint; neither are the sets $\{F_i\}_1^\infty$.

* an operation of addition of two functions from Φ_i, Φ_j: $i, j = 1, 2, \ldots, \forall a(x) \in \Phi_i$ for $x \in X_i$, $b(x) \in \Phi_j$ for $x \in X_j$: $\forall x \in X_i \cap X_j$, $\exists!(a+b)(x) \equiv a(x) + b(x) \in \Phi_k$, $k = 1, 2, \ldots$, is introduced with the following properties ($\forall a(x) \in \Phi_i$, $b(x) \in \Phi_j$, $c(x) \in \Phi_y$, $i, j, y = 1, 2, \ldots$, on the common definition set X):

$$5.1.\ a(x) + (b(x) + c(x)) = (a(x) + b(x)) + c(x); \quad (50)$$

$$5.2.\ a(x) + b(x) = b(x) + a(x); \quad (51)$$

$$5.3.\ \forall i\colon \forall a(x) \in \Phi_i,\ \exists 0 \in \Phi_i\colon a(x) + 0 = a(x); \quad (52)$$

$$5.4.\ \forall i\colon \forall a(x) \in \Phi_i,\ \exists(-a(x)) \in \Phi_i\colon \\ a(x) + (-a(x)) = 0; \quad (53)$$

* an operation of multiplication of two functions from Φ_i, Φ_j: $i, j = 1, 2, \ldots, \forall a(x) \in \Phi_i$ for $x \in X_i$, $b(x) \in \Phi_j$ for $x \in X_j$: $\forall x \in X_i \cap X_j$, $\exists! a(x) \cdot b(x) \in \Phi_k$, $k = 1, 2, \ldots$, is introduced with the following properties ($\forall a(x) \in \Phi_i$, $b(x) \in \Phi_j$, $c(x) \in \Phi_y$, $i, j, y = 1, 2, \ldots$, on the common definition set X):

$$5.5.\ a(x) \cdot (b(x) \cdot c(x)) = (a(x) \cdot b(x)) \cdot c(x); \quad (54)$$

* on the set of functions Φ_i, $i = 1, 2, \ldots$, an operation of multiplication by elements of field R is introduced $\forall \alpha \in R$, $a(x) \in \Phi_i$ for $x \in X_i$, $i = 1, 2, \ldots$: $\exists! \alpha a(x) \in \Phi_i$ for $x \in X_i$, $i = 1, 2, \ldots$, with the following properties ($\forall a(x) \in \Phi_i$, $b(x) \in \Phi_j$, $c(x) \in \Phi_y$, $i, j, y = 1, 2, \ldots$, on the common definition set X, $\forall \alpha, \beta \in R$):

$$5.6.\ (\alpha a(x) + \beta b(x))c(x) \\ = \alpha a(x)c(x) + \beta b(x)c(x); \quad (55)$$

$$5.7.\ \alpha(\beta a(x)) = \alpha\beta a(x); \quad (56)$$

$$5.8.\ (a + \beta)a(x) = \alpha a(x) + \beta a(x); \quad (57)$$

$$5.9.\ \alpha(a(x) + b(x)) = \alpha a(x) + \alpha b(x). \quad (58)$$

V. Statement and proof of the theorem on validity/invalidity of the properties of DIA for the construction considered in the example

Proposition 5.

Let

* R be the field of real numbers;

* $I = \{(x, f(x)), x \in X, f(x) \in F\}$ ($X \in \{X_i\}_1^\infty$, $F \in \{F_i\}_1^\infty$, where F is the set of values taken by of image I on set X and $f(x) \in \{\Phi_i\}_1^\infty$) are elements of the set U;

* $I_1 = \{(x, \mathrm{a}(x)), x \in X_1, a(x) \in F_1\}$, $I_2 = \{(x, \mathrm{b}(x)), x \in X_2, b(x) \in F_2\}$, ($F_1, F_2 \in \{F_i\}_1^\infty$, $X_1, X_2 \in \{X_i\}_1^\infty$, $a(x), b(x) \in \{\Phi_i\}_1^\infty$).

We introduce

* an operation of addition of two images I_1 and I_2:

$$I_1 + I_2 = \begin{cases} (x, a(x) + b(x)), & x \in X_1 \cap X_2 \\ (x, a(x)), & x \in X_1 \backslash X_2 \\ (x, b(x)), & x \in X_2 \backslash X_1 \end{cases};$$

* an operation of multiplication of two images I_1 and I_2:

$$I_1 \cdot I_2 = \begin{cases} (x, a(x) \cdot b(x)), & x \in X_1 \cap X_2 \\ (x, a(x)), & x \in X_1 \backslash X_2 \\ (x, b(x)), & x \in X_2 \backslash X_1 \end{cases};$$

* an operation of multiplication of image I by real number $\alpha \in R$:

$$\alpha \cdot I = \{(x, \alpha \cdot f(x)), x \in X\}.$$

Then, the *set* U with these operations is *neither algebra nor group with any one of these operations.*

Proof.

The proof is based on testing whether the set U with the introduced operations of addition, multiplication, and multiplication by real numbers satisfies the properties of an algebra (see Section 3, Definition 1) and whether the set U with the operation of addition satisfies the properties of a group (see Section 3, Definition 2).

I. By assumption of the theorem, R is the field of real numbers.

II. Verification of the properties of ring U ($I_1, I_2, I_3 \in U$, $I_1 = \{(x, a(x)), x \in X, a(x) \in X\}$, $I_2 = \{(x, b(x)), x \in X, b(x) \in X\}$, $I_3 = \{(x, c(x)), x \in X, c(x) \in X\}$).

1. Verification of the associative property of the operation of addition.

$$\forall I_1, I_2 \in U,\ \exists!(I_1 + I_2) \in U;$$

$$I_1 + (I_2 + I_3) = I_1 + \begin{cases} (x, b(x) + c(x)), \ x \in X_2 \cap X_3 \\ (x, b(x)), \ x \in X_2 \backslash X_3 \\ (x, c(x)), \ x \in X_3 \backslash X_2 \end{cases}$$

$$= \begin{cases} (x, a(x) + b(x) + c(x)), \ x \in X_1 \cap X_2 \cap X_3 \\ (x, a(x)), \ x \in X_1 \backslash \{X_2 \cap X_3\} \\ (x, b(x) + c(x)), \ x \in \{X_2 \cap X_3\} \backslash X_1 \\ (x, a(x) + b(x)), \ x \in X_1 \backslash \{X_2 \backslash X_3\} \\ (x, a(x)), \ x \in X_1 \cap \{X_2 \backslash X_3\} \\ (x, b(x)), \ x \in \{X_2 \backslash X_3\} \backslash X_1 \\ (x, a(x) + c(x)), \ x \in X_1 \cap \{X_3 \backslash X_2\} \\ (x, a(x)), \ x \in X_1 \backslash \{X_3 \backslash X_2\} \\ (x, c(x)), \ x \in \{X_3 \backslash X_2\} \backslash X_1 \end{cases}$$

$$= \begin{cases} (x, a(x) + b(x) + c(x)), \ x \in X_1 \cap X_2 \cap X_3 \\ (x, a(x) + b(x)), \ x \in X_1 \cap \{X_2 \backslash X_3\} \\ (x, b(x) + c(x)), \ x \in \{X_2 \cap X_3\} \backslash X_1 \\ (x, a(x) + c(x)), \ x \in X_1 \cap \{X_3 \backslash X_2\} \\ (x, a(x)), \ x \in X_1 \\ (x, b(x)), \ x \in \{X_2 \backslash X_3\} \backslash X_1 \\ (x, c(x)), \ x \in \{X_3 \backslash X_2\} \backslash X_1 \end{cases} ;$$

$$(I_1 + I_2) + I_3 = \begin{cases} (x, a(x) + b(x)), \ x \in X_1 \cap X_2 \\ (x, a(x)), \ x \in X_1 \backslash X_2 \\ (x, b(x)), \ x \in X_2 \backslash X_1 \end{cases} + I_3$$

$$= \begin{cases} (x, a(x) + b(x) + c(x)), \ x \in X_1 \cap X_2 \cap X_3 \\ (x, a(x) + b(x)), \ x \in \{X_1 \cap X_2\} \backslash X_3 \\ (x, c(x)), \ x \in X_3 \backslash \{X_1 \cap X_2\} \\ (x, a(x) + c(x)), \ x \in \{X_1 \backslash X_2\} \cap X_3 \\ (x, a(x)), \ x \in \{X_1 \backslash X_2\} \backslash X_3 \\ (x, c(x)), \ x \in X_3 \backslash \{X_1 \backslash X_2\} \\ (x, b(x) + c(x)), \ x \in \{X_2 \backslash X_1\} \cap X_3 \\ (x, b(x)), \ x \in \{X_2 \backslash X_1\} \backslash X_3 \\ (x, c(x)), \ x \in X_3 \backslash \{X_2 \backslash X_1\} \end{cases}$$

$$= \begin{cases} (x, a(x) + b(x) + c(x)), \ x \in X_1 \cap X_2 \cap X_3 \\ (x, a(x) + b(x)), \ x \in \{X_1 \cap X_2\} \backslash X_3 \\ (x, b(x) + c(x)), \ x \in \{X_2 \backslash X_1\} \cap X_3 \\ (x, a(x) + c(x)), \ x \in \{X_1 \backslash X_2\} \backslash X_3 \\ (x, a(x)), \ x \in \{X_1 \backslash X_2\} \backslash X_3 \\ (x, b(x)), \ x \in \{X_2 \backslash X_1\} \backslash X_3 \\ (x, c(x)), \ x \in X_3 \end{cases} .$$

One can easily see that the identity $I_1 + (I_2 + I_3) = (I_1 + I_2) + I_3$ holds only if the condition $X_1 \cap X_2 = X_2 \cap X_3 = X_3 \cap X_1 = \varnothing$ is satisfied. This contradicts the definition of the sum, because the sum is defined on the union of the sets.

Similarly, the associative property does not hold for the operation of multiplication.

The set *is not a group for any one of the operations introduced.*

The proposition is proved.

3.6. Construction of Descriptive Image Algebras with the Help of Operations of Ritter's Image Algebra

The above examples demonstrate that not every operation generates a DIA with one ring. In this connection, a question arises about the conditions that should be imposed on the operations and operands in order that these operations and operands may be used for constructing algorithmic schemes based on DIA with one ring.

In distinguishing the operations, which generate and do not generate a DIA with one ring, we essentially apply the apparatus of the standard image algebra (Ritter's image algebra) as an element of formalization of an example. Since, in a certain sense, DIAs are more specialized structures (because, along with the closeness condition imposed on operations of Ritter's image algebras, the conditions of a standard algebra are imposed on the operations of DIAs), we may claim that not every operation of Ritter's image algebras generate a DIA with one ring over images.

Consider in more detail the operations of the standard image algebra in the context of construction of a DIA. The conditions determining the subsets of operations of the standard image algebra (described in [25]), which make possible the construction of a DIA with one ring, are established in Theorem 6.

Theorem 6.

Suppose that

* F is a field;

* elements of ring U are images in Ritter's sense; i.e., $I = \{(x, a(x)), x \in X, a(x) \in F\}$, where F is a set of values and X is a set of points;

* operations ⊕, ⊗, and multiplication by elements of the field are introduced. These operations belong to the set of operations of the standard image algebra.

Then, the following conditions are necessary and sufficient for construction of a basic DIA with one ring:

(1) $I \in (R^X)$ or $I \in (R^n)^X$, then

* the role of operation ⊕ is played by the operation of addition of two images;

* the role of operation ⊗ is played by the operation of multiplication of two images;

* as multiplication by elements of the field, we take the scalar multiplication of the element of set F and the image;

(2) $I \in (2^F)^X$, then

* the role of operation ⊗ is played by the operation of union of two images or the operation of intersection of two images;

* as multiplication by elements of the field, we take the scalar multiplication of the element of set F and the image;

(3) $I \in (R^2)^X$, then, in addition to the operations described in item (1),

* the role of operation ⊕ is played by the operation of addition of two images;

* the role of operation ⊗ is played by the operation of product of two images or the following operation:

Suppose that γ_1 and γ_2 are binary operations $R^2 \times R^2 \longrightarrow R$ defined as follows:

$$(x_1, x_2)\gamma_1(y_1, y_2) = x_1y_1 - x_2y_2, \quad (59)$$

$$(x_1, x_2)\gamma_2(y_1, y_2) = x_1y_2 + x_2y_1, \quad (60)$$

if I_1 and $I_2 \in (R^2)^X$ represent two complex-valued images, then the product $I_3 = I_1\gamma I_2$ represents the complex product

$$I_3 = \{(x, c(x)), c(x) = (a_1(x)b_1(x) - a_2(x)b_2(x), a_1(x)b_2(x) + a_2(x)b_1(x)), x \in X\}; \quad (61)$$

* as multiplication by elements of the field, we take the scalar multiplication of the element of set F and the image.

Proof.

The proof is based on verification of the properties of an algebra (Definition 1 in Section 3) for the operations of the standard image algebra and their combinations.

As the operation of multiplication by elements of the field, we consider two operations induced by operations in the algebraic system F:

for $k \in F$ and $a \in F^X$,

$$K\gamma a = \{(x, c(x)): c(x) = k\gamma a(x), x \in X\} \quad (62)$$

and

$$A\gamma k = \{(x, c(x)): c(x) = a(x)\gamma k, x \in X\}. \quad (63)$$

Binary operations on images are also uniquely determined by the operations of the algebraic system F. For instance, if γ is a binary operation on the set F and $a, b \in F^X$, then

$$A\gamma b = \{(x, c(x)): c(x) = a(x)\gamma b(x), x \in X\}. \quad (64)$$

This operation may be considered as the operation of addition in ring U only if it satisfies property 1 of a ring and, together with the operations of multiplication and multiplication by elements of the field, satisfies property 3 of a ring and the properties of a vector space.

This operation may be considered as the operation of multiplication in ring U only if it satisfies property 2 of a ring and, together with the operations of multiplication and multiplication by elements of the field, satisfies property 3 of a ring and the properties of a vector space.

Let us test whether properties 1 and 2 of ring U hold for the sets of operations of image algebra of Ritter.

* Let $I_1, I_2 \in (R^X)$.

Replacing γ with particular operations +, ·, ∨ (operation of taking the maximum), and ∧ (operation of taking the minimum) on real-valued images, we obtain the following:

$$I_1 + I_2 = \{(x, c(x)): c(x) = a(x) + b(x), x \in X\} \quad (65)$$

(properties 1 and 2 of a ring hold for the operation of addition);

$$I_1 \cdot I_2 = \{(x, c(x)): c(x) = a(x) \cdot b(x), x \in X\} \quad (66)$$

(properties 1 and 2 of a ring hold for the operation of multiplication);

$$I_1 \vee I_2 = \{(x, c(x)): c(x) = a(x) \vee b(x), x \in X\} \quad (67)$$

(the operation of taking the maximum satisfies property 2 of a ring and does not satisfy properties 1(c), 1(d));

$$I_1 \wedge I_2 = \{(x, c(x)): c(x) = a(x) \wedge b(x), x \in X\} \quad (68)$$

(the operation of taking the minimum satisfies property 2 of a ring and does not satisfy properties 1(c), 1(d)).

* Let $I_1, I_2 \in (2^F)^X$.

Let 2^X be the power set, i.e., the set of all subsets of set X. Suppose that an image I is such that $I: X \longrightarrow 2^F$. In this case, the following binary operations may be introduced:

$$I_1 \cup I_2 = \{(x, c(x)): c(x) = a(x) \cup b(x), x \in X\} \quad (69)$$

(the operation of union satisfies property 2 of a ring and does not satisfy properties 1(c), 1(d));

$$I_1 \cap I_2 = \{(x, c(x)): c(x) = a(x) \cap b(x), x \in X\} \quad (70)$$

(the operation of intersection satisfies property 2 of a ring and does not satisfy properties 1(c), 1(d)).

* Let $I_1, I_2 \in (R^{\geq 0})^X$

$$I_1^{I_2} = \{(x, c(x)): c(x) = a(x)^{b(x)}, x \in X\} \quad (71)$$

(the operation of exponentiation does not satisfy the properties of associativity (1(a), 2)).

* Let $I_1, I_2 \in (R^{+})^X$

$$\log_{I_2} I_1 = \{(x, c(x)): c(x) = \log_{b(x)} a(x), x \in X\} \quad (72)$$

(the operation of taking the logarithm does not satisfy the properties of associativity (1(a), 2)).

* Let $I_1 \in (F)^X$, $I_2 \in (F)^Y$, X and Y be subsets of a topological space.

By the extension $A|^b(x)$ of an image $a \in F^X$ with an image $b \in F^Y$ on a set Y, where X and Y are subsets of a topological space, we mean:

$$A|^b(x) = \begin{cases} a(x), & \text{if } x \in X \\ b(x), & \text{if } x \in Y \backslash X \end{cases}. \quad (73)$$

We also introduce an operation of concatenation by a series of images $a \in F^{Z_m \times Z_k}$ and $b \in F^{Z_m \times Z_n}$:

$$(a|b) \equiv a|^{b + (0, k)}. \quad (74)$$

Using the notion of matrix transpose, we similarly introduce the column concatenation:

$$\begin{pmatrix} a \\ b \end{pmatrix} = (a|b)' \quad (75)$$

(the concatenation operations do not satisfy the associative or commutative properties).

* Let $I_1, I_2 \in (R^n)^X$

$$I_1 + I_2 = (I_1^1 + I_2^1, \ldots, I_1^n + I_2^n) \quad (76)$$

(properties 1 and 2 of a ring hold for the operation of addition);

$$I_1 \cdot I_2 = (I_1^1 \cdot I_2^1, \ldots, I_1^n \cdot I_2^n) \quad (77)$$

(properties 1 and 2 of a ring hold for the operation of multiplication);

$$I_1 \vee I_2 = (I_1^1 \vee I_2^1, \ldots, I_1^n \vee I_2^n) \quad (78)$$

(the operation of taking the maximum satisfies property 2 of a ring and does not satisfy properties 1(c), 1(d));

$$I_1 \wedge I_2 = (I_1^1 \wedge I_2^1, \ldots, I_1^n \wedge I_2^n) \quad (79)$$

(the operation of taking the minimum satisfies property 2 of a ring and does not satisfy properties 1(c), 1(d)).

Let a binary operation γ be such that γ_j: $R^n \times R^n$ R, where $j = 1, \ldots, n$, and be defined as follows:

$$I_1 \gamma I_2 = (I_1 \gamma_1 I_2, \ldots, I_1 \gamma_n I_2). \quad (80)$$

For example, if γ_j: $R^n \times R^n \longrightarrow R$ are such that $(x_1, \ldots, x_n)\gamma(y_1, \ldots, y_n) = \max\{x_i \vee y_j: 1 \leq i \leq j\}$, then, for $I_1, I_2 \in (R^n)^X$ and $I_3 = I_1 \gamma I_2$, the components $c(x) = (c_1(x), \ldots, c_n(x))$ have the values

$$\begin{aligned} c_j(x) &= a(x)\gamma_j b(x) \\ &= \max\{a_i(x) \vee a_j(x): 1 \leq i \leq j\} \text{ for } j = 1, \ldots, n \end{aligned} \quad (81)$$

(the operation does not satisfy the properties of associativity (1(a), 2)).

Consider another example of a binary operation γ. Let γ_1 and γ_2 be binary operations $R^2 \times R^2 \longrightarrow R$ defined as follows (see (77, 78)):

$$(x_1, x_2)\gamma_1(y_1, y_2) = x_1 y_1 - x_2 y_2; \quad (82)$$

$$(x_1, x_2)\gamma_2(y_1, y_2) = x_1 y_2 + x_2 y_1. \quad (83)$$

If $I_1, I_2 \in (R^2)^X$ represent two complex-valued images, then the product $I_3 = I_1 \gamma I_2$ represents the complex product $c(x) = (a_1(x)b_1(x) - a_2(x)b_2(x), a_1(x)b_2(x) + a_2(x)b_1(x))$

(the operation satisfies properties 1 and 2 of a ring).

Consider several other examples of operations of image algebra of Ritter (let $I_1, I_2 \in (R^n)^X$, i.e., $I_1 = \{(x, a(x)): a(x) = (a_1(x), a_2(x), \ldots, a_n(x))\}$ and $I_2 = \{(x, b(x)): b(x) = (b_1(x), b_2(x), \ldots, b_n(x))\}$). For any subscript j, we can introduce the following operations of taking the maximum and taking the minimum in the jth component of the image representation:

$$\begin{aligned} I_1 \vee|_j I_2 &= \{(x, c(x)): c(x) = a(x), \\ &\text{if } a_j(x) \geq b_j(x), \text{ otherwise } c(x) = b(x)\} \end{aligned} \quad (84)$$

(the operation of taking the maximum in the jth component satisfies property 2 of a ring and does not satisfy properties 1(c), 1(d));

$$\begin{aligned} I_1 \wedge|_j I_2 &= \{(x, c(x)): c(x) = a(x), \\ &\text{if } a_j(x) \leq b_j(x), \text{ otherwise } c(x) = b(x)\} \end{aligned} \quad (85)$$

(the operation of taking the maximum in the jth component satisfies property 2 of a ring and does not satisfy properties 1(c), 1(d));

For the sake of visualization, we present operations satisfying property 1 or property 2 of a ring in the table. Columns 1 and 2 of this table show that properties 1 and 2 of a ring hold if this operation is used as the operation of addition or multiplication, respectively, over the given operands.

Considering different combinations of the operations, we conclude that property 3 of ring U and the properties of a vector space over field F hold only for operations mentioned in Theorem 1.

The proposition is proved.

Operands and operations of the standard image algebra, which may be used in construction of DAIs with one ring

Operands	Operations	1	2
$I_1, I_2 \in (R^X)$	$I_1 + I_2 = \{(x, c(x)): c(x) = a(x) + b(x), x \in X\}$	+	+
	$I_1 \cdot I_2 = \{(x, c(x)): c(x) = a(x) \cdot b(x), x \in X\}$	+	+
	$I_1 \vee I_2 = \{(x, c(x)): c(x) = a(x) \vee b(x), x \in X\}$	–	+
	$I_1 \wedge I_2 = \{(x, c(x)): c(x) = a(x) \wedge b(x), x \in X\}$	–	+
$I_1, I_2 \in (2^F)^X$	$I_1 \cup I_2 = \{(x, c(x)): c(x) = a(x) \cup b(x), x \in X\}$	–	+
	$I_1 \cap I_2 = \{(x, c(x)): c(x) = a(x) \cap b(x), x \in X\}$	–	+
$I_1, I_2 \in (R^n)^X$	$I_1 + I_2 = (I_1^1 + I_2^1, \ldots, I_1^n + I_2^n)$	+	+
	$I_1 \cdot I_2 = (I_1^1 \cdot I_2^1, \ldots, I_1^n \cdot I_2^n)$	+	+
	$I_1 \vee I_2 = (I_1^1 \vee I_2^1, \ldots, I_1^n \vee I_2^n)$	–	+
	$I_1 \wedge I_2 = (I_1^1 \wedge I_2^1, \ldots, I_1^n \wedge I_2^n)$	–	+
$I_1, I_2 \in (R^2)^X$	Complex product	+	+
$I_1, I_2 \in (R^n)^X$	$I_1 \vee\|_j I_2 = \{(x, c(x)): c(x) = a(x)$ if $a_j(x) \ge b_j(x)$, otherwise $c(x) = b(x)\}$	–	+
	$I_1 \wedge\|_j I_2 = \{(x, c(x)): c(x) = a(x)$ if $a_j(x) \le b_j(x)$, otherwise $c(x) = b(x)\}$	–	+

3.7. Example of Algorithmic Scheme

3.7.1. Scheme of description of the example

Since a DIA is intended to use as a language for recording, comparing, and standardizing different algorithms for image analysis, understanding, and processing, we describe, using the apparatus of basic DIAs with one ring, the algorithmic scheme of solving an illustrative problem.

By an algorithmic scheme, we mean a sequence of algorithmic procedures for passing from initial images to the desired solution of the problem.

In essence, the description of the scheme is reduced to the description of a special DIA with one ring. An example of the use of this DIA for design of an algorithm for solving the stated problem is presented in the subsection "Algorithmic Scheme." To demonstrate that the operations used in constructing the DIA can be written in terms of Ritter's image algebra, we have included a subsection "Description of This Operation in Terms of the Standard Image Algebra" in the scheme description.

The scheme is described in accordance with the following stages.

1. Statement of the problem.

2. Description of the special DIA with one ring:

* elements of the special DIA with one ring;

* elements of the field;

* operations introduced in the ring of the special DIA:

(1) operation of addition of two elements of set U (description, interpretation of this operation, and description of this operation in terms of the standard image algebra);

(2) operation of multiplication (description, interpretation of this operation, and description of this operation in terms of the standard image algebra);

(3) operation of multiplication by elements of the field (description, interpretation of this operation, and description of this operation in terms of the standard image algebra).

3. Algebraic scheme:

Let us pass to a consideration of individual elements of the scheme of description of the algorithmic example.

3.7.2. Statement of the problem

Suppose a distorted image I is given. It is required to obtain, using some filtering, an image closest to the ideal one, i.e., the visually most acceptable image (see Fig. 2).

Let $I_1 = \Phi_1(I)$, $I_2 = \Phi_2(I)$, $I_3 = \Phi_3(I)$, … be images obtained by applying filters Φ_1, Φ_2, Φ_3, to image I. The choice of an acceptability criterion for selection of the most appropriate filter depends on the problem and is considered in this example in the general form.

Remark. Without loss of generality, we may take as the filters, for instance, linear filters. Suppose that an image is represented in the form of brightness of pixels

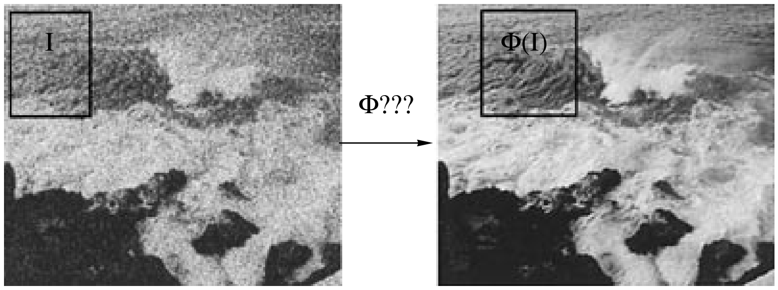

Fig. 2. To the statement of the problem: illustration of the filter choice.

with coordinates (x, y). If $f(x, y)$ is the initial image, then the image at the output of a linear filter takes the form $g(x, y) = \sum_{i=-N}^{N}\sum_{j=-N}^{N} h(x, y, i, j)f(x+i, y+j)$, where $h(x, y, i, j)$ is a weight function.

3.7.3. Description of a special DIA with one ring

I. Elements of the special DIA with one ring (elements of set U)

Suppose that images are represented by a function whose values are pixel intensities. Let U be the set of operations of application of filters to the initial image I: the operation of application of filter Φ_i to image I produces an image I_i represented as a function whose values are pixel intensities $f_i(x, y)$ $(i = 1, 2, \ldots)$.

II. Elements of the field

The set U is a DIA with one ring over the field of real numbers.[3]

III. Operations in the ring of the special DIA.

Operation of addition of two elements of set U.

Description

$$\Phi_1(I) + \Phi_2(I) = ([I_1/2] + [I_2/2]). \qquad (86)$$

Interpretation of this operation.

The physical meaning of the operation. By the operation of addition of two *operations of filter application* to the initial image I, we mean the half-sum of two functions of pixel brightness of images obtained by applying this filter. Note that the division (in the half-sum) is integer (i.e., after dividing the brightness function by two, we take only the integer part of the result) and the addition is modulo the maximal value of the tone level plus 1. For instance, if the pixel brightness varies from 0 to 255, then the addition is performed modulo 256.

Addition introduced in this way does not go outside the set of operations of filter application to the initial image I, because, adding two filtering operations, we obtain a new filtering operation.

The goal of introduction of the operation. The operation introduced in this way allows us to combine several different filtering methods by applying different filters to the initial image and averaging the results of filtering.

This operation is *neither physically nor visually interpretable.*

Description of the operation in terms of the standard image algebra.

In terms of the standard image algebra, any image is represented in the form $I = \{(x, a(x)), x \in X, a(x) \in F\}$; i.e., the image is an element of the set F^X. (Obviously, this definition does not contradict the above definition and representation of the image by a function of pixel brightness.) On images represented in this form, we may introduce so-called forced operations on images, i.e., operations certainly induced by operations of algebraic system F (see (82)). (If γ is a binary operation on set F and $a, b \in F^X$, then $a\gamma b = \{(x, c(x)): c(x) = a(x)\gamma b(x), x \in X\}$.)

In the case considered, F is the set of integers Z (moreover, Z is bounded by the maximal value of the pixel brightness plus 1; without loss of generality, $F = [0\ldots255]$).

Let $I_1 = \{(x, g_1(x)), x \in Z^2\}$ and $I_2 = \{(x, g_2(x)), x \in Z^2\}$. Then,

$$I_1 + I_2 = \{\{(x, c(x)): c(x) = [(g_1(x)/2] + [g_2(x)/2])\mathrm{mod}\,256, x \in Z^2\}\}. \qquad (87)$$

The square brackets in the expression $[(g_i(x)/2]$ $(i = 1, 2)$ denote the operation of taking the integer part of function $g_i(x)$ divided by two.

Operation of multiplication of two elements of set U.

Description:

$$\Phi_1(I)\cdot\Phi_2(I) = \mathrm{Compare}(\Phi_1(I), \Phi_2(I)), \qquad (88)$$

[3] Without loss of generality, we assume that there exist images with negative brightness of pixels. These images are not used in solving the problem, but they are necessary in order that the properties of classical algebras (see Definition 1 in Section 2) hold on the operations introduced below.

where the function Compare($\Phi_1(I)$, $\Phi_2(I)$) = $\Phi_i(I)$, i = 1, 2; the subscript i refers to the filter whose application to image I gives the best result for this problem. Note that this function is the acceptability criterion mentioned in the statement of the problem.

Interpretation of this operation.

The physical sense of the operation is as follows: the operation of multiplication of two operations of filtering of images is reduced to the comparison of the filtering results. As has been said above, the choice of the comparison criterion depends on the problem and is considered in this example in the most general form. The result of the multiplication operation is the filter that gives the best result for this problem.

The multiplication introduced in this way does not go outside the set of operations of filter application to the initial image I, because, multiplying two filtering operations, we obtain a new filtering operation.

The goal of introduction of the operation. The operation introduced in this way allows us to evaluate the quality of the filter applied and to choose automatically the most appropriate filter.

This operation is *neither physically nor visually interpretable.*

Description of the operation in terms of the standard image algebra.

In terms of image algebras, this operation is referred to as the binary operation induced by the unary operation and has the following form.

Let $I_1 = \{(x, g_1(x)), x \in Z^2\}$ and $I_2 = \{(x, g_2(x)), x \in Z^2\}$. Then,

$$I_1 \cdot I_2 = \{(x, c(x)): c(x) = g_1(x), \text{ if Compare}(\Phi_1(I), \Phi_2(I)) = \Phi_1(I), \text{ and } c(x) = g_2(x), \text{ if Compare}(\Phi_1(I), \Phi_2(I)) = \Phi_2(I), x \in Z^2\}. \quad (89)$$

Operation of multiplication of an element of set U by an element of the field.

Description.

Suppose that $\Phi(I) = g(x, y)$ and, by applying the operation of multiplication by a real number $\alpha \in R$, we obtain an image with a brightness function $G(x, y)$. We introduce the operation as follows:

$$\alpha\Phi(I) = \begin{cases} G(x, y) = 0, \text{ if } \alpha \le 0 \\ G(x, y) = [\alpha g(x, y)], \text{ if } [\alpha g(x, y)] < 256. \\ G(x, y) = 255, \text{ otherwise} \end{cases} \quad (90)$$

Interpretation of this operation.

The physical sense of the operation is as follows: the operation of multiplication by a real number corresponds to a proportional increase in the image brightness after applying the corresponding filter. If the multiplier is negative, then this operation is meaningless, because the value of the pixel brightness cannot be negative. Since the value of the pixel brightness is an integer, it is necessary to take into account only the integer part of the result obtained after multiplication by a real number. If the increased brightness of a pixel goes out the bounded set of intensities, the result of the operation is equal to the maximal value of brightness.

The multiplication introduced in this way does not go outside the set of operations of filter application to the initial image I, because, multiplying a filtering operation by a real number, we obtain a new filtering operation.

The goal of introduction of the operation is as follows: the operation introduced in this way allows us to proportionally increase the image brightness after filtering.

This operation is *neither physically nor visually interpretable* and is *not interpretable* in the context of image processing, because, applying it, we obtain a new filtering function for the initial image.

Description of this operation in terms of the standard image algebra.

In terms of image algebras, this operation is referred to as the unary operation induced by a unary operation and has the following form. Let $I = \{(x, g(x)), x \in Z^2\}$, $\alpha \in R$. Then,

$$\alpha I = \{\{(x, c(x)): c(x) = \begin{cases} 0, \text{ if } \alpha \le 0 \\ [\alpha g(x)], \text{ if } [\alpha g(x)] < 256, \quad x \in Z^2\}. \\ 255, \text{ otherwise} \end{cases} \quad (91)$$

3.7.4. Algorithmic scheme

Operations of the described DIA with one ring are used for solving the stated problem. The algorithmic scheme of the solution is constructed by the following principle: $\forall\alpha, \beta \in R$,[4] $\forall i, j \in [1...n]$, where n is the number of filters, and the best filter is chosen from the combinations $\alpha\Phi_i + \beta\Phi_j$ of filters Φ_i and Φ_j with the help of operation (38) of multiplication of two filters. Note that such a combination of filters is constructed in turn by operation (86) of addition of two filters and operation (90) of multiplication of a filter by an element of the field. The flow chart of the algorithm is presented in Fig. 3.

[4] In the algorithm written in the C language, α and β vary with a certain step h and are bounded from above by a real number maxAlpha.

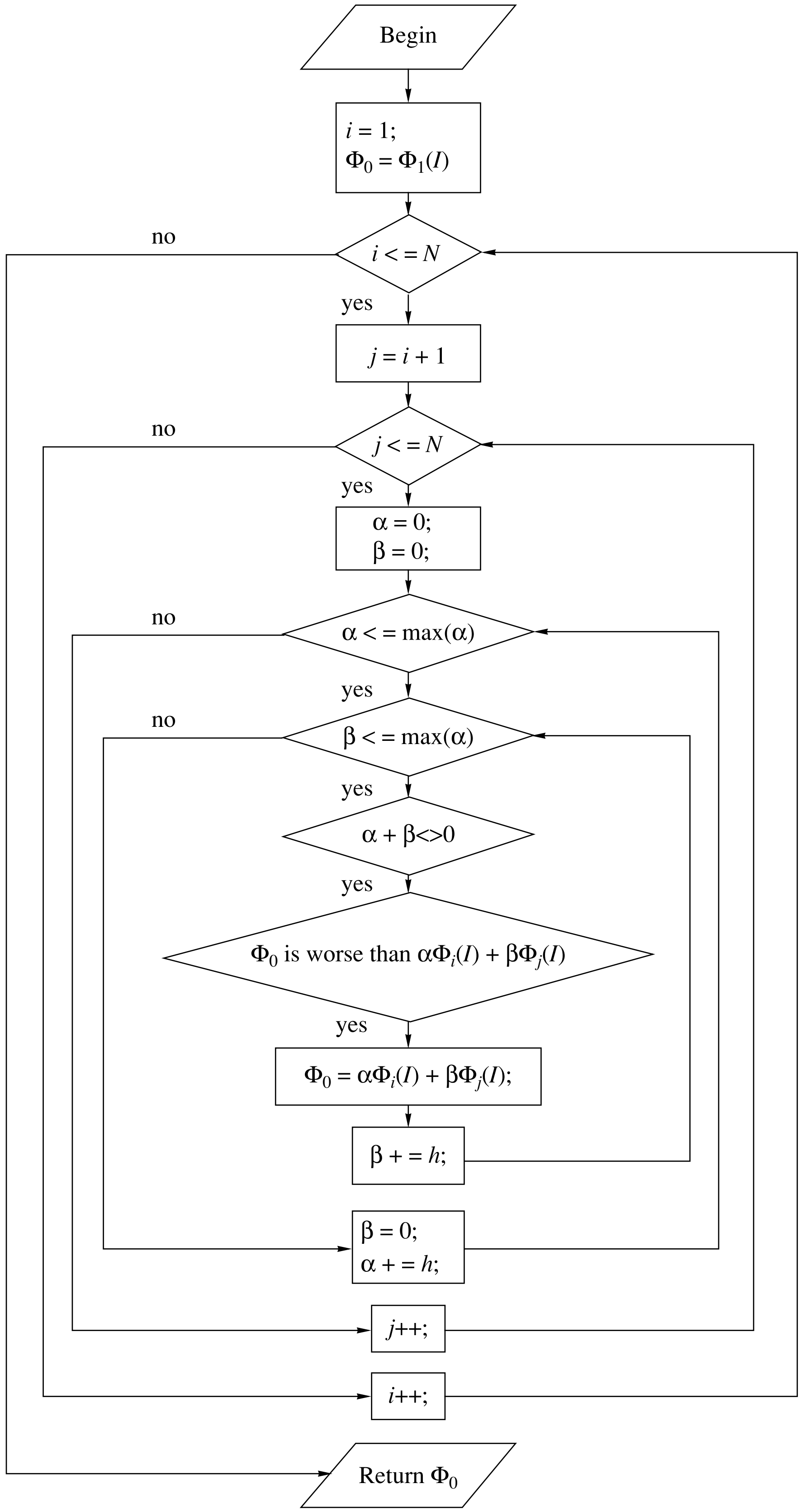

Fig. 3. Flow chart of the solution of the problem.

The algorithmic scheme is described in C++.

```
// An example of an algorithm using the DIA with one ring
// described above.
public class algorithm
{
// the step of application of operation (91)
double h;
// the maximal real number used in (91)
double maxAlpha;

// image
Image I;
// the number of input filters
int N;
// an array of input filters
ArrayList F0;
// constructor of the algorithm class
public algorithm()
        {InitiateData();}
// data initialization
public void InitiateData()
        {/*input of all data*/}
// Implementation of the operations
// Operation of addition
public Filter Plus(Filter F1, Filter F2)
        {/*Implementation of the operation of addition */}
// Operation of multiplication (comparison of two filters)
public Filter Compare(Filter F1, Filter F2)
        {/*Implementation of the operation of multiplication*/}
// Operation of multiplication of a filtering operation
// by a real number alpha
public Filter MultiplyByAlpha(double alpha, Filter F)
{/*Implementation of the operation of multiplication by a real number*/}
// main function: the choice of the best filter
public Filter TheBestFilter(int M)
{
        double alpha, better;
        Filter F = new Filter;
        F(I)=F0[0](I);
        // comparison of all linear combinations of input filters
        for(int i=0;i<N;i++)
        for(int j=i+1;j<N;j++)
        {
                alpha = 0;
```

```
          better = 0;
          while(alpha<=maxAlpha)
          {
                    while(better<=maxAlpha)
                    {
                              if(!((alpha=0)&(better=0)))
                              {
F(I) = Compare(F(I),Plus(MultiplyByAlpha(alpha, F0[i](I)), MultiplyByAlpha(better, F0[j](I)));
                              }
                              better+=h;
                    }
                    alpha+=h;
          }
     }
     return F;
     }
}
```

4. CONCLUSIONS

In this paper, we justify the choice of a new algebraic apparatus for description of algorithms for image processing. Along with theoretical results on structuring the concept of DIA and selection of operations of the standard image algebra that generate the construction of a DIA with a ring from images, an example of an algorithmic scheme of solving a problem of image processing in terms of DIA is given.

The main results of the paper are the following:

* a hierarchical scheme of heterogeneous algebras is constructed, which includes different versions of specialized image algebras;

* operands and operations of DIA with one ring are introduced and investigated;

* a concept of interpretability of operations of DIA is introduced and studied;

* examples of sets of operations generating DIA are presented;

* examples of sets of operations, which do not generate DIA, are presented;

* conditions of generating DIA by operations of the standard Ritter image algebra are obtained;

* an algorithmic scheme for the problem of image filtering based on DIA with one ring is designed.

In the framework of the theory, the following stages of investigation are planned.

(1) Investigation of possible image representations.

Construction of a DIA with one ring whose result or operands are different special classes of image models. The main practical value of the possibility of formalization of different image representations (models) is the extension of the algebraic concept of recognition to images. This, in turn, leads to the design of a standard language for description of algorithms for image processing, understanding, and analysis.

(2) Continuation of the search for necessary and sufficient conditions imposed on the set of operations of image processing, which generate DIAs with one or several rings.

(3) Continuation of the investigation of interpretability of operations generating DIA.

(4) Investigation of DIAs with several rings (the use of the apparatus of graded algebras).

(5) Construction of DIAs based on the properties of equivalence and invariance in the explicit form.

(6) Design, investigation, and implementation of algebraic schemes intended for solving model and applied problems of analysis and evaluation of information presented in the form of images.

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