An Efficient Technique for Calculating Proximity Functions in the 2D Family of Algorithms Based on Estimate Calculations with Rectangular Support Sets[1]

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Abstract—A fast method for calculating estimates in the case when a system of support sets in the algorithm is a family of rectangles of a fixed size is proposed. Here, the characteristic matrix of proximity for the object and its precedent is used instead of the characteristic vector of proximity. For faster calculations, a preprocessing is performed. At this step, some characteristics of the matrix are calculated that make it possible to verify the necessary conditions required for the rectangles of unit elements with fixed size to occur in the matrix.

A model of the algorithms based on estimate calculations (AEC) is one of the most important classes of algorithms used in recognition tasks [5]. It is known that a recognition algorithm contains a recognizing operator and decision rule [4]. In AEC, a recognizing operator converts a standard description of object S subject to recognition into a set of numerical estimates $(\Gamma_1(S), \Gamma_2(S), \ldots, \Gamma_l(S))$, where l is a number of classes. A decision rule helps us to construct the information vector $(\alpha_1^A, \alpha_2^A, \ldots, \alpha_l^A)$, $\alpha_j^A \in \{0, 1, \Delta\}$ from this set. Here, $\alpha_j^A = 0$ if an algorithm does not assign object S to jth class; $\alpha_j^A = 1$, if an algorithm assigns object S to jth class; and $\alpha_j^A = \Delta$ if an algorithm cannot classify object S.

The estimate $\Gamma_j(S)$ of the object S in regard to jth class is calculated as follows:

$$\Gamma_j(S) = \frac{1}{N}\frac{1}{|W_j|}\sum_{S' \in W_j}\gamma(S')\sum_{\tilde{\omega} \leftrightarrow \Omega \in \Omega_A} p(\tilde{\omega})B_{\tilde{\omega}}(S, S'), \quad j = \overline{1, l}, \tag{1}$$

where N is the normalizing factor; W_j is a set of objects from jth class; Ω_A is a system of support sets; $B_{\tilde{\omega}}(S, S')$ is the proximity function; $\gamma(S')$ is a weight of the precedent S'; and $p(\tilde{\omega})$ is a weight of the support set Ω with characteristic vector $\tilde{\omega}$.

The setting of all these parameters—a system of support sets, a proximity function, weights of precedents and features, and a decision rule—specifies the recognition algorithm in the AEC model.

In the applied tasks, where the number of precedents is large and the power of a system of support sets is high, the calculation of estimates by Eq. (1) may be very laborious and sometimes impracticable. The most complicated task here is the calculation of the sum

$$\sum_{\tilde{\omega} \leftrightarrow \Omega \in \Omega_A} p(\tilde{\omega})B_{\tilde{\omega}}(S, S'). \tag{2}$$

This complexity depends on the choice of the system of support sets Ω_A and on the type of the proximity function $B_{\tilde{\omega}}(S, S')$.

As a result, the challenge arises to get efficient formulas for estimate calculations without exhaustive search for all support sets of Eq. (2) and, thus, a combinatorial complexity of calculating the value of Eq. (2) is replaced by the complexity which is proportional to the size of the learning table.

There are a lot of works dealing with generation of efficient formulas for AEC. They differ both in approach to the task and in degree of generality.

In [5], efficient formulas are obtained for particular, frequently used systems of the support sets and proximity functions. In addition, a case is considered where the feature values in object description are the probability measures and there are gaps in the object description, which means that information on the value of some features is not available.

In [2], the technique is proposed for constructing efficient formulas by using DNF of a characteristic function of the system of support sets wherein the elementary conjunctions are mutually orthogonal. The formula complexity is directly proportional to the number of conjunctions in this DNF. It should be noted that the construction of a simple DNF with orthogonal elementary conjunctions presents a separate task of a complexity, which is equivalent to constructing the minimum (shortest) DNF.

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The same paper suggests a technique for constructing efficient formulas by using DNF, which contains mutually non-orthogonal elementary conjunctions. In this case, the complexity of a formula exponentially depends on the number of mutually non-orthogonal pairs of conjunctions in the DNF.

The author of [3] continues to investigate concerning the ways of developing efficient formulas by using DNF of the characteristic function of the system of support sets.

It is shown in [4], that generally

$$\Gamma_j(S) = \sum_{S' \in W_j} \gamma(S') \sum_{i=1}^{n} p_i V_i(S, S'), \quad j = \overline{1, l}, \qquad (3)$$

where $V_i(S, S')$ is a number of support sets Ω from Ω_A, which contain feature i, such that $B_{\tilde{\omega}}(S, S') = 1$. If the number $V_i(S, S')$ is small (relatively to n), then Eq. (3) is efficient. The classes of proximity functions and the systems of support sets with small number $V_i(S, S')$ are also considered in [4].

In [1], the maximum number of different values of $V_i(S, S')$ is considered to be the characteristic of a system of support sets Ω_A. In addition, two quantities are introduced which characterize such properties of the system of support sets as *simplicity* and *symmetry*. The relations of the introduced characteristics are analyzed together with their changes when the isometric group of permutations affects Ω_A and when set-theoretic operations are applied to the systems of support sets.

The results obtained in [1] show the ways of transforming the systems of support sets with a limited (not very large) number of values $V_i(S, S')$ into new support sets with the same property.

The author of [7] introduces a notion of atomic subset for the Boolean cube E^n and defines it as follows: the n-dimensional binary vector is partitioned into subvectors and weights are assigned to each subvector. The binary vector belongs to the atomic set if and only if the weights of the given subvectors coincide with the specified ones. A layer of a cube, an intersection of a cube with an interval, a sphere, and a ball can serve as examples of such atomic sets. There are also atomic sets of a more complicated structure.

Efficient formulas are derived for the systems of support sets which are the union of a finite number of disjoint atomic sets.

Derivation of efficient formulas for AEC becomes especially important in the case when recognition objects are images and object descriptions are large-sized 2D matrices. In addition, the system of support sets is subject to restrictions which are specified by matching two or more images. Thus, a necessity arises to derive the efficient formulas for those systems of support sets which have not been considered earlier, first of all, for 2D (spatial) support sets.

One of the ways for efficient calculation of estimate (1) is proposed in [6]. The author considers a case where an object is described by the matrix of the elements of a K-valued alphabet and the support set is a rectangular submatrix of this matrix. However, the author considers neither the possibilities of efficient estimate calculations for such a system of the support sets nor the possibility of obtaining this formula with a minimum complexity.

The same problems arise if we attempt to use other systems of 2D support sets.

Here, we present an approach to this problem. We consider the following system of support sets: a totality of fixed-sized rectangles (the lengths of the sides of the rectangles are determined) which constitute an image-describing matrix. The feature weights are considered to be equal to some p. In this case, for the considered systems of support sets, we can carry out the constant factor $p(\tilde{\omega})$ beyond the summation sign. After that, it is sufficient to calculate the value of the sum

$$\sum_{\tilde{\omega} \leftrightarrow \Omega \in \Omega_A} B_{\tilde{\omega}}(S, S'). \qquad (4)$$

In the case of 2D support sets, the characteristic proximity matrix $C = C(S, S')$ is analogous to $\tilde{\delta} = \tilde{\delta}(S, S')$, the characteristic proximity matrix of object S and precedent S' [5]. The value of expression (4) coincides with the number of rectangles of fixed size which consist of unities taken from matrix $C = C(S, S')$.

Computational speedup for Eq. (4) is based on preliminary computation of the values of some characteristics of the matrix C, which allow us to verify the fulfillment of necessary conditions for matrix C: the presence of rectangles of fixed size which consist of unities.

Suppose that in the standard classification task, a set of admissible objects M is a set of matrices of size $u \times v$ with the elements from the set D. We identify a standard description $I(S)$ of the object S with the object itself and suppose that $I(S) = S = (a_{ij})_{u \times v}$ and $I(S') = S' = (b_{ij})_{u \times v}$.

Consider a family of the AECs with the following parameters:

(1) A system of the support sets Ω_A is a totality of rectangles $\Pi_{R_1 \times R_2}$ with the sides R_1 and R_2, where $1 \le R_1 \le u$, $1 \le R_2 \le v$, and $R_1 R_2 > 1$.

(2) A proximity function:

(a) let a metric (semimetric) $\rho(x, y)$ be set and the numbers $\varepsilon_{ij} > 0$, $i = \overline{1, u}$, $j = \overline{1, v}$, $\Omega = \{(i_1, j_1), \ldots, (i_1, j_1 + 1), \ldots, (i_1, j_1 + R_2 - 1), \ldots, (i_1 + R_1 - 1, j_1 + R_2 - 1)\}$ be given on the set D;

(b) consider a system of equations

$$\begin{aligned}&\rho(a_{i_1, j_1}, b_{i_1, j_1}) \le \varepsilon_{i_1, j_1},\\ &\rho(a_{i_1, j_1+1}, b_{i_1, j_1+1}) \le \varepsilon_{i_1, j_1+1}, \ldots,\\ &\rho(a_{i_1, j_1+R_2-1}, b_{i_1, j_1+R_2-1}) \le \varepsilon_{i_1, j_1+R_2-1}, \ldots,\\ &\rho(a_{i_1+R_1-1, j_1+R_2-1}, b_{i_1+R_1-1, j_1+R_2-1})\\ &\le \varepsilon_{i_1+R_1-1, j_1+R_2-1}.\end{aligned} \tag{5}$$

(c) let us define

$$B_{\tilde{\omega}}(S, S') = \begin{cases} 1, & \text{if all the inequalities in (5) are fulfilled} \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

The feature weights are positive; therefore, $p(\tilde{\omega}) = pR_1R_2$, $\forall \tilde{\omega} \in \Omega_A$. The weights of the precedents and decision rule are arbitrary. This completes the description of the AEC family.

Consider the matrix $C = C(S, S') = (c_{ij})_{u \times v}$ defined according to the following rule:

$$c_{ij} = \begin{cases} 1, & \rho(a_{ij}, b_{ij}) \le \varepsilon_{ij} \\ 0, & \rho(a_{ij}, b_{ij}) > \varepsilon_{ij}. \end{cases} \tag{7}$$

Suppose that

$$h_{1, i} = \bigvee_{j=1}^{v-R_2+1} c_{i, j} c_{i, j+1} \ldots c_{i, j+R_2-1}, \quad i = \overline{1, u} \tag{8}$$

and

$$h_{2, j} = \bigvee_{i=1}^{u-R_1+1} c_{i, j} c_{i+1, j} \ldots c_{i+R_1-1, j}, \quad j = \overline{1, v}. \tag{9}$$

As the figure shows, $h_{1,i} = 1$ ($h_{2j} = 1$) if and only if matrix C contains a continuous sequence of at least R_2 (R_1) unities in the ith line (in the jth column).

Let $H_1(C(S, S') = H_1(S') = \left(\bigvee_{i=1}^{u} h_{1, i}\right) \cdot \left(\bigvee_{j=1}^{v} h_{2, j}\right)$.

The equation $H_1 = 1$ holds if and only if there is a continuous sequence of minimum R_2 unities in at least one row and there is a continuous sequence of minimum R_1 unities in at least one column. Thus, if $H_1(S') = 0$, then matrix C *a priori* does not contain the rectangle comprised of unities with the sides R_1 and R_2 and, therefore, $\sum_{\tilde{\omega} \leftrightarrow \Omega \in \Omega_A} B_{\tilde{\omega}}(S, S') = 0$. The following assertion is proven.

	1	2	3	4	5	$h_{1,i}$
1	1	1	1		1	1
2		**<u>1</u>**	1	1	1	1
3	1	**<u>1</u>**	1	1	1	1
4		1		1		
5		1				
$h_{j,2}$		1	1	1	1	1 (H_2)

$u = v = 5$, $R_1 = R_2 = 3$; the empty cells correspond to zeroes.

Assertion 1.

$$\Gamma_j(S) = \frac{pR_1R_2}{N|W_j|} \sum_{\substack{S' \in W_j \\ H_1(S') = 1}} \gamma(S') \sum_{\tilde{\omega} \leftrightarrow \Omega \in \Omega_A} B_{\tilde{\omega}}(S, S'), \quad j = \overline{1, l}. \tag{10}$$

Suppose that

$$H_2(C(S, S')) = H_2(S') = \left(\bigvee_{i=1}^{u-R_1+1} h_{1, i} h_{1, i+1} \ldots \times h_{1, i+R_1-1}\right)\left(\bigvee_{j=1}^{v-R_2+1} h_{2, j} h_{2, j+1} \ldots h_{2, j+R_2-1}\right). \tag{11}$$

The equation $H_2 = 1$ holds if and only if in matrix C

(a) there are sequences of R_2 unities located in at least R_1 adjacent rows;

(b) there are sequences of R_1 unities located in at least R_2 adjacent columns.

Therefore, if $H_2(S') = 0$, then $\sum_{\tilde{\omega} \leftrightarrow \Omega \in \Omega_A} B_{\tilde{\omega}}(S, S') = 0$.

In addition, note that if $H_2(S') = 1$, then $H_1(S') = 1$; if $H_1(S') = 0$, then $H_2(S') = 0$. Thus, the following assertion is proven.

Assertion 2.

$$\Gamma_j(S) = \frac{pR_1R_2}{N|W_j|} \sum_{\substack{S' \in W_j \\ H_2(S') = 1}} \gamma(S') \sum_{\tilde{\omega} \leftrightarrow \Omega \in \Omega_A} B_{\tilde{\omega}}(S, S'), \quad j = \overline{1, l}. \tag{12}$$

Consider the matrices $C_1 = (c^1_{ij})_{(u-R_1+1) \times (v-R_2+1)}$ and $C_2 = (c^2_{ij})_{(u-R_1+1) \times (v-R_2+1)}$ defined as follows: $c^1_{i,j} = c_{i,j} c_{i,j+1} \ldots c_{i,j+R_2-1}$, $c^2_{i,j} = c_{i,j} c_{i+1,j} \ldots c_{i+R_1-1,j}$, $i = \overline{1, u-R_1+1}$, $j = \overline{1, v-R_2+1}$.

If $c_{ij}^1 = 1$ ($c_{ij}^2 = 1$), then in matrix C, the sequence of at least $R_2(R_1)$ unities starts in ith row (jth column) from jth column (ith row). Suppose that $C' = (c'_{ij})_{(u-R_1+1)\times(v-R_2+1)} = C_1C_2$, $H_3(C(S, S')) = H_3(S') = \|C'\|$, where C_1C_2 is a coordinatewise multiplication of the matrices C_1 and C_2 and $\|C'\|$ is the number of unities in the matrix C'. Suppose that $c_{ij}^1 = 1$; this means that in the point (i, j) of the matrix C, a vertex of the rectangle is situated with its sides in the ith row (the length of the side is at least R_2 unities) and in the jth column (the length of the side is at least R_1 unities). Matrix C' corresponding to the matrix C (see figure) contains unities in the positions (2, 3) and (3, 2).

Then, if the equation $H_3(S') = 0$ holds, $\sum\limits_{\tilde{\omega} \leftrightarrow \Omega \in \Omega_A} B_{\tilde{\omega}}(S, S') = 0$. Therefore, the following assertion is valid.

Assertion 3.

$$\Gamma_j(S) = \frac{pR_1R_2}{N|W_j|} \sum_{\substack{S' \in W_j \\ H_2(S') = 1 \\ H_3(S') \geq 1}} \gamma(S') \sum_{\tilde{\omega} \leftrightarrow \Omega \in \Omega_A} B_{\tilde{\omega}}(S, S'), \tag{13}$$

$$j = \overline{1, l}.$$

If $H_2(S') = 1$, $H_3(S') \geq 1$, then the value of summation (4) can be calculated with the help of the following fast method (note that the value may be both zero and nonzero): In the inner sum of Eq. (13), we substitute the summation over all $\tilde{\omega} \longleftrightarrow \Omega \in \Omega_A$ by the summation over all $\tilde{\omega} \longleftrightarrow \Omega \in \Omega_A$ such that the index of the first unity in $\tilde{\omega}$ coincides with the index of some unity in the matrix C'.

Thus, the following technique is suggested for the fast calculation of estimate (1).

(1) The matrix $C(S, S')$ is constructed for the current precedent $S' \in W_j$.

(2) The value $H_1(C(S, S'))$ is calculated; if $H_1(C(S, S')) = 0$, proceed to the next precedent $S' \in W_j$.

(3) The value $H_2(C(S, S'))$ is calculated; if $H_2(C(S, S')) = 0$, proceed to the next precedent $S' \in W_j$.

(4) The matrix $C(S, S')$ is constructed; the value $H_3(C(S, S'))$ is calculated; if $H_3(C(S, S')) = 0$, proceed to the next precedent $S'' \in W_j$; otherwise, while calculating the sum in Eq. (3), the summation should be done over all $\tilde{\omega} \longleftrightarrow \Omega \in \Omega_A$ such that the index of the first unity coincides with the index of some unity in the matrix C'.

In the following research, we propose to investigate the possibility of obtaining efficient formulas for the subclasses of AEC with other types of 2D support sets.

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