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# Completeness Criteria for Classification Problems with Set-Theoretic Constraints

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**Abstract**—Within the framework of the algebraic approach to the synthesis of correct algorithms, a class of problems is described and analyzed in which, for each element of the initial-information space, a corresponding subset of the final-information set (problems with set-theoretic constraints) is defined. The concept of completeness is introduced for models of algorithms and algorithmic operators and for decision rule classes and correcting operations. Relevant completeness criteria are derived and proved.

**Keywords:** algebraic approach, algorithmic operator, classification, correcting operation, model of algorithms, completeness, decision rule, set-theoretic constraints

Within the framework of the algebraic approach to the synthesis of correct algorithms for pattern recognition, classification, and prediction [1, 2], we consider a class of problems characterized by explicit set-theoretic constraints imposed on the admissible output space of an algorithm.

Following [3–6], the classification problem is described as the problem of designing a data-transformation algorithm. Consider a set $\mathfrak{S} = \{S\}$, whose elements are called objects. The descriptions $D(S)$ of the objects form the initial-information space $\mathfrak{I}_i = \{D(S) \mid S \in \mathfrak{S}\}$, whose elements are denoted by $I_i$, so $\mathfrak{I}_i = \{I_i\}$.

Consider the problem of designing algorithms $A$ that implement mappings from $\mathfrak{I}_i$ to a final-information space $\mathfrak{I}_f = \{I_f\}$. In what follows, we do not distinguish algorithms and the mappings they implement. A solutions is synthesized within the framework of a model $\mathfrak{M}$ of algorithms, where $\mathfrak{M} \subseteq \{A \mid A : \mathfrak{I}_i \longrightarrow \mathfrak{I}_f\}$. An individual problem is defined by structural information $I_S$ that singles out from $\mathfrak{M}$ a subset of admissible mappings, designated as $\mathfrak{M}[I_S]$. Any algorithm $A$ implementing an arbitrary admissible mapping is called correct for the problem defined by $I_S$ and is its solution.

Constructions based on the algebraic approach to the synthesis of correct algorithms make use of an estimate space $\mathfrak{I}_e = \{I_e\}$ that is intermediate between $\mathfrak{I}_i$ and $\mathfrak{I}_f$. Correct algorithms are synthesized on the basis of heuristic information models, i.e., parametric classes of mappings from $\mathfrak{I}_i$ to $\mathfrak{I}_f$, which are special superpositions of algorithmic operators (mappings from $\mathfrak{I}_i$ to $\mathfrak{I}_e$) and decision rules (mappings from $\mathfrak{I}_e^p$ to $\mathfrak{I}_f$, where $p$ is the arity of a decision rule).

Recall that, for arbitrary sets $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{U}'$, and $\mathfrak{V}'$ and arbitrary mappings $u$ from $\mathfrak{U}$ and $\mathfrak{V}$ and $u'$ from $\mathfrak{U}'$ and $\mathfrak{V}'$, the product $u \times u'$ is a mapping $v$ from $\mathfrak{U} \times \mathfrak{U}'$ to $\mathfrak{V} \times \mathfrak{V}'$ such that $v(U, U') = (u(U), u'(U'))$ for any pair $(U, U')$ from $\mathfrak{U} \times \mathfrak{U}'$ [4]. For an arbitrary mapping $u$ from $\mathfrak{U}^p$ to $\mathfrak{V}$ with $p \geq 1$, diagonalization is a mapping $u_\Delta$ from $\mathfrak{U}$ to $\mathfrak{V}$ such that $u_\Delta(U) = u(U, \ldots, U)$ for any $U$ from $\mathfrak{U}$.

The models $\mathfrak{M}$ are defined by models of algorithmic operators $\mathfrak{M}^0$, where

$$\mathfrak{M}^0 \subseteq \mathfrak{M}_* \stackrel{\text{df}}{=} \{B \mid B : \mathfrak{I}_i \longrightarrow \mathfrak{I}_e\},$$

and by decision rules $\mathfrak{M}^1$, where

$$\mathfrak{M}^1 \subseteq \bigcup_{p=0}^{\infty} \{C \mid C : \mathfrak{I}_e^p \longrightarrow \mathfrak{I}_f\},$$

as follows:

$$\mathfrak{M} = \mathfrak{M}^1 \circ \mathfrak{M}^0 = \{C \circ (B_1 \times \ldots \times B_p)_\Delta \mid C \in \mathfrak{M}^1, B_1, \ldots, B_p \in \mathfrak{M}^0\}.$$

In the synthesis of correct algorithms, we also use sets $\mathfrak{F}$ of correcting operations defined over the set of mappings $\mathfrak{M}_*$. The correcting operations $F$ considered in this paper are induced by operations $\tilde{F}$ over $\mathfrak{I}_e$:

$$F(B_1, \ldots, B_p)(I_i) \stackrel{\text{df}}{=} \tilde{F}(B_1(I_i), \ldots, B_p(I_i)),$$

where $I_i$ ranges over $\mathfrak{I}_i$; the algorithmic operators $B_1, \ldots, B_p$ are arbitrary mappings from $\mathfrak{I}_i$ to $\mathfrak{I}_e$; and $\tilde{F}$ is an operation over $\mathfrak{I}_e$.

The construction scheme for an algorithm model $\mathfrak{M}$ is shown in the following commutative diagram (see [3–6]):

$$\begin{array}{ccc} \mathfrak{I}_i & \xrightarrow{\mathfrak{M}} & \mathfrak{I}_f \\ {\scriptstyle \mathfrak{M}^0}\downarrow & & \uparrow{\scriptstyle \mathfrak{M}^1} \\ \mathfrak{I}_e^p & \xrightarrow{\mathfrak{I}} & \mathfrak{I}_e \end{array}.$$

For the problems with set-theoretic constraints considered here, algorithm models $\mathfrak{M}$ are constructed on the basis of parametric classes of models of algorithmic operators and correcting operations. It is assumed that $\mathfrak{M}^0 = \{\mathfrak{M}^0_{\lambda, \omega} \mid \lambda \in L, \omega \in W(\lambda)\}$ and $\mathfrak{F} = \{\mathfrak{F}^\lambda \mid \lambda \in L\}$, where $W(\lambda)$ and $L$ are sets of structural indices. A model $\mathfrak{M}$ is constructed in the form

$$\mathfrak{M} = \bigcup_{\lambda \in L} \bigcup_{\omega \in W(\lambda)} \mathfrak{M}^1 \circ \mathfrak{F}^\lambda(\mathfrak{M}^0_{\lambda, \omega}),$$

where

$$\mathfrak{M}^1 \circ \mathfrak{F}^\lambda(\mathfrak{M}^0_{\lambda, \omega}) = \{C \circ (F_1(B_1^1, \ldots, B_{r(1)}^1) \times \ldots \times F_p(B_1^p, \ldots, B_{r(p)}^p))_\Delta \big| C \in \mathfrak{M}^1,$$
$$(F_1, \ldots, F_p) \in (\mathfrak{F}^\lambda)^p, B_1^1, \ldots, B_{r(1)}^1 \in \mathfrak{M}^0_{\lambda, \omega(1)}, \ldots, B_{r(p)}^p, \ldots, B_{r(p)}^p \in \mathfrak{M}^0_{\lambda, \omega(p)}\}$$

for all $\lambda \in L$ and $\omega \in W$.

To formalize the concept of set-theoretic constraints, we introduce a set $\Pi = \{\pi_1, \ldots, \pi_k\}$ of predicates $\pi_i$ : $\mathfrak{I}_i \times \mathfrak{I}_f \longrightarrow \{0, 1\}$.

Let $I_i$ be an arbitrary element of $\mathfrak{I}_i$. Let

$$\Pi(I_i) = \left\{ I_f \mid I_f \in \mathfrak{I}_f, \quad \underset{1, 2, \ldots, k}{\forall} j : \pi_j(I_i, I_f) = 1 \right\}$$

be the set of all admissible values of correct algorithms for the initial information $I_i$.

A set $\Pi$ is called covering if $\Pi(I_i) \neq \varnothing$ for any $I_i$ in $\mathfrak{I}_i$, i.e., if for any element, there exists at least one admissible value.

In what follows, we consider an arbitrary fixed covering set $\Pi$.

Denote the set of positive integers by $\mathbb{N}$, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

**Definition 1.** The set

$$\text{Prec} = \{((I_i^1, \ldots, I_i^q), (I_f^1, \ldots, I_f^q)) \mid q \in \mathbb{N}, (I_i^1, \ldots, I_i^q) \in \mathfrak{I}_i^q, I_i^j \neq I_i^k$$
$$\text{for } j \neq k, (I_f^1, \ldots, I_f^q) \in \mathfrak{I}_f^q, I_f^i \in \Pi(I_i^j) \text{ for } j = 1, 2, \ldots, q\}$$

is called *the set of collections of admissible precedents*.

For an arbitrary set $\mathfrak{I}$ and $q \in \mathbb{N}$, the symbol $(\mathfrak{I}^q)^*$ stands for the set $\{(I^1, \ldots, I^q)(I^1, \ldots, I^q) \in \mathfrak{I}^q, I^k \neq I^j$ for $k \neq j\}$.

Note that

$$\text{Prec} = \bigcup_{q \in \mathbb{N}} \bigcup_{(I_i^1, \ldots, I_i^q) \in (\mathfrak{I}_i^q)^*} \{(I_i^1, \ldots, I_i^q), \Pi(I_i^1) \times \ldots \times \Pi(I_i^q)\}.$$

**Definition 2.** A model $\mathfrak{M}$ is called $\Pi$-complete if

$$\underset{\mathfrak{I}_i}{\forall} I_i : \mathfrak{M}(I_i) = \{A(I_i) \mid A \in \mathfrak{M}\} \subseteq \Pi(I_i), \tag{1}$$

$$\underset{\text{Prec}}{\forall} ((I_i^1, \ldots, I_i^q), (I_f^1, \ldots, I_f^q)) \underset{\mathfrak{M}}{\exists} A : \underset{\{1, 2, \ldots, q\}}{\forall} j : A(I_i^j) = I_f^j. \tag{2}$$

Note that conditions (1) and (2) are independent. Moreover, under condition (2), condition (1) is equivalent to

$$\underset{\mathfrak{I}_i}{\forall} I_i : \mathfrak{M}(I_i) = \{A(I_i) \mid A \in \mathfrak{M}\} = \Pi(I_i).$$

The goal of this study is to characterize the conditions on $\mathfrak{M}^1$, $\mathfrak{F}$, and $\mathfrak{M}^0$ under which the model

$$\mathfrak{M} = \bigcup_{\lambda \in L} \bigcup_{\omega \in W(\lambda)} \mathfrak{M}^1 \circ \mathfrak{F}^{\lambda}(\mathfrak{M}^0_{\lambda, \omega}) \tag{3}$$

is complete.

It can easily be seen that the completeness problem for $\mathfrak{M}$ can be analyzed under the assumption that $q = 1$. Indeed, to this end, it suffices to proceed from $\mathfrak{I}_i$ to $\bigcup_{q=1}^{\infty} \mathfrak{I}_i^q$, from $\mathfrak{I}_f$ to $\bigcup_{q=1}^{\infty} \mathfrak{I}_f^q$, from $\mathfrak{I}_e$ to $\bigcup_{q=1}^{\infty} \mathfrak{I}_e^q$, and from the original mappings (say, $A \in \mathfrak{M}$, $A : \mathfrak{I}_i \longrightarrow \mathfrak{I}_f$) to

$$A^* : \bigcup_{q=1}^{\infty} \mathfrak{I}_i^q \longrightarrow \bigcup_{q=1}^{\infty} \mathfrak{I}_f^q, \text{ where } A^*(I_i^1, \ldots, I_i^q) \stackrel{\text{df}}{=} (A(I_i^1), \ldots, A(I_i^q)).$$

**Definition 3.** A decision rule class $\mathfrak{M}^1$ is called $\Pi$-*complete* if there exists a model of algorithmic operators $\mathfrak{M}^0$ and a class of correcting operations $\mathfrak{F}$ such that model (3) is $\Pi$-complete.

**Definition 4.** For a fixed $\Pi$-complete decision rule class $\mathfrak{M}^1$, a family $\mathfrak{F}$ of correcting operations is called $\mathfrak{M}^1$-$\Pi$-*complete* if there exists a model of algorithmic operators $\mathfrak{M}^0$ such that model (3) is $\Pi$-complete.

**Definition 5.** For a fixed $\Pi$-complete decision rule class $\mathfrak{M}^1$ and a fixed $\mathfrak{M}^1$-$\Pi$-complete family $\mathfrak{F}$ of correcting operations, a model of algorithmic operators $\mathfrak{M}^0$ is called $\mathfrak{F}$ – $\mathfrak{M}^1$-$\Pi$-*complete* if model (3) is $\Pi$-complete.

Consider a nonempty decision rule class $\mathfrak{M}^1 = \bigcup_{p=0}^{\infty} \mathfrak{M}_p^1$, where $\mathfrak{M}_p^1 \subseteq \{C \mid C : \mathfrak{I}_e^p \longrightarrow \mathfrak{I}_f\}$ for any $p$ in $\mathbb{N}_0$. For any $X \subseteq \mathfrak{I}_e$, it turns out that

$$\mathfrak{M}^1(X) = \bigcup_{p=0}^{\infty} \mathfrak{M}_p^1(X^p) = \bigcup_{p=0}^{\infty} \bigcup_{C \in \mathfrak{M}_p^1} \bigcup_{\bar{x} \in X^p} C(\bar{x}).$$

**Definition 6.** Let $p \in \mathbb{N}_0$. For an arbitrary $I_i$ in $\mathfrak{I}_i$, *the set* $\alpha_p(\mathfrak{M}^1, I_i)$ is defined as the intersection, in the $p$th Cartesian power of $\mathfrak{I}_e$, of all complete preimages of $\Pi(I_i)$ with respect to the decision rules of arity $p$:

$$\alpha_p(\mathfrak{M}^1, I_i) = \bigcap_{C \in \mathfrak{M}_p^1} C^{-1}(\Pi(I_i)) = \left\{\bar{I}_e \mid \bar{I}_e \in \mathfrak{I}_e^p, \underset{\mathfrak{M}_p^1}{\forall} C : C(\bar{I}_e) \in \Pi(I_i)\right\}.$$

**Definition 7.** Let $p \in \mathbb{N}_0$. For a class $\mathfrak{M}^1$ and an element $I_i$ of $\mathfrak{I}_i$, a subset $X(I_i)$ of $\mathfrak{I}_e$ is called *an admissible p projection* if

$$(X(I_i))^p \subseteq \alpha_p(\mathfrak{M}^1, I_i),$$

$$\neg\exists Z \subseteq \mathfrak{I}_e : (X(I_i) \subset Z) \wedge (Z^p \subseteq \alpha_p(\mathfrak{M}^1, I_i)).$$

The set of all admissible $p$ projections for $\mathfrak{M}^1$ and $I_i$ is denoted by $\xi_p(\mathfrak{M}^1, I_i)$.

For an arbitrary $I_i$ in $\mathfrak{I}_i$, we introduce the set $\Phi(\mathfrak{M}^1, I_i)$ of choice functions of admissible projections:

$$\Phi(\mathfrak{M}^1, I_i) = \Big\{\varphi \mid \varphi : \mathbb{N}_0 \longrightarrow \mathrm{B}(\mathfrak{I}_e), \underset{\mathbb{N}_0}{\forall} p : ((\mathfrak{M}^1_p = \varnothing) \Rightarrow (\varphi(p) = \mathfrak{I}_e)) \wedge$$

$$\wedge ((\mathfrak{M}^1_p \neq \varnothing) \Rightarrow (\varphi(p) \in \xi_p(\mathfrak{M}^1, I_i)))\Big\},$$

where $\mathrm{B}(\mathfrak{I}_e)$ is the set of all subsets of $\mathfrak{I}_e$.

For each choice function of admissible projections $\varphi$ in $\Phi(\mathfrak{M}^1, I_i)$, we set

$$X(I_i, \varphi) = \bigcap_{p=0}^{\infty} \varphi(p).$$

Note that

$$\mathfrak{M}^1(X(I_i, \varphi)) = \bigcup_{r=0}^{\infty} \bigcup_{C \in \mathfrak{M}^1_r} C\left(\left(\bigcap_{p=0}^{\infty} \varphi(p)\right)^r\right).$$

Let $\tilde{\Phi}(\mathfrak{M}^1, I_i) = \{\varphi \mid \varphi \in \Phi(\mathfrak{M}^1, I_i), X(I_i, \varphi) \neq \varnothing\}$.

**Theorem 1.** *For all $I_i$ in $\mathfrak{I}_i$,*

$$\bigcup_{\varphi \in \tilde{\Phi}(\mathfrak{M}^1, I_i)} \mathfrak{M}^1(X(I_i, \varphi)) \subseteq \Pi(I_i). \tag{4}$$

**Proof.** For an arbitrary $I_i$ in $\mathfrak{I}_i$, any $\varphi$ in $\tilde{\Phi}(\mathfrak{M}^1, I_i)$, and any $p$ in $\mathbb{N}_0$, if $\mathfrak{M}^1_p((X(I_i, \varphi))^p)$ is empty, then $\mathfrak{M}^1_p$ is obviously empty as well. However, if $\mathfrak{M}^1_p$ is not empty, then $\mathfrak{M}^1_p((X(I_i, \varphi))^p) \subseteq \mathfrak{M}^1_p(\varphi(p))$ because $X(I_i, \varphi) = \bigcap_{p=0}^{\infty} \varphi(p)$. Since $\varphi(p)$ for a nonempty $\mathfrak{M}^1_p$ belongs to $\xi_p(\mathfrak{M}^1, I_i)$, it follows from the definitions of $\alpha_p(\mathfrak{M}^1, I_i)$ and $\xi_p(\mathfrak{M}^1, I_i)$ that $\mathfrak{M}^1_p((X(I_i, \varphi))^p) \subseteq \Pi(I_i)$. Therefore,

$$\mathfrak{M}^1(X(I_i, \varphi)) = \bigcup_{p=0}^{\infty} \mathfrak{M}^1_p((X(I_i, \varphi))^p) \subseteq \Pi(I_i).$$

Since this holds true for all $\varphi$, we conclude that (4) is always true, which was to be proved.

**Theorem 2** ($\Pi$-completeness criterion for decision rule classes). *A decision rule class $\mathfrak{M}^1$ is $\Pi$-complete if and only if*

$$\bigcup_{\varphi \in \tilde{\Phi}(\mathfrak{M}^1, I_i)} \mathfrak{M}^1(X(I_i, \varphi)) = \Pi(I_i) \tag{5}$$

*for any $I_i$ in $\mathfrak{I}_i$.*

**Proof.** *Necessity.* Assume that $\bigcup_{\varphi \in \tilde{\Phi}(\mathfrak{M}^1, I_i)} \mathfrak{M}^1(X(I_i, \varphi))$ does not coincide with $\Pi(I_i)$. Then, there exists $I_f^0$ such that

$$I_f^0 \in \Pi(I_i) - \bigcup_{\varphi \in \tilde{\Phi}(\mathfrak{M}^1, I_i)} \mathfrak{M}^1(X(I_i, \varphi)).$$

At the same time, let model (3) be $\Pi$-complete. Then a $p_0$-ary decision rule $C_0 \in \mathfrak{M}^1$; parameter values $\lambda \in L$ and $\omega(1), \ldots, \omega(p_0) \in W(\lambda)$ correcting $F_1, \ldots, F_{p_0} \in \mathfrak{F}^\lambda$; and algorithmic operators $B_1^1, \ldots, B_{r(1)}^1 \in \mathfrak{M}^0_{\lambda, \omega(1)}, \ldots, B_1^{p_0}, \ldots, B_{r(p_0)}^{p_0} \in \mathfrak{M}^0_{\lambda, \omega(p_0)}$ exist such that

$$C_0 \circ (F_1(B_1^1, \ldots, B_{r(1)}^1) \times \ldots \times F_{p_0}(B_1^{p_0}, \ldots, B_{r(p_0)}^{p_0}))_\Delta(I_i) = I_f^0.$$

Let $\bar{I}_e^0 = (F_1(B_1^1, \ldots, B_{r(1)}^1)(I_i), \ldots, F_{p_0}(B_1^{p_0}, \ldots, B_{r(p_0)}^{p_0})(I_i)) \in \mathfrak{I}_e^{p_0}$. Then $C_0(\bar{I}_e^0) = I_f^0$.

If $\mathfrak{M}^1_{p_0}(\bar{I}_e^0)$ is not a subset of $\Pi(I_i)$, then $\mathfrak{M}(I_i)$ is also not a subset of $\Pi(I_i)$, so $\mathfrak{M}$ is not $\Pi$-complete.

It remains to consider the case where $\mathfrak{M}^1_{p_0}(\bar{I}_e^0) \subseteq \Pi(I_i)$, which is equivalent to $\bar{I}_e^0 \in \alpha_{p_0}(\mathfrak{M}^1, I_i)$.

For any vector $\bar{I}_e^p = (I_e^1, \ldots, I_e^p) \in \mathfrak{I}_e^p$, we set $Q(\bar{I}_e^p) = \{I_e^1, \ldots, I_e^p\}$.

Suppose that there exists $p$ such that $(Q(\bar{I}_e^0))^p$ is not a subset of $\alpha_p(\mathfrak{M}^1, I_i)$. Then, there exists $C_0$ in $\mathfrak{M}^1_p$ and an index set $(i_1, \ldots, i_p) \in \{1, 2, \ldots, p_0\}^p$ such that

$$C_0 \circ (F_{i_1}(B_1^{i_1}, \ldots, B_{r(i_1)}^{i_1}) \times \ldots \times F_{i_p}(B_1^{i_p}, \ldots, B_{r(i_p)}^{i_p}))_\Delta \notin \Pi(I_i),$$

so that $\mathfrak{M}(I_i)$ is not a subset of $\Pi(I_i)$.

Finally, let either $\mathfrak{M}^1_p = \varnothing$ or $(Q(\bar{I}_e^0))^p \subseteq \alpha_p(\mathfrak{M}^1, I_i)$ for all $p$. Then, for each $p$, $\mathfrak{M}^1_p$ is either empty or there exists $X_p$ in $\xi_p(\mathfrak{M}^1, I_i)$ such that $Q(\bar{I}_e^0) \subseteq X_p$. Consequently, there is a function $\varphi_0$ in $\tilde{\Phi}(\mathfrak{M}^1, I_i)$ such that $Q(\bar{I}_e^0) \subseteq \varphi_0(p)$ for all $p$, so that $Q(\bar{I}_e^0) \subseteq X(I_i, \varphi)$. Therefore,

$$\mathfrak{M}^1(Q(\bar{I}_e^0)) \subseteq \bigcup_{\varphi \in \tilde{\Phi}(\mathfrak{M}^1, I_i)} \mathfrak{M}^1(X(I_i, \varphi)),$$

which contradicts the assumption that

$$C_0(\bar{I}_e^0) = I_f^0 \notin \bigcup_{\varphi \in \tilde{\Phi}(\mathfrak{M}^1, I_i)} \mathfrak{M}^1(X(I_i, \varphi)).$$

The necessity is proved.

*Sufficiency.* Let condition (5) be fulfilled. We set

$$W = \prod_{I_i \in \mathfrak{I}_i} \tilde{\Phi}(\mathfrak{M}^1, I_i) \overset{\text{df}}{=} \left\{\omega | \omega : \mathfrak{I}_i \longrightarrow \bigcup_{I_i \in \mathfrak{I}_i} \tilde{\Phi}(\mathfrak{M}^1, I_i), \ \ \underset{\mathfrak{I}_i}{\forall} I_i : \omega(I_i) \in \tilde{\Phi}(\mathfrak{M}^1, I_i)\right\}.$$

As a class of correcting operations, we use the singleton set $\mathfrak{F}$ consisting of the identity mapping of $\mathfrak{I}_e$ into itself. For notational definiteness, we set $L = \{0\}$.

The models $\mathfrak{M}^0_{0, \omega}$ are defined as

$$\mathfrak{M}^0_{0, \omega} = \left\{B \mid B : \mathfrak{I}_i \longrightarrow \mathfrak{I}_e, \ \ \underset{\mathfrak{I}_i}{\forall} I_i : B(I_i) \in X(I_i, \omega(I_i))\right\}.$$

Let $(I_i^0, I_f^0)$ be an arbitrary admissible precedent. The assumption made implies that there exist $p_0 \in \mathbb{N}_0$ and $\varphi_0 \in \tilde{\Phi}(\mathfrak{M}^1, I_i^0)$ that satisfy $I_f^0 \in \mathfrak{M}^1_{p_0}((X(I_i^0, \varphi_0))^{p_0})$. Therefore, there exists $C_0 \in \mathfrak{M}^1_p$ such that, for some $(I_e^1, \ldots, I_e^{p_0}) \in (X(I_i^0, \varphi_0))^{p_0}$, we have $C_0(I_e^1, \ldots, I_e^{p_0}) = I_f^0$.

Let ω for $\mathfrak{M}^0_{0,\omega}$ be an arbitrary function satisfying $\omega(I_i^0) = \varphi_0$. In $\mathfrak{M}^0_{0,\omega}$, there are operators $B_1, \ldots, B_{p_0}$ such that $B_k(I_i^0) = I_e^k \in X(I_i^0, \varphi_0)$ for all $k = 1, 2, \ldots, p_0$. Thus, we have $C_0(B_1 \times \ldots \times B_{p_0})_\Delta(I_i^0) = I_f^0$, which completes the proof of the theorem.

In what follows, we assume that $\mathfrak{M}^1$ is an arbitrary fixed Π-complete decision rule class.

**Definition 8.** Let $I_i \in \mathfrak{I}_i$. The system of subsets $G(I_i) = \{X(I_i, \gamma) \mid X(I_i, \gamma) \subseteq \mathfrak{I}_e, \gamma \in \Gamma(I_i)\}$, where $\Gamma(I_i)$ is an index set, is called $\mathfrak{M}^1$-*complete* for $I_i$ if

$$\begin{gathered}\underset{\Gamma(I_i)}{\forall \gamma} \; \underset{\tilde{\Phi}(\mathfrak{M}^1, I_i)}{\exists} \varphi : X(I_i, \gamma) \subseteq X(I_i, \varphi), \\ \bigcup_{\gamma \in \Gamma(I_i)} \mathfrak{M}^1(X(I_i, \gamma)) = \Pi(I_i).\end{gathered} \tag{6}$$

Now, we consider classes of correcting operators $\mathfrak{F} = \{\mathfrak{F}^\lambda \mid \lambda \in L\}$, assuming that $\mathfrak{F}^\lambda = \bigcup_{p=0}^{\infty} \mathfrak{F}_p^\lambda$ for all λ in $L$. Here, $\mathfrak{F}_p^\lambda$ is defined as $\mathfrak{F}_p^\lambda = \mathfrak{F}^\lambda \cap \{F \mid F : \mathfrak{I}_e^p \longrightarrow \mathfrak{I}_e\}$ for all $p$ in $\mathbb{N}_0$.

**Definition 9.** Let $p \in \mathbb{N}_0$. Given arbitrary λ in $L$, $I_i$ in $\mathfrak{I}_i$, and φ in $\tilde{\Phi}(\mathfrak{M}^1, I_i)$, *the set* $\beta_p^\lambda(I_i, \varphi)$ is defined as the intersection, in the $p$th Cartesian power of $\mathfrak{I}_e$, of all the complete preimages of $X(I_i, \varphi)$ with respect to correcting operations in $\mathfrak{F}_p^\lambda$:

$$\beta_p^\lambda(I_i, \varphi) = \bigcap_{F \in \mathfrak{F}_p^\lambda} F^{-1}(X(I_i, \varphi)) = \left\{ \bar{I}_e \mid \bar{I}_e \in \mathfrak{I}_e^p, \underset{\mathfrak{F}_p^\lambda}{\forall} F : F(\bar{I}_e) \in X(I_i, \varphi) \right\}. \tag{7}$$

**Definition 10.** Let $p \in \mathbb{N}_0$. Given arbitrary λ in $L$, $I_i$ in $\mathfrak{I}_i$, and φ in $\tilde{\Phi}(\mathfrak{M}^1, I_i)$, the subset $Y(I_i, \varphi, \lambda)$ of $\mathfrak{I}_e$ is called an $\mathfrak{F}^\lambda - \mathfrak{M}^1$-*admissible p projection* if

$$(Y(I_i, \varphi, \lambda))^p \subseteq \beta_p^\lambda(I_i, \varphi),$$

$$\neg \exists Z \subseteq \mathfrak{I}_e : (Y(I_i, \varphi, \lambda) \subset Z) \wedge (Z^p \subseteq \beta_p^\lambda(I_i, \varphi)).$$

The set of all $\mathfrak{F}^\lambda - \mathfrak{M}^1$-admissible $p$ projections for $\lambda \in L$, $I_i \in \mathfrak{I}_i$, and φ in $\tilde{\Phi}(\mathfrak{M}^1, I_i)$ is denoted by $\zeta_p(I_i, \varphi, \lambda)$.

For arbitrary $\lambda \in L$ in $L$, $I_i$ in $\mathfrak{I}_i$, and φ in $\tilde{\Phi}(\mathfrak{M}^1, I_i)$, the set $\Psi(I_i, \varphi, \lambda)$ of choice functions of $\mathfrak{F}^\lambda - \mathfrak{M}^1$-admissible projections is defined as

$$\Psi(I_i, \varphi, \lambda) = \left\{ \psi \mid \psi : \mathbb{N}_0 \longrightarrow \mathrm{B}(\mathfrak{I}_e), \bigcup_{\mathbb{N}_0} p : ((\mathfrak{F}_p^\lambda = \varnothing) \Rightarrow (\psi(p) = \mathfrak{I}_e)) \right.$$
$$\left. \wedge ((\mathfrak{F}_p^\lambda \neq \varnothing) \Rightarrow (\psi(p) \in \zeta_p(I_i, \varphi, \lambda))) \right\}.$$

For each choice function of $\mathfrak{F}^\lambda - \mathfrak{M}^1$-admissible projections $\psi$ in $\Psi(I_i, \varphi, \lambda)$, we set $Y(I_i, \varphi, \lambda, \psi) = \bigcap_{p=0}^{\infty}\psi(p)$. Note that

$$\mathfrak{F}^\lambda(Y(I_i, \varphi, \lambda, \psi)) = \bigcup_{r=0}^{\infty}\bigcup_{F \in \mathfrak{F}_r^\lambda} F\left(\left(\bigcap_{p=0}^{\infty}\psi(p)\right)^r\right).$$

Let $\tilde{\Psi}(I_i, \varphi, \lambda) = \{\psi \mid \psi \in \Psi(I_i, \varphi, \lambda), Y(I_i, \varphi, \lambda, \psi) \neq \varnothing\}$.

**Theorem 3** ($\mathfrak{M}^1$–$\Pi$-completeness criterion for classes of correcting operations). *A class of correcting operations $\mathfrak{F} = \{\mathfrak{F}^\lambda \mid \lambda \in L\}$ is $\mathfrak{M}^1$-complete if and only if, for any $I_i$ in $\mathfrak{I}_i$, there exists an $\mathfrak{M}^1$-complete system of subsets $G(I_i) = \{X(I_i, \gamma) \mid X(I_i, \gamma) \subseteq \mathfrak{I}_e, \gamma \in \Gamma(I_i)\}$ such that, for any $\gamma$ in $\Gamma(I_i)$, one can find $\lambda$ in $L$ for which*

$$\bigcup_{\psi \in \tilde{\Psi}(I_i, \varphi, \lambda)} \mathfrak{F}^\lambda(Y(I_i, \varphi, \lambda, \psi)) = X(I_i, \gamma). \tag{8}$$

**Remark.** It follows from (6) and (7) that

$$\bigcup_{\psi \in \tilde{\Psi}(I_i, \varphi, \lambda)} \mathfrak{F}^\lambda(Y(I_i, \varphi, \lambda, \psi)) \subseteq X(I_i, \gamma).$$

for all $I_i$ in $\mathfrak{I}_i$, $\lambda$ in $L$, $\varphi$ in $\tilde{\Phi}(\mathfrak{M}^1, I_i)$, and $\psi$ in $\tilde{\Psi}(I_i, \varphi, \lambda)$. Indeed, for all $\lambda \in L$, all $\varphi \in \tilde{\Phi}(\mathfrak{M}^1, I_i)$, all $\psi \in \tilde{\Psi}(I_i, \varphi, \lambda)$, and all $p$ in $\mathbb{N}_0$, we either have $\mathfrak{F}_p^\lambda = \varnothing$ or $\psi(p) \in \zeta_p(I_i, \varphi, \lambda)$. In the former case, naturally, $\mathfrak{F}_p^\lambda((Y(I_i, \varphi, \lambda, \psi))^p) = \varnothing$. In the latter case, $Y(I_i, \varphi, \lambda, \psi) \subseteq \psi(p)$, so $\mathfrak{F}_p^\lambda((Y(I_i, \varphi, \lambda, \psi))^p) \subseteq \mathfrak{F}_p^\lambda(\psi(p))$. Since $\psi(p) \in \zeta_p(I_i, \varphi, \lambda)$, we have $(\psi(p))^p \in \beta_p^\lambda(I_i, \varphi)$, which implies that $\mathfrak{F}_p^\lambda(\psi(p)) \subseteq X(I_i, \varphi)$ and $\mathfrak{F}_p^\lambda((Y(I_i, \varphi, \lambda, \psi))^p) \subseteq X(I_i, \varphi)$.

**Proof.** *Necessity.* Suppose that there exists $I_i^0$ in $\mathfrak{I}_i$ for which there is no $\mathfrak{M}^1$-complete (for $I_i$) system of subsets $G(I_i) = \{X(I_i, \gamma) \mid X(I_i, \gamma) \subseteq \mathfrak{I}_e, \gamma \in \Gamma(I_i)\}$ such that, for all $\gamma$ in $\Gamma(I_i)$, one can find $\lambda$ in $L$ for which (8) holds.

At the same time, let the class of correcting operations $\mathfrak{F} = \{\mathfrak{F}^\lambda \mid \lambda \in L\}$ be $\mathfrak{M}^1$-complete. This means that there exists a model of algorithmic operators $\mathfrak{M}^0$ such that model (3) is $\Pi$-complete, so that $\mathfrak{M}(I_i^0) = \Pi(I_i^0)$.

Consider the system $G^0(I_i^0)$ of subsets of $\mathfrak{I}_e$ defined as

$$G^0(I_i^0) = \{X(I_i^0, \gamma) \mid X(I_i^0, \gamma) \subseteq \mathfrak{I}_e, \gamma \in \Gamma^0(I_i^0)\},$$

where

$$\Gamma^0(I_i^0) = \bigcup_{\lambda \in L}\{\lambda\} \times W(\lambda)$$

and, for all $\lambda \in L$ and $\omega \in W(\lambda)$,

$$X(I_i^0, \gamma) = X(I_i^0, (\lambda, \omega)) = \mathfrak{F}^\lambda(\mathfrak{M}_{\lambda, \omega}^0)(I_i^0).$$

Obviously, for all $\gamma$ in $\Gamma^0(I_i^0)$, we have $\mathfrak{F}^\lambda(\mathfrak{M}_{\lambda, \omega}^0)(I_i^0) \subseteq X(I_i, \varphi)$ (for some $\varphi$ in $\tilde{\Phi}(\mathfrak{M}^1, I_i)$) and

$$\bigcup_{\gamma \in \Gamma^0} \mathfrak{M}^1(\mathfrak{F}^\lambda(\mathfrak{M}_{\lambda, \omega}^0)(I_i^0)) = \Pi(I_i^0).$$

Thus, $G^0(I_i^0)$ is $\mathfrak{M}^1$-complete for $I_i^0$, which contradicts the assumption made. The necessity is proved.

*Sufficiency.* For each $I_i$ in $\Im_i$, suppose that there exists an $\mathfrak{M}^1$-complete (for $I_i$) system of subsets $G^0(I_i) = \{X^0(I_i, \gamma) \mid X^0(I_i, \gamma) \subseteq \Im_e, \gamma \in \Gamma^0(I_i)\}$ such that, for all $\gamma$ in $\Gamma(I_i)$, one can find $\lambda$ in $L$ for which (8) holds.

For each $\lambda \in L$, we set

$$W(\lambda) = \prod_{I_i \in \Im_i}\left(\bigcup_{\varphi \in \tilde{\Phi}(\mathfrak{M}^1, I_i)} \tilde{\Psi}(I_i, \varphi, \lambda)\right)$$

$$= \left\{\omega \mid \omega : \Im_i \longrightarrow \bigcup_{I_i \in \Im_i} \bigcup_{\varphi \in \tilde{\Phi}(\mathfrak{M}^1, I_i)} \tilde{\Psi}(I_i, \varphi, \lambda), \quad \underset{\Im_i}{\forall} I_i \underset{\tilde{\Phi}(\mathfrak{M}^1, I_i)}{\exists} \varphi : \omega(I_i) \in \tilde{\Psi}(I_i, \varphi, \lambda)\right\}.$$

The models $\mathfrak{M}^0_{\lambda, \omega}$ are defined as

$$\mathfrak{M}^0_{\lambda, \omega} = \left\{\mathrm{B} \mid \mathrm{B} : \Im_i \longrightarrow \Im_e, \underset{\Im_i}{\forall} I_i : \mathrm{B}(I_i) \in Y(I_i, \varphi, \lambda, \omega(I_i))\right\}.$$

For all $I_i$ in $\Im_i$, we have $\mathfrak{M}^0_{\lambda, \omega}(I_i) = \{\mathrm{B}(I_i) \mid \mathrm{B} \in \mathfrak{M}^0_{\lambda, \omega}\} = Y(I_i, \varphi, \lambda, \omega(I_i))$, so that $\mathfrak{F}^\lambda(\mathfrak{M}^0_{\lambda, \omega})(I_i) = \mathfrak{F}^\lambda(Y(I_i, \varphi, \lambda, \omega(I_i)))$. It follows that

$$\bigcup_{\psi \in \tilde{\Psi}(I_i, \varphi, \lambda)} \mathfrak{F}^\lambda(\mathfrak{M}^0_{\lambda, \omega})(I_i) = \bigcup_{\psi \in \tilde{\Psi}(I_i, \varphi, \lambda)} \mathfrak{F}^\lambda(Y(I_i, \varphi, \lambda, \psi)) = X^0(I_i, \gamma)$$

for all $\gamma$ in $\Gamma^0(I_i)$.

Since $G^0$ is $\mathfrak{M}^1$-complete for $I_i$ by assumption, we obtain

$$\mathfrak{M}(I_i) = \bigcup_{\lambda \in L} \bigcup_{\omega \in W(\lambda)} \mathfrak{M}^1 \circ \mathfrak{F}^\lambda(\mathfrak{M}^0_{\lambda, \omega})(I_i) = \Pi(I_i),$$

which completes the proof of the theorem.

In what follows, it is assumed that $\mathfrak{F} = \{\mathfrak{F}^\lambda \mid \lambda \in L\}$ is an arbitrary fixed $\mathfrak{M}^1 - \Pi$-complete class of correcting operations.

**Definition 11.** Let $I_i \in \Im_i$, and let $G(I_i) = \{X(I_i, \gamma) \mid X(I_i, \gamma) \subseteq \Im_e, \gamma \in \Gamma(I_i)\}$ be a fixed $\mathfrak{M}^1$-complete (for $I_i$) system of subsets. A system of subsets $H(I_i, G) = \{Y(I_i, \gamma, \lambda, \delta) \mid Y(I_i, \gamma, \lambda, \delta) \subseteq \Im_e, \gamma \in \Gamma(I_i), \delta \in \Delta(I_i, G)\}$ is called $\mathfrak{F} - \mathfrak{M}^1$*-complete* for $I_i$ if

$$\underset{\Delta(I_i, G)}{\forall} \delta \underset{L}{\forall} \lambda \underset{\tilde{\Phi}(\mathfrak{M}^1, I_i)}{\exists} \varphi \underset{\tilde{\Psi}(I_i, \varphi, \lambda)}{\exists} \psi : Y(I_i, \gamma, \lambda, \delta) \subseteq Y(I_i, \varphi, \lambda, \psi),$$

$$\underset{\Gamma(I_i)}{\forall} \gamma \underset{L}{\exists} \lambda \bigcup_{\delta \in \Delta(I_i, G)} \mathfrak{F}^\lambda(Y(I_i, \gamma, \lambda, \delta)) = X(I_i, \gamma).$$

**Theorem 4** ($\mathfrak{F} - \mathfrak{M}^1$-$\Pi$-completeness criterion for models of algorithmic operators). *A model of algorithmic operators* $\mathfrak{M}^0 = \{\mathfrak{M}^0_{\lambda, \omega} \mid \lambda \in L, \omega \in W(\lambda)\}$ *is* $\mathfrak{F} - \mathfrak{M}^1 - \Pi$*-complete if and only if*

$$\underset{L}{\forall} \lambda \underset{W(\lambda)}{\forall} \omega \underset{\tilde{\Phi}(\mathfrak{M}^1, I_i)}{\exists} \varphi \underset{\tilde{\Psi}(I_i, \varphi, \lambda)}{\exists} \psi : \mathfrak{M}^0_{\lambda, \omega}(I_i) \subseteq Y(I_i, \varphi, \lambda, \psi) \tag{9}$$

*for all* $I_i$ *in* $\Im_i$ *and there exists an* $\mathfrak{M}^1$*-complete system of subsets* $G(I_i) = \{X(I_i, \gamma) \mid X(I_i, \gamma) \subseteq \Im_e, \gamma \in \Gamma(I_i)\}$ *and an* $\mathfrak{F} - \mathfrak{M}^1$*-complete system of subsets* $H(I_i, G) = \{Y(I_i, \gamma, \lambda, \delta) \mid Y(I_i, \gamma, \lambda, \delta) \subseteq \Im_e, \gamma \in \Gamma(I_i), \delta \in \Delta(I_i, G)\}$

*such that*

$$\underset{\Gamma(I_i)}{\forall} \gamma \underset{L}{\exists} \lambda \underset{\Delta(I_i, G)}{\forall} \delta \underset{W(\lambda)}{\exists} \omega : Y(I_i, \gamma, \lambda, \delta) \subseteq \mathfrak{M}^0_{\lambda, \omega}(I_i). \tag{10}$$

**Proof.** *Necessity of condition* (9). For some $I_i^0$ in $\mathfrak{I}_i$, $\lambda_0 \in L$, and $\omega_0 \in W(\lambda_0)$, suppose that none of the sets $Y(I_i^0, \varphi, \lambda_0, \psi)$ with $\varphi \in \tilde{\Phi}\{\mathfrak{M}^1, I_i^0\}$ and $\psi \in \tilde{\Psi}(I_i^0, \varphi, \lambda_0)$ covers $\mathfrak{M}^0_{\lambda_0, \omega_0}(I_i^0)$.

If there is no $\varphi_0$ in $\tilde{\Phi}(\mathfrak{M}^1, I_i^0)$ such that $\mathfrak{F}^{\lambda_0}(\mathfrak{M}^0_{\lambda_0, \omega_0}(I_i^0)) \subseteq X(I_i^0, \varphi_0)$, then $\mathfrak{M}^1(\mathfrak{F}^{\lambda_0}(\mathfrak{M}^0_{\lambda_0, \omega_0}(I_i^0))) \subseteq \Pi(I_i^0)$ does not hold. However, if $\tilde{\Phi}(\mathfrak{M}^1, I_i^0)$ for some $\varphi_0$ in $\mathfrak{F}^{\lambda_0}(\mathfrak{M}^0_{\lambda_0, \omega_0}(I_i^0)) \subseteq X(I_i^0, \varphi_0)$, then in $\tilde{\Psi}(I_i^0, \varphi, \lambda_0)$ there exists a choice function of $\mathfrak{F}^{\lambda_0} - \mathfrak{M}^1$-admissible projections $\psi_0$ such that $\mathfrak{M}^0_{\lambda_0, \omega_0}(I_i^0) \subseteq Y(I_i^0, \varphi, \lambda_0, \psi)$, which proves the necessity of condition (9).

The necessity of condition (10) follows from the fact that $\{\mathfrak{M}^0_{\lambda, \omega}(I_i) \mid \lambda \in L, \omega \in W(\lambda)\}$ can be used as an $\mathfrak{F} - \mathfrak{M}^1$-complete system of subsets $H(I_i, G)$ and $\{\mathfrak{F}^\lambda(\mathfrak{M}^0_{\lambda, \omega}(I_i) \mid \lambda \in L, \omega \in W(\lambda)\}$ can be used as $G(I_i)$.

*Sufficiency of condition.* Suppose that $\mathfrak{M}^0$ satisfies (9) and (10). It follows from (9) that $\mathfrak{M}^1 \circ \mathfrak{F}^\lambda(\mathfrak{M}^0_{\lambda, \omega})(I_i) \subseteq \Pi(I_i)$ for all $I_i \in \mathfrak{I}_i$, $\lambda \in L$, and $\omega \in W(\lambda)$.

It follows from (10) that, for any $I_i$ and any $I_f \in \Pi(I_i)$, there exist $\lambda$ and $\omega$ such that $I_f \in \mathfrak{M}^1 \circ \mathfrak{F}^\lambda(\mathfrak{M}^0_{\lambda, \omega})(I_i)$, which was to be proved. The proof of Theorem 4 is completed.


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