Matrix method of polynomial solutions to constant coefficient PDE's

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- The problem to find a polynomial solution to constant coefficient PDE('s) is a familiar problem, see for example [1–6]. And the well-known methods to find polynomial solutions are based on (complicated) general algebra approaches (like the using of the Macaulay matrices, primary decompositions of ideals, and so on).

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Preface

- **The problem to find a polynomial solution to constant coefficient PDE('s) is a familiar problem. And the well-known methods to find polynomial solutions are based on (complicated) general algebra approaches (like the using of the Macaulay matrices, primary decompositions of ideals, and so on).**
- **On the other hand, the finding of a polynomial that the polynomial is solution to constant coefficient PDE('s) can be, equaling the same degree terms, easily reduced to solving to a linear algebraic system.**
- So, the linear algebra approach to find a polynomial solution to constant coefficient PDE('s) is elementary and, probably, more effective.
- In this report, see also [7], we present a constructive method to obtain the matrix of the linear algebraic system.
- Further, our matrix approach allows to investigate in more detail some linear algebra properties of the polynomial space, for example, dimension, basis, affine invariance etc.
- Moreover, our method allows to use elementary computer code (in Mathematica, for example) to obtain a polynomial basis for any constant coefficient PDE('s).

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Notations

Multi-indices and polynomial spaces

- d-tuple $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d_{\geq 0}$ is ***multi-index***;
- $|\alpha| := \alpha_1 + \cdots + \alpha_d$ ***length*** of a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$;
- for $\alpha, \beta \in \mathbb{Z}^d_{\geq 0}$, we write $\beta \leq \alpha$ if $\beta_j \leq \alpha_j$ for all $j = 1, \dots, d$;
- $\alpha! := \alpha_1! \cdots \alpha_d!$;
- ***binomial coefficient*** for multi-indices α, β are:
  $\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d} = \frac{\alpha!}{\beta!(\alpha-\beta)!}, \qquad \binom{\alpha}{\beta} = 0 \quad \text{if } \beta \not\leq \alpha;$
- *monomial*: $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $x := (x_1, \dots, x_d)$, $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d_{\geq 0}$;
- $l \in \mathbb{Z}_{\geq 0}$, polynomial spaces are:
  - $\Pi_l := \operatorname{span}\left\{x^\alpha : \alpha \in \mathbb{Z}^d_{\geq 0}, |\alpha| = l\right\}$ total degrees of polynomials are equal to l;
  - $\Pi_{\leq l} := \operatorname{span}\left\{x^\alpha : \alpha \in \mathbb{Z}^d_{\geq 0}, |\alpha| \leq l\right\}$ total degrees of polynomials are less than or equal to l;
  - $\Pi := \bigcup_{l=0}^{\infty} \Pi_l$ *all* polynomials of d variables with coefficients from $\mathbb{C}$;

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Remark

- *Since the linear algebra definitions and assertions are valid for* any *field; we can consider polynomials with coefficients from an arbitrary field.*
- *On the other hand, considering d-variate polynomials in the space $\mathbb{C}^d$, we have* all *zeroes of the multivariate polynomials.*

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Definition

$f, g \in L^2(\mathbb{C}^d)$, *an* inner product *in the space* $L^2(\mathbb{C}^d)$ *is:*

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- ***S′ space of tempered distributions*;**
- ***S Schwartz space* (of test functions);**

Definition ([1,2])

Let $\phi \in S(\mathbb{C}^d)$ *be a* complex *test function. Let* $f = f(x)$, $x \in \mathbb{C}^d$, *be a locally integrable on* $\mathbb{C}^d$ complex *function. Then the function* f *induces some distribution (continuous* linear *functional) on* $S(\mathbb{C}^d)$ *as follows*

$$\int_{\mathbb{C}^d} \overline{f(x)}\phi(x)\,dx = \overline{\langle f, \phi\rangle}, \qquad (**)$$

where $\langle\cdot,\cdot\rangle$, *in the right-hand side of* $(**)$, *is the inner product defined by* $(*)$.

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- I ***identity operator***;
- $\delta_{jk} := \begin{cases} 1, & j = k, \\ 0, & j \neq k; \end{cases} \quad j, k \in \mathbb{Z}_{\geq 0}$, ***Kronecker delta***;
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Remark

if $x, y \in \mathbb{C}^d$, *then* $x \cdot y$ *defined by* $(*)$ *is, in fact, a* <u>semi–dot</u> product: $\boxed{y \cdot x \neq \overline{x \cdot y}}$;

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- if $x, y \in \mathbb{C}^d$, then genuine dot product is:

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$$f(x) \mapsto \hat{f}(\xi) = (\mathfrak{F} f)(\xi) := (2\pi)^{-d/2} \int_{\mathbb{C}^d} f(x) e^{-i\xi \cdot x} \, dx, \quad x \in \mathbb{C}^d, \ \xi \in \hat{\mathbb{C}}^d;$$

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- $D^\alpha := D_1^{\alpha_1} \cdots D_d^{\alpha_d}$ *differential operator*, D_n, $n = 1, \ldots, d$, *partial derivative* with respect to the nth coodrinate;
  - $x_0 \in \mathbb{C}^d, \alpha \in \mathbb{Z}^d_{\geq 0}$: $\left(\mathfrak{F} e^{ix_0 \cdot x} x^\alpha\right)(\xi) = i^{|\alpha|} D^\alpha \delta(\xi - x_0).$
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Remark

Below, abusing notations, for a function $\mathbf{f} = \mathbf{f}(\mathbf{x})$ *and* constant *point* $\mathbf{x}_0$, *we shall write everywhere* $\mathbf{D}^\alpha \mathbf{f}(\mathbf{x}_0)$, *meaning, in fact,* $\mathbf{D}^\alpha \mathbf{f}(\mathbf{x})|_{x=x_0}$.

- $<_{\mathrm{lex}}$ a *lexicographical order*;
- $\mathcal{A}_k$, $k \in \mathbb{Z}_{\geq 0}$, *lexicographically ordered set* of all multi-indices of length k:

$$\mathcal{A}_k := \left({}^1\alpha, {}^2\alpha, \ldots, {}^{d(k)}\alpha\right), \qquad \begin{array}{l} {}^q\alpha \in \mathbb{Z}^d_{\geq 0},\ |{}^q\alpha| = k,\ q = 1, \ldots, d(k), \\ {}^q\alpha <_{\mathrm{lex}} {}^{q'}\alpha \iff q < q'; \end{array}$$

- $d(k)$ *number of k-combinations with repetition from the d elements:*

$$d(k) := \binom{d+k-1}{k} = \frac{(d+k-1)!}{k!(d-1)!};$$

- $\widetilde{\mathcal{A}}_k$ *concatenated set of multi-indices*:

$$\widetilde{\mathcal{A}}_k := (\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_k);$$

- $\widetilde{d}(k)$ *length of a concatenated set* like $\widetilde{\mathcal{A}}_k$:

$$\widetilde{d}(k) := d(0) + d(1) + \cdots + d(k) = \frac{(d+k)!}{k!d!}.$$

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- $\mathcal{P}_k$, $k \in \mathbb{Z}_{\geq 0}$, *lexicographically ordered set of all monomials of total degree k:*
$$\mathcal{P}_k(x) := \left(x^{{}^1\alpha}, \ldots, x^{{}^{d(k)}\alpha}\right), \quad x = (x_1, \ldots, x_d), \; \left({}^1\alpha, \ldots, {}^{d(k)}\alpha\right) = \mathcal{A}_k;$$
- $\mathcal{P}_k^\beta$, $\beta \in \mathbb{Z}^d_{\geq 0}$, the *set of monomials:*
$$\mathcal{P}_k^\beta(x) := \left(\binom{{}^1\alpha}{\beta} x^{{}^1\alpha - \beta}, \ldots, \binom{{}^{d(k)}\alpha}{\beta} x^{{}^{d(k)}\alpha - \beta}\right), \qquad \left({}^1\alpha, \ldots, {}^{d(k)}\alpha\right) = \mathcal{A}_k.$$
- $\mathcal{D}_k$, $\mathcal{D}_k^\beta$, $\beta \in \mathbb{Z}^d_{\geq 0}$, *ordered sets of differential operators:*
$$\mathcal{D}_k := \left((-i)^k D^{{}^1\alpha}, \ldots, (-i)^k D^{{}^{d(k)}\alpha}\right),$$
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$$\mathcal{D}_k := \left((-i)^k D^{{}^1\alpha}, \dots, (-i)^k D^{{}^{d(k)}\alpha}\right),$$

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Remark

If $\beta \not\leq {}^q\alpha$, *then* $\left[\mathcal{P}_k^\beta(x)\right]_q = \left[\mathcal{D}_k^\beta\right]_q = 0$. *If* $|\beta| > k$, *then* $\mathcal{P}_k^\beta(x)$, $\mathcal{D}_k^\beta$ *are zero sets.*

- *concatenated sets of monomials:*

$$\widetilde{\mathcal{P}}_k := (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_k), \quad \widetilde{\mathcal{P}}_k^\beta := \left(\mathcal{P}_0^\beta, \mathcal{P}_1^\beta, \dots, \mathcal{P}_k^\beta\right);$$

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- $\mathcal{D}_k$, $\mathcal{D}_k^\beta$, $\beta \in \mathbb{Z}^d_{\geq 0}$, *ordered sets of differential operators:*
$$\mathcal{D}_k := \left((-i)^k D^{{}^1\alpha}, \dots, (-i)^k D^{{}^{d(k)}\alpha}\right),$$
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- *concatenated sets of derivatives:*
$$\widetilde{\mathcal{D}}_k := (\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_k), \quad \widetilde{\mathcal{D}}_k^\beta := \left(\mathcal{D}_0^\beta, \mathcal{D}_1^\beta, \dots, \mathcal{D}_k^\beta\right).$$

Null-space definition

Definition

Suppose $\mathbf{A} := [a_{ij}]$ $(1 \le i \le n, 1 \le j \le m)$, $a_{ij} \in \mathbb{C}$, *is an* $n \times m$ *matrix. By definition, put*

$$\ker \mathbf{A} := \{ v \in \mathbb{C}^m : \mathbf{A}v = 0 \} \, .$$

We say that the linear space $\ker \mathbf{A}$ *is the (right)* null-space *of the matrix* $\mathbf{A}$.

Remark

Sometimes we shall treat the null-space $\ker \mathbf{A}$ *as a* $(\dim \ker \mathbf{A})$-column matrix *of vectors that the vectors form a basis for* $\ker \mathbf{A}$.

For $l \in \mathbb{Z}_{\geq 0}$, define a $\widetilde{d}(l) \times \widetilde{d}(l)$ matrix $\widetilde{\mathrm{D}}_l$ as

$$\widetilde{\mathrm{D}}_l := \begin{bmatrix} \mathrm{D}_0 & \mathrm{D}_1 & \dots & \mathrm{D}_{l-1} & \mathrm{D}_l \end{bmatrix} = \begin{bmatrix} \mathrm{D}_0^0 & \mathrm{D}_1^0 & \dots & \mathrm{D}_{l-1}^0 & \mathrm{D}_l^0 \\ 0 & \mathrm{D}_1^1 & \dots & \mathrm{D}_{l-1}^1 & \mathrm{D}_l^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathrm{D}_{l-1}^{l-1} & \mathrm{D}_l^{l-1} \\ 0 & 0 & \dots & 0 & \mathrm{D}_l^l \end{bmatrix}, \qquad (\spadesuit)$$

where $k \in \mathbb{Z}_{\geq 0}$, $k \leq l$, and a $\widetilde{d}(l) \times d(k)$ matrix D_k defined as follows:

$$\mathrm{D}_k := \begin{bmatrix} \mathrm{D}_k^0 \\ \mathrm{D}_k^1 \\ \vdots \\ \mathrm{D}_k^k \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} \widetilde{d}(l) - \widetilde{d}(k) \end{bmatrix},$$

D_k^r, $r = 0, \dots, k$, are $d(r) \times d(k)$ submatrices defined as

$$\mathrm{D}_k^r := \begin{bmatrix} \begin{bmatrix} \mathcal{D}_k^{{}^1\beta} \end{bmatrix} \\ \begin{bmatrix} \mathcal{D}_k^{{}^2\beta} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \mathcal{D}_k^{{}^{d(r)}\beta} \end{bmatrix} \end{bmatrix}$$

$$\left({}^1\beta, \dots, {}^{d(r)}\beta\right) = \mathcal{A}_r.$$

For $l \in \mathbb{Z}_{\geq 0}$, define a $\widetilde{d}(l) \times \widetilde{d}(l)$ matrix $\widetilde{\mathrm{D}}_l$ as

$$\widetilde{\mathrm{D}}_l := \begin{bmatrix} \mathrm{D}_0 & \mathrm{D}_1 & \dots & \mathrm{D}_{l-1} & \mathrm{D}_l \end{bmatrix} = \begin{bmatrix} \mathrm{D}_0^0 & \mathrm{D}_1^0 & \dots & \mathrm{D}_{l-1}^0 & \mathrm{D}_l^0 \\ 0 & \mathrm{D}_1^1 & \dots & \mathrm{D}_{l-1}^1 & \mathrm{D}_l^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathrm{D}_{l-1}^{l-1} & \mathrm{D}_l^{l-1} \\ 0 & 0 & \dots & 0 & \mathrm{D}_l^l \end{bmatrix}, \qquad (\spadesuit)$$

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$$\mathrm{D}_k := \begin{bmatrix} \mathrm{D}_k^0 \\ \mathrm{D}_k^1 \\ \vdots \\ \mathrm{D}_k^k \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \left.\right\} \widetilde{d}(l) - \widetilde{d}(k) \end{matrix}, \qquad \mathrm{D}_k^r := (-i)^{k-r} \begin{bmatrix} \left[\binom{{}^1\alpha}{{}^1\beta} D^{{}^1\alpha - {}^1\beta} \dots \binom{{}^{d(k)}\alpha}{{}^1\beta} D^{{}^{d(k)}\alpha - {}^1\beta} \right] \\ \left[\binom{{}^1\alpha}{{}^2\beta} D^{{}^1\alpha - {}^2\beta} \dots \binom{{}^{d(k)}\alpha}{{}^2\beta} D^{{}^{d(k)}\alpha - {}^2\beta} \right] \\ \dots\dots\dots\dots \\ \left[\binom{{}^1\alpha}{{}^{d(r)}\beta} D^{{}^1\alpha - {}^{d(r)}\beta} \dots \binom{{}^{d(k)}\alpha}{{}^{d(r)}\beta} D^{{}^{d(k)}\alpha - {}^{d(r)}\beta} \right] \end{bmatrix}$$

$$\left({}^1\beta, \dots, {}^{d(r)}\beta\right) = \mathcal{A}_r, \qquad \left({}^1\alpha, \dots, {}^{d(k)}\alpha\right) = \mathcal{A}_k,$$

$$r = 0, 1, \dots, k \qquad k = 0, 1, \dots, l$$

Component-wise form of the matrix $\widetilde{\mathrm{D}}_l$, $l \in \mathbb{Z}_{\geq 0}$ is:

$$\left[\widetilde{\mathrm{D}}_l\right]_{qr,\ 1\leq q,r\leq \widetilde{d}(l)} = \begin{cases} (-i)^{|{}^r\alpha - {}^q\beta|} \binom{{}^r\alpha}{{}^q\beta} D^{{}^r\alpha - {}^q\beta}, & {}^q\beta \leq {}^r\alpha\,, \\ 0, & \text{otherwise}, \end{cases}$$

where $\left({}^1\alpha, \ldots, {}^{\widetilde{d}(l)}\alpha,\right) = \left({}^1\beta, \ldots, {}^{\widetilde{d}(l)}\beta,\right) = \widetilde{\mathcal{A}}_l$.

2D $\widetilde{d}(3) \times \widetilde{d}(3) = 10 \times 10$ matrix $\widetilde{\mathrm{D}}_3$ is of the form:

$$\widetilde{\mathrm{D}}_3 := \left[\begin{array}{c|cc|ccc|cccc}
\mathrm{I} & -i\partial_x & -i\partial_y & -\partial_{xx} & -\partial_{xy} & -\partial_{yy} & i\partial_{xxx} & i\partial_{xxy} & i\partial_{xyy} & i\partial_{yyy} \\ \hline
0 & \mathrm{I} & 0 & -2i\partial_x & -i\partial_y & 0 & -3\partial_{xx} & -2\partial_{xy} & -\partial_{yy} & 0 \\
0 & 0 & \mathrm{I} & 0 & -i\partial_x & -2i\partial_y & 0 & -\partial_{xx} & -2\partial_{xy} & -3\partial_{yy} \\ \hline
0 & 0 & 0 & \mathrm{I} & 0 & 0 & -3i\partial_x & -i\partial_y & 0 & 0 \\
0 & 0 & 0 & 0 & \mathrm{I} & 0 & 0 & -2i\partial_x & -2i\partial_y & 0 \\
0 & 0 & 0 & 0 & 0 & \mathrm{I} & 0 & 0 & -i\partial_x & -3i\partial_y \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I}
\end{array}\right],$$

where I is the identity operator.

Properties of the matrix $\widetilde{\mathrm{D}}_l$

$$\widetilde{\mathrm{D}}_l := \begin{bmatrix} \mathrm{D}_0^0 & \mathrm{D}_1^0 & \dots & \mathrm{D}_{l-1}^0 & \mathrm{D}_l^0 \\ 0 & \mathrm{D}_1^1 & \dots & \mathrm{D}_{l-1}^1 & \mathrm{D}_l^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathrm{D}_{l-1}^{l-1} & \mathrm{D}_l^{l-1} \\ 0 & 0 & \dots & 0 & \mathrm{D}_l^l \end{bmatrix}. \qquad (\spadesuit)$$

Proposition

Let $l \in \mathbb{Z}_{\geq 0}$. The submatrix

$$\mathrm{D}_m^{m'} := (-i)^{m-m'} \begin{bmatrix} \left[\binom{{}^1\alpha}{{}^1\beta} D^{{}^1\alpha - {}^1\beta} \quad \dots \quad \binom{{}^{d(m)}\alpha}{{}^1\beta} D^{{}^{d(m)}\alpha - {}^1\beta} \right] \\ \left[\binom{{}^1\alpha}{{}^2\beta} D^{{}^1\alpha - {}^2\beta} \quad \dots \quad \binom{{}^{d(m)}\alpha}{{}^2\beta} D^{{}^{d(m)}\alpha - {}^2\beta} \right] \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \left[\binom{{}^1\alpha}{{}^{d(m')}\beta} D^{{}^1\alpha - {}^{d(m')}\beta} \quad \dots \quad \binom{{}^{d(m)}\alpha}{{}^{d(m')}\beta} D^{{}^{d(m)}\alpha - {}^{d(m')}\beta} \right] \end{bmatrix},$$

$m' = 0, \dots, m$, $m = 0, \dots, l$, *contains only the derivatives of order* $m - m'$.

$$\widetilde{D}_l := \begin{bmatrix} D_0^0 & D_1^0 & \dots & D_{l-1}^0 & D_l^0 \\ 0 & D_1^1 & \dots & D_{l-1}^1 & D_l^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & D_{l-1}^{l-1} & D_l^{l-1} \\ 0 & 0 & \dots & 0 & D_l^l \end{bmatrix}. \qquad (\spadesuit)$$

Proposition

Let $l \in \mathbb{Z}_{\geq 0}$. The submatrix $D_m^{m'}$ $m' = 0, \dots, m$, $m = 0, \dots, l$, contains only the derivatives of order $m - m'$.

Corollary

1. *All the submatrices on the mth block diagonal of matrix $(\spadesuit)$, i. e., the submatrices $D_m^0, D_{m+1}^1, \dots, D_l^{l-m}$, contain the derivatives of order m;*

Properties of the matrix $\widetilde{\mathrm{D}}_l$

$$\widetilde{\mathrm{D}}_l := \begin{bmatrix} \mathrm{I} & \mathrm{D}_1^0 & \dots & \mathrm{D}_{l-1}^0 & \mathrm{D}_l^0 \\ 0 & \mathrm{I} & \dots & \mathrm{D}_{l-1}^1 & \mathrm{D}_l^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathrm{I} & \mathrm{D}_l^{l-1} \\ 0 & 0 & \dots & 0 & \mathrm{I} \end{bmatrix}. \qquad (\spadesuit)$$

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2. $\mathrm{D}_m^m = \mathrm{I}$, *where* I *is the* $d(m) \times d(m)$ *identity matrix.*

$$\widetilde{\mathrm{D}}_l f(x_0) := \begin{bmatrix} \mathrm{I}f(x_0) & \mathrm{D}_1^0 f(x_0) & \dots & \mathrm{D}_{l-1}^0 f(x_0) & \mathrm{D}_l^0 f(x_0) \\ 0 & \mathrm{I}f(x_0) & \dots & \mathrm{D}_{l-1}^1 f(x_0) & \mathrm{D}_l^1 f(x_0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathrm{I}f(x_0) & \mathrm{D}_l^{l-1} f(x_0) \\ 0 & 0 & \dots & 0 & \mathrm{I}f(x_0) \end{bmatrix}. \qquad (\spadesuit\spadesuit)$$

Corollary

1. *All the submatrices on the* m*th block diagonal of the matrix* $\widetilde{\mathrm{D}}_l$*, i. e., the submatrices* $\mathrm{D}_m^0, \mathrm{D}_{m+1}^1, \dots, \mathrm{D}_l^{l-m}$*, contain the derivatives of order* m*;*
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Corollary

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2. $\mathbf{D}_m^m f(x_0) = \mathbf{I}f(x_0)$*, where* $\mathbf{I}$ *is* $d(m) \times d(m)$ *identity matrix.*

Theorem

The matrix $\widetilde{\mathbf{D}}_l f(x_0)$*,* $l \in \mathbb{Z}_{\geq 0}$*, see* $(\spadesuit\spadesuit)$*, is singular iff* $f(x_0) = 0$*.*

Properties of the matrix $\widetilde{\mathrm{D}}_l f(x_0)$

$$\widetilde{\mathrm{D}}_l f(x_0) := \begin{bmatrix} \mathrm{I}f(x_0) & \mathrm{D}_1^0 f(x_0) & \dots & \mathrm{D}_{l-1}^0 f(x_0) & \mathrm{D}_l^0 f(x_0) \\ 0 & \mathrm{I}f(x_0) & \dots & \mathrm{D}_{l-1}^1 f(x_0) & \mathrm{D}_l^1 f(x_0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathrm{I}f(x_0) & \mathrm{D}_l^{l-1} f(x_0) \\ 0 & 0 & \dots & 0 & \mathrm{I}f(x_0) \end{bmatrix}. \qquad (\spadesuit\spadesuit)$$

Corollary

1. *All the submatrices on the* m*th block diagonal of the matrix* $\widetilde{\mathrm{D}}_l$*, i. e., the submatrices* $\mathrm{D}_m^0, \mathrm{D}_{m+1}^1, \dots, \mathrm{D}_l^{l-m}$*, contain the derivatives of order* m*;*
2. $\mathrm{D}_m^m f(x_0) = \mathrm{I}f(x_0)$*, where* I *is* $d(m) \times d(m)$ *identity matrix.*

Theorem

The matrix $\widetilde{\mathrm{D}}_l f(x_0)$*,* $l \in \mathbb{Z}_{\geq 0}$*, see* $(\spadesuit\spadesuit)$*, is singular iff* $f(x_0) = 0$*.*

Theorem

The $d(m') \times d(m)$ *submatrix* $\mathrm{D}_m^{m'} f(x_0)$*,* $m' = 0, \dots, m$*,* $m = 0, \dots, l$*,* $l \in \mathbb{Z}_{\geq 0}$*, has full rank if and only if there exists at least one non-zero derivative* $D^\gamma f(x_0)$*,* $|\gamma| = m - m'$*.*

$$\widetilde{\mathrm{D}}_l f(x_0) := \begin{bmatrix} \mathrm{I}f(x_0) & \mathrm{D}_1^0 f(x_0) & \dots & \mathrm{D}_{l-1}^0 f(x_0) & \mathrm{D}_l^0 f(x_0) \\ 0 & \mathrm{I}f(x_0) & \dots & \mathrm{D}_{l-1}^1 f(x_0) & \mathrm{D}_l^1 f(x_0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathrm{I}f(x_0) & \mathrm{D}_l^{l-1} f(x_0) \\ 0 & 0 & \dots & 0 & \mathrm{I}f(x_0) \end{bmatrix}. \qquad (\spadesuit\spadesuit)$$

Theorem

The matrix $\widetilde{\mathrm{D}}_l f(x_0)$, $l \in \mathbb{Z}_{\geq 0}$, see $(\spadesuit\spadesuit)$, is singular iff $f(x_0) = 0$.

Remark

Each of the submatrices $\mathrm{D}_m^{m'} f(x_0)$, $m' = 0, \dots, m$, $m = 0, \dots, l$, is either a full rank matrix or zero matrix.

Let $P \in \Pi$, let $x_0 \in \mathbb{C}^d$, $l \in \mathbb{Z}_{\geq 0}$, then

$$\widetilde{\mathrm{D}}_l P(x_0) := \begin{bmatrix} \mathrm{I}P(x_0) & \mathrm{D}_1^0 P(x_0) & \dots & \mathrm{D}_{l-1}^0 P(x_0) & \mathrm{D}_l^0 P(x_0) \\ 0 & \mathrm{I}P(x_0) & \dots & \mathrm{D}_{l-1}^1 P(x_0) & \mathrm{D}_l^1 P(x_0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathrm{I}P(x_0) & \mathrm{D}_l^{l-1} P(x_0) \\ 0 & 0 & \dots & 0 & \mathrm{I}P(x_0) \end{bmatrix}. \qquad (\spadesuit\spadesuit)$$

Theorem

Let $l \in \mathbb{Z}_{\geq 0}$, $\widetilde{\mathcal{P}}_l$ be the lexicographically ordered set of monomials.

Let $P \in \Pi$, $x_0 \in \mathbb{C}^d$ be a given point, $\widetilde{\mathrm{D}}_l P(x_0)$ be matrix $(\spadesuit\spadesuit)$, $v \in \mathbb{C}^{\widetilde{d}(l)}$ be a vector. Then

$$\left.\begin{array}{l} e^{ix_0 \cdot x} \left[\widetilde{\mathcal{P}}_l(x)\right] v \neq 0, \\ e^{ix_0 \cdot x} \left[\widetilde{\mathcal{P}}_l(x)\right] v \in e^{ix_0 \cdot x} \Pi_{\leq l} \cap \ker P(-iD) \end{array}\right\} \iff \left\{\begin{array}{l} P(x_0) = 0, \\ v \in \ker \widetilde{\mathrm{D}}_l P(x_0) \end{array}\right.$$

Let $P \in \Pi$, let $x_0 \in \mathbb{C}^d$, $l \in \mathbb{Z}_{\geq 0}$, then

$$\widetilde{\mathrm{D}}_l P(x_0) := \begin{bmatrix} \mathrm{I}P(x_0) & \mathrm{D}_1^0 P(x_0) & \dots & \mathrm{D}_{l-1}^0 P(x_0) & \mathrm{D}_l^0 P(x_0) \\ 0 & \mathrm{I}P(x_0) & \dots & \mathrm{D}_{l-1}^1 P(x_0) & \mathrm{D}_l^1 P(x_0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathrm{I}P(x_0) & \mathrm{D}_l^{l-1} P(x_0) \\ 0 & 0 & \dots & 0 & \mathrm{I}P(x_0) \end{bmatrix}. \qquad (\spadesuit\spadesuit)$$

Theorem

Let $l \in \mathbb{Z}_{\geq 0}$, $\widetilde{\mathcal{P}}_l := (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_l)$,

$$\mathcal{P}_k(x) := \left(x^{{}^1\alpha}, \dots, x^{{}^{d(k)}\alpha}\right), \quad \left({}^1\alpha, \dots, {}^{d(k)}\alpha\right) = \mathcal{A}_k, \; k = 0, \dots, l.$$

Let $P \in \Pi$, $x_0 \in \mathbb{C}^d$ be a given point, $\widetilde{\mathrm{D}}_l P(x_0)$ be matrix $(\spadesuit\spadesuit)$, $v \in \mathbb{C}^{\widetilde{d}(l)}$ be a vector. Then

$$\left.\begin{matrix} e^{ix_0 \cdot x}\left[\widetilde{\mathcal{P}}_l(x)\right] v \neq 0, \\ e^{ix_0 \cdot x}\left[\widetilde{\mathcal{P}}_l(x)\right] v \in e^{ix_0 \cdot x}\Pi_{\leq l} \cap \ker P(-iD) \end{matrix}\right\} \iff \begin{cases} P(x_0) = 0, \\ v \in \ker \widetilde{\mathrm{D}}_l P(x_0) \end{cases}$$

Lemma

Let $l \in \mathbb{Z}_{\geq 0}$, $\widetilde{\mathcal{D}}_l$ be the lexicographically ordered set of derivatives, $\widetilde{\mathrm{D}}_l$ be matrix $(\spadesuit)$, and functions f, g be sufficiently differentiable. Then

$$\boxed{\left[\widetilde{\mathcal{D}}_l(fg)\right] = \left[\widetilde{\mathcal{D}}_l f\right] \widetilde{\mathrm{D}}_l g = \left[\widetilde{\mathcal{D}}_l g\right] \widetilde{\mathrm{D}}_l f} \qquad (\bigstar)$$

Remark

The proof of the previous lemma can be based on the Leibniz rule.

Sketch of the proof

- $\det \left[\widetilde{\mathrm{D}}_l P(x_0)\right] = 0 \iff P(x_0) = 0$
- *Consider an expression*

$$P(-iD) \left(e^{ix_0 \cdot x} \left[\widetilde{\mathcal{P}}_l(x)\right] v\right) \stackrel{\mathfrak{F}}{\Longrightarrow} P(\xi) \left[\widetilde{\mathcal{D}}_l \delta(\xi - x_0)\right] v.$$

For $\forall \phi \in S(\mathbb{C}^d)$, *the functional* $\overline{\left\langle P \left[\widetilde{\mathcal{D}}_l \delta(\cdot - x_0)\right] v, \phi \right\rangle}$ *is of the form*

$$\overline{\left\langle P \left[\widetilde{\mathcal{D}}_l \delta(\cdot - x_0)\right] v, \phi \right\rangle} = \overline{\left\langle \delta(\cdot - x_0), \left[\widetilde{\mathcal{D}}_l \left(\overline{P}\phi\right)\right] \overline{v} \right\rangle}$$

$$= \left[\widetilde{\mathcal{D}}_l \left(P(x_0)\overline{\phi(x_0)}\right)\right] v \stackrel{\text{by } (\bigstar)}{=} \left[\widetilde{\mathcal{D}}_l \overline{\phi(x_0)}\right] \left[\widetilde{\mathrm{D}}_l P(x_0)\right] v.$$

Sketch of the proof

- $\det \left[\widetilde{\mathrm{D}}_l P(x_0) \right] = 0 \iff P(x_0) = 0$
- *Consider an expression*

$$P(-iD) \left(e^{ix_0 \cdot x} \left[\widetilde{\mathcal{P}}_l(x) \right] v \right) \stackrel{\mathfrak{F}}{\Longrightarrow} P(\xi) \left[\widetilde{\mathcal{D}}_l \delta(\xi - x_0) \right] v.$$

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$$\overline{\left\langle P \left[\widetilde{\mathcal{D}}_l \delta(\cdot - x_0) \right] v, \phi \right\rangle} = \overline{\left\langle \delta(\cdot - x_0), \left[\widetilde{\mathcal{D}}_l \left(\overline{P} \phi \right) \right] \overline{v} \right\rangle}$$
$$= \left[\widetilde{\mathcal{D}}_l \left(P(x_0) \overline{\phi(x_0)} \right) \right] v = \left[\widetilde{\mathcal{D}}_l \overline{\phi(x_0)} \right] \left[\widetilde{\mathrm{D}}_l P(x_0) \right] v.$$

Thus

$$\left. \begin{array}{l} e^{ix_0 \cdot x} \left[\widetilde{\mathcal{P}}_l(x) \right] v \neq 0, \\ e^{ix_0 \cdot x} \left[\widetilde{\mathcal{P}}_l(x) \right] v \in e^{ix_0 \cdot x} \Pi_{\leq l} \cap \ker P(-iD) \end{array} \right\} \iff \left\{ \begin{array}{l} P(x_0) = 0, \\ v \in \ker \widetilde{\mathrm{D}}_l P(x_0) \end{array} \right. \quad \square$$

Sketch of the proof

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- *For* $\forall\phi \in \boldsymbol{S}(\mathbb{C}^d)$, *the functional* $\overline{\left\langle \boldsymbol{P}\left[\widetilde{\mathcal{D}}_l \delta(\cdot - x_0)\right] \boldsymbol{v}, \phi \right\rangle}$ *is of the form*

$$\overline{\left\langle \boldsymbol{P}\left[\widetilde{\mathcal{D}}_l \delta(\cdot - x_0)\right] \boldsymbol{v}, \phi \right\rangle} = \overline{\left\langle \delta(\cdot - x_0), \left[\overline{\widetilde{\mathcal{D}}_l} \left(\overline{\boldsymbol{P}}\phi\right)\right] \overline{\boldsymbol{v}} \right\rangle}$$
$$= \left[\widetilde{\mathcal{D}}_l \left(\boldsymbol{P}(x_0)\overline{\phi(x_0)}\right)\right] \boldsymbol{v} = \left[\widetilde{\mathcal{D}}_l \overline{\phi(x_0)}\right] \left[\widetilde{\mathrm{D}}_l \boldsymbol{P}(x_0)\right] \boldsymbol{v}.$$

Thus

$$\left.\begin{array}{l} e^{ix_0\cdot x}\left[\widetilde{\mathcal{P}}_l(x)\right] \boldsymbol{v} \neq 0, \\ e^{ix_0\cdot x}\left[\widetilde{\mathcal{P}}_l(x)\right] \boldsymbol{v} \in e^{ix_0\cdot x}\Pi_{\leq l} \cap \ker \boldsymbol{P}(-iD) \end{array}\right\} \iff \left\{\begin{array}{l} \boldsymbol{P}(x_0) = 0, \\ \boldsymbol{v} \in \ker \widetilde{\mathrm{D}}_l \boldsymbol{P}(x_0) \end{array}\right. \quad \square$$

1. **А.Н. Колмогоров, С.В. Фомин. Элементы теории функций и функционального анализа. М.: Наука, 2004.**

Remark

The previous proof can be transformed for other (linear and linear-conjugate, see for example [1]) functionals like $\overline{\langle \boldsymbol{f}, \phi \rangle}$.

Sketch of the proof

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- *For* $\forall\phi \in \boldsymbol{S}(\mathbb{C}^d)$, *the functional* $\overline{\left\langle P\left[\widetilde{\mathcal{D}}_l\delta(\cdot - x_0)\right] v, \phi\right\rangle}$ *is of the form*

$$\overline{\left\langle P\left[\widetilde{\mathcal{D}}_l\delta(\cdot - x_0)\right] v, \phi\right\rangle} = \overline{\left\langle \delta(\cdot - x_0), \left[\overline{\widetilde{\mathcal{D}}_l}\left(\overline{P}\phi\right)\right]\overline{v}\right\rangle}$$
$$= \left[\widetilde{\mathcal{D}}_l\left(P(x_0)\overline{\phi(x_0)}\right)\right] v = \left[\widetilde{\mathcal{D}}_l\overline{\phi(x_0)}\right]\left[\widetilde{\mathrm{D}}_l P(x_0)\right] v.$$

Thus

$$\left.\begin{array}{l} e^{ix_0\cdot x}\left[\widetilde{\mathcal{P}}_l(x)\right] v \neq 0, \\ e^{ix_0\cdot x}\left[\widetilde{\mathcal{P}}_l(x)\right] v \in e^{ix_0\cdot x}\Pi_{\leq l} \cap \ker P(-iD) \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} P(x_0) = 0, \\ v \in \ker \widetilde{\mathrm{D}}_l P(x_0) \end{array}\right. \quad \square$$

1. Kolmogorov A.N., Fomin S.V. Elements of the Theory of Functions and Function Analysis. Dover Books, New York (1957)

Remark

The previous proof can be transformed for other (linear and linear-conjugate, see for example [1]) functionals like $\overline{\langle \boldsymbol{f}, \phi\rangle}$.

Lemma (almost obvious)

Let $\mathbf{l} \in \mathbb{Z}_{\geq 0}$, $\widetilde{\mathcal{P}}_l$ *be the lexicographically ordered set of monomials.*

Let $\mathcal{V}_l \subseteq \Pi_{\leq l}$ *be a space of polynomials. Then there is the linear space* $\mathbf{V}_l \subseteq \mathbb{C}^{\tilde{d}(l)}$ *such that*

$$\mathcal{V}_l = \left\{ \left[\widetilde{\mathcal{P}}_l \right] v : v \in \mathbf{V}_l \right\}. \qquad (*)$$

And vice versa.

Theorem (unproved yet)

Let $\mathbf{l} \in \mathbb{Z}_{\geq 0}$, $\mathbf{P} \in \Pi$, $\mathbf{x_0} \in \mathbb{C}^d$ *be a given point. Let* $\mathbf{V}_l$ *be the linear space and* $\mathcal{V}_l$ *be the polynomial space defined in the previous lemma. Then*

$$\boxed{e^{ix_0 \cdot x}\mathcal{V}_l = e^{ix_0 \cdot x}\Pi_{\leq l} \cap \ker P(-iD) \Longleftrightarrow V_l = \ker \widetilde{\mathrm{D}}_l P(x_0).} \qquad (**)$$

Lemma (almost obvious)

Let $l \in \mathbb{Z}_{\geq 0}$, $\widetilde{\mathcal{P}}_l := (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_l)$,

$$\mathcal{P}_k(x) := \left(x^{{}^1\alpha}, \dots, x^{{}^{d(k)}\alpha}\right),$$

$$\left({}^1\alpha, \dots, {}^{d(k)}\alpha\right) = \mathcal{A}_k, \ k = 0, \dots, l.$$

Let $\mathcal{V}_l \subseteq \Pi_{\leq l}$ *be a space of polynomials. Then there is the linear space* $V_l \subseteq \mathbb{C}^{\tilde{d}(l)}$ *such that*

$$\mathcal{V}_l = \left\{ \left[\widetilde{\mathcal{P}}_l\right] v : v \in V_l \right\}. \qquad (*)$$

And vice versa.

Theorem (unproved yet)

Let $l \in \mathbb{Z}_{\geq 0}$, $P \in \Pi$, $x_0 \in \mathbb{C}^d$ *be a given point. Let* V_l *be the linear space and* $\mathcal{V}_l$ *be the polynomial space defined in the previous lemma. Then*

$$\boxed{e^{ix_0 \cdot x}\mathcal{V}_l = e^{ix_0 \cdot x}\Pi_{\leq l} \cap \ker P(-iD) \iff V_l = \ker \widetilde{\mathrm{D}}_l P(x_0).} \qquad (**)$$

Theorem

Let $l \in \mathbb{Z}_{\geq 0}$. Let $P \in \Pi$, let x_0 be a root of P. Let $D^\alpha P(x_0)$, $\alpha \in \mathbb{Z}^d_{\geq 0}$, be a nonzero derivative of the least order. Then we have

$$\dim \left(e^{ix_0 \cdot x} \Pi_{\leq l} \cap \ker P(-iD) \right) = \begin{cases} \tilde{d}(l) - \tilde{d}(l - |\alpha|) & \text{if } l \geq |\alpha| > 0; \\ \tilde{d}(l) & \text{if } l < |\alpha|. \end{cases}$$

Moreover, if $l < |\alpha|$; then

$$e^{ix_0 \cdot x} \Pi_{\leq l} \cap \ker P(-iD) = e^{ix_0 \cdot x} \Pi_{\leq l}.$$

Let $P_{n'} \in \Pi$, $n' = 1, 2, \dots, n$. Then the *system of PDE's* is

$$\begin{cases} P_1(-iD)\cdot = 0, \\ \vdots \\ P_n(-iD)\cdot = 0. \end{cases} \qquad (*)$$

Theorem

Let $l \in \mathbb{Z}_{\geq 0}$, $v \in \mathbb{C}^{\widetilde{d}(l)}$, $x_0 \in \mathbb{C}^d$. Then

$$\left. \begin{aligned} & e^{ix_0 \cdot x} \left[\widetilde{\mathcal{P}}_l\right] v \neq 0, \\ & e^{ix_0 \cdot x} \left[\widetilde{\mathcal{P}}_l\right] v \\ & \qquad \in e^{ix_0 \cdot x} \Pi_{\leq l} \cap \bigcap_{n'=1}^{n} \ker P_{n'}(-iD) \end{aligned} \right\} \Longleftrightarrow \begin{cases} v \in \ker \begin{bmatrix} \widetilde{\mathrm{D}}_l P_1(x_0) \\ \vdots \\ \widetilde{\mathrm{D}}_l P_n(x_0) \end{bmatrix}, \\ P_1(x_0) \\ \quad = \cdots = P_n(x_0) = 0 \end{cases}$$

Theorem (based on Kronecker–Capelli (Rouché—Capelli) theorem)

- $P, F \in \Pi$; $x_0 \in \mathbb{C}^d$; $\underline{P(x_0) = 0}$; $F(x) := \left[\widetilde{\mathcal{P}}_{\deg F}(x)\right] w$, $w \in \mathbb{C}^{\widetilde{d}(\deg F)}$.
- *Let* D^α, $\alpha \in \mathbb{Z}^d_{\geq 0}$, *be the least order derivative such that* $D^\alpha P(x_0) \neq 0$.
- $l \in \mathbb{Z}_{\geq 0}$, $\boxed{l \geq \deg F + |\alpha|;}$
- *let* $V_l \subseteq \mathbb{C}^{\widetilde{d}(l)}$ *be a* linear space *and* $\mathcal{V}_l \subseteq \Pi_{\leq l}$ *be the corresponding polynomial space:* $\mathcal{V}_l := \left\{\left[\widetilde{\mathcal{P}}_l\right] v : v \in V_l\right\}$.

Then

$$\forall p \in \mathcal{V}_l : \quad P(-iD)\left(e^{ix_0\cdot x} p\right) = e^{ix_0\cdot x} F(x) \qquad (*)$$

$$\Updownarrow$$

$$\forall v \in V_l : \quad \left[\widetilde{\mathrm{D}}_l P(x_0)\right] v = \begin{bmatrix} w \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} \widetilde{d}(l) - \widetilde{d}(\deg F) \end{bmatrix}. \qquad (**)$$

Moreover,

1. $\dim \mathcal{V}_l = \widetilde{d}(l) - \widetilde{d}(l - |\alpha|)$;
2. *for any polynomial* F *such that* $\deg F \leq l - |\alpha|$, *linear system* $(**)$ *is* consistent;
3. *there exists a polynomial of* arbitrary large degree *to satisfy PDE* $(*)$.

Theorem (based on Kronecker–Capelli (Rouché—Capelli) theorem)

- $P, F \in \Pi$; $x_0 \in \mathbb{C}^d$; $\underline{P(x_0) \neq 0}$; $F(x) := \left[\widetilde{\mathcal{P}}_{\deg F}(x)\right] w$, $w \in \mathbb{C}^{\widetilde{d}(\deg F)}$.
- $l \in \mathbb{Z}_{\geq 0}$, $\boxed{l \geq \deg F;}$
- *let* $V_l \subseteq \mathbb{C}^{\widetilde{d}(l)}$ *be a* linear space *and* $\mathcal{V}_l \subseteq \Pi_{\leq l}$ *be the corresponding polynomial space:* $\mathcal{V}_l := \left\{\left[\widetilde{\mathcal{P}}_l\right] v : v \in V_l\right\}$.

Then

$$\forall p \in \mathcal{V}_l : \quad P(-iD)\left(e^{ix_0\cdot x} p\right) = e^{ix_0\cdot x} F(x) \qquad (*)$$

$$\Updownarrow$$

$$\forall v \in V_l : \quad \left[\widetilde{\mathrm{D}}_l P(x_0)\right] v = \begin{bmatrix} w \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} \widetilde{d}(l) - \widetilde{d}(\deg F) \end{bmatrix}. \qquad (**)$$

Remark

If $P(x_0) \neq 0$, *since the matrix* $\widetilde{\mathrm{D}}_l P(x_0)$ *is not singular, it follows that the polynomial* p *is defined* uniquely and does not depend on the choice of l.

Definition

- *Let the space* $\mathbb{C}^{\widetilde{d}(l)}$, $l \in \mathbb{Z}_{\geq 0}$, *be the space with a* Cartesian coordinate system:

$$\boxed{\mathbb{C}^{\widetilde{d}(l)} := \operatorname{span}\left\{e_q : 1 \leq q \leq \widetilde{d}(l)\right\},}$$

where e_q *is a* q*th basis vector:* $e_q := \left(\delta_{q1}, \dots, \delta_{q,\widetilde{d}(l)}\right)$.

We can decompose $\mathbb{C}^{\widetilde{d}(l)}$ *as follows*

$$\begin{aligned} &\mathbb{C}^{\widetilde{d}(l)} = {}^{0}\mathbb{C}^{\widetilde{d}(l)} \oplus {}^{1}\mathbb{C}^{\widetilde{d}(l)} \oplus \dots \oplus {}^{l}\mathbb{C}^{\widetilde{d}(l)}, \\ &{}^{j}\mathbb{C}^{\widetilde{d}(l)} \perp {}^{j'}\mathbb{C}^{\widetilde{d}(l)} \quad \text{if } j \neq j', \end{aligned} \qquad (*)$$

where

$$\begin{aligned} &{}^{0}\mathbb{C}^{\widetilde{d}(l)} := \operatorname{span}\{e_1\}, \\ &{}^{m}\mathbb{C}^{\widetilde{d}(l)} := \operatorname{span}\left\{e_q : \widetilde{d}(m-1)+1 \leq q \leq \widetilde{d}(m)\right\}, \qquad m = 1, \dots, l. \end{aligned}$$

Remark

Decomposition $(*)$ *corresponds to the block structure of matrix* $(\spadesuit)$ *(as well as structures of* $\widetilde{\mathcal{P}}_l$ *and* $\widetilde{\mathcal{D}}_l$*).*

Definition

- *Let the space* $\mathbb{C}^{\widetilde{d}(l)}$, $l \in \mathbb{Z}_{\geq 0}$, *be the space with a* Cartesian coordinate system:

$$\boxed{\mathbb{C}^{\widetilde{d}(l)} := \operatorname{span}\left\{e_q : 1 \leq q \leq \widetilde{d}(l)\right\},}$$

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- *By* $\mathfrak{P}_m$ *denote the* orthogonal projection *on the subspace* ${}^{m}\mathbb{C}^{\widetilde{d}(l)}$, $m = 0, \ldots, l$; *and define subspaces of the null-space* $V_l := \ker \widetilde{\mathrm{D}}_l(x_0)$ *as follows*

$$\boxed{{}^{m}V_l := \mathfrak{P}_m V_l, \qquad m = 0, 1, \ldots, l.} \qquad (**)$$

Remark

The linear space V_l *can be presented as a sum, like* $(*)$, *of the subspaces* ${}^{m}V_l$ *only in the case of scale invariance of the corresponding polynomial space* $\mathcal{V}_l$.

Definition

$$\mathbb{C}^{\widetilde{d}(l)} := \operatorname{span}\left\{ e_q : 1 \le q \le \widetilde{d}(l) \right\},$$

$$e_q := \left(\delta_{q1}, \ldots, \delta_{q,\widetilde{d}(l)} \right).$$

$$\mathbb{C}^{\widetilde{d}(l)} = {}^{0}\mathbb{C}^{\widetilde{d}(l)} \oplus {}^{1}\mathbb{C}^{\widetilde{d}(l)} \oplus \cdots \oplus {}^{l}\mathbb{C}^{\widetilde{d}(l)}. \qquad (*)$$

$${}^{m}V_l := \mathfrak{P}_m V_l, \qquad m = 0, 1, \ldots, l. \qquad (**)$$

Theorem ($\diamondsuit$)

Let $l \in \mathbb{Z}_{\ge 0}$. Let the matrix $\widetilde{\mathrm{D}}_l P(x_0)$ be singular and $V_l := \ker \widetilde{\mathrm{D}}_l P(x_0)$. Then the subspace ${}^{l}V_l := \mathfrak{P}_l V_l$ is nonzero.

1. **Abramov S., Petkovšek M. On polynomial solutions of linear partial differential and (q-)diference equations. In: Computer Algebra in Scientific Computing. 14th Internetional workshop LNCS 7442, 1–11 (2012)**

Corollary (of Theorem $\diamondsuit$)

The null-space of the operator $P(-iD)$ contains polynomials (multiplied by the exponential $e^{ix_0 \cdot x}$, where $P(x_0) = 0$) of any *degree. Thus we have a fundamental property, see [1], of polynomial solutions to a single constant coefficient PDE.*

2D Laplace operator

$$L_1 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad x, y \in \mathbb{R}.$$

The polynomial that induces the operator is

$$P_1(\xi, \eta) := -\xi^2 - \eta^2, \qquad \xi, \eta \in \mathbb{C}.$$

15×15 matrix $\widetilde{\mathrm{D}}_4 P_1(0,0)$ is of the form

$$\widetilde{\mathrm{D}}_4 P_1(0,0) := \left[\begin{array}{c|cc|ccc|cccc|ccccc}
0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 12 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

$$\left[\widetilde{\mathcal{P}}_4\right]\left[\ker \widetilde{\mathrm{D}}_4 P_1(0,0)\right]$$

$$= \left[\begin{array}{c|cc|ccc|cccc|ccccc} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^4 & x^3y & x^2y^2 & xy^3 & y^4 \end{array}\right]$$

$$\times \left[\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \boxed{\left[\begin{array}{ccccccccc} 1 & x & y & xy & y^2 - x^2 & 3xy^2 - x^3 & y^3 - 3x^2y & xy^3 - x^3y & x^4 - 6x^2y^2 + y^4 \end{array}\right]}$$

$$L_1 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad x, y \in \mathbb{R}.$$

$$P_1(\xi, \eta) := -\xi^2 - \eta^2, \qquad \xi, \eta \in \mathbb{C},$$

$$P_1(1, i) = 0.$$

1. Zakharov V.G. Reproducing solutions to PDEs by scaling functions. Int. J. Wavelets Multiresolut. Inf. Process. 3, 2050017 (2020)

$$L_1 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad x, y \in \mathbb{R}.$$

$$P_1(1, i) = 0.$$

1. Zakharov V.G. Reproducing solutions to PDEs by scaling functions. Int. J. Wavelets Multiresolut. Inf. Process. 3, 2050017 (2020)

15×15 matrix $\widetilde{\mathrm{D}}_4 P_1(1, i)$ is of the form

$$\begin{bmatrix}
0 & 2i & -2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4i & -2 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2i & -4 & 0 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6i & -2 & 0 & 0 & 12 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4i & -4 & 0 & 0 & 6 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2i & -6 & 0 & 0 & 2 & 0 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8i & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6i & -4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4i & -6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2i & -8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$L_1 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad x, y \in \mathbb{R}.$$

$$P_1(1, i) = 0.$$

1. Zakharov V.G. Reproducing solutions to PDEs by scaling functions. Int. J. Wavelets Multiresolut. Inf. Process. 3, 2050017 (2020)

$$e^{ix-y}\Pi_{\le 4} \cap \ker L_1 = e^{ix-y}\,\mathrm{span}\left\{1, y - ix, -x^2 - 2ixy + y^2, ix^3 - 3x^2y - 3ixy^2 + y^3, x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4\right\}. \qquad (*)$$

Remark

If real or imaginary parts of the polynomials in $(*)$ (compare with

$$\Pi_{\le 4} \cap \ker L_1 = \Pi_{\le 1} \oplus \mathrm{span}\left\{xy, y^2 - x^2, 3xy^2 - x^3, y^3 - 3x^2y, xy^3 - x^3y, x^4 - 6x^2y^2 + y^4\right\})$$

multiplied by an exponential, then these parts are not solutions to the Laplace operator.

1. Zakharov V.G. Reproducing solutions to PDEs by scaling functions. Int. J. Wavelets Multiresolut. Inf. Process. 3, 2050017 (2020)

$$L_{\mathrm{sh}} = \left(\frac{\partial}{\partial x} - \mathrm{I}\right)^2 + \left(\frac{\partial}{\partial y} - i\mathrm{I}\right)^2, \quad \text{where I is the identity operator} \; .$$

$$P_{\mathrm{sh}}(\xi, \eta) := (i\xi - 1)^2 + (i\eta - i)^2$$

15×15 matrix $\widetilde{\mathrm{D}}_4 P_{\mathrm{sh}}(-i, 1)$ is of the form

$$\begin{bmatrix}
0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$L_{\mathrm{sh}} = \left(\frac{\partial}{\partial x} - \mathrm{I}\right)^2 + \left(\frac{\partial}{\partial y} - i\mathrm{I}\right)^2, \quad \text{where I is the identity operator} \; .$$

$$P_{\mathrm{sh}}(\xi,\eta) := (i\xi - 1)^2 + (i\eta - i)^2\,, \qquad P_1(\xi,\eta) := -\xi^2 - \eta^2.$$

$$\widetilde{\mathrm{D}}_4 P_{\mathrm{sh}}(-i,1) := \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\widetilde{\mathrm{D}}_4 P_1(0,0) := \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. **Zakharov V.G. Reproducing solutions to PDEs by scaling functions. Int. J. Wavelets Multiresolut. Inf. Process. 3, 2050017 (2020)**

$$L_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

$$L_{\rm sh} = \left(\frac{\partial}{\partial x} - \mathrm{I}\right)^2 + \left(\frac{\partial}{\partial y} - i\mathrm{I}\right)^2, \quad \text{where I is the identity operator}.$$

$$P_{\rm sh}(\xi, \eta) := (i\xi - 1)^2 + (i\eta - i)^2$$

$$\widetilde{\mathrm{D}}_4 P_{\rm sh}(-i, 1) := \left[\begin{array}{c|cc|ccc|cccc|ccccc}
0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & & & & & & & & & & & & & & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

$$\boxed{e^{x+iy}\Pi_{\le 4} \cap L_{\rm sh} = e^{x+iy}\left(\Pi_{\le 4} \cap L_1\right)}$$

$$\begin{cases} \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} = 0, \\ \dfrac{\partial^2}{\partial x \partial y} + \dfrac{\partial^2}{\partial x \partial z} + \dfrac{\partial^2}{\partial y \partial z} = 0. \end{cases}$$

1. Pedersen P. A basis for polynomial solutions to systems of linear constant coefficient PDE's. Adv. Math. 117, 157–163 (1996)

$$\begin{cases} \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} = 0, \\ \dfrac{\partial^2}{\partial x \partial y} + \dfrac{\partial^2}{\partial x \partial z} + \dfrac{\partial^2}{\partial y \partial z} = 0. \end{cases}$$

Polynomials: $P_1(\xi, \eta, \zeta) := -\xi^2 - \eta^2 - \zeta^2, \quad P_2(\xi, \eta, \zeta) := -\xi\eta - \xi\zeta - \eta\zeta.$

40×20 matrix $\left[\dfrac{\widetilde{D}_3 P_1(0,0,0)}{\widetilde{D}_3 P_2(0,0,0)}\right]$ is of the form

$$\left[\begin{array}{c|ccc|cccccc|cccccccccc}
0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 6 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

$$\begin{cases} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = 0, \\ \frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial x \partial z} + \frac{\partial^2}{\partial y \partial z} = 0. \end{cases}$$

1. Pedersen P. A basis for polynomial solutions to systems of linear constant coefficient PDE's. Adv. Math. 117, 157–163 (1996)

Polynomials: $P_1(\xi, \eta, \zeta) := -\xi^2 - \eta^2 - \zeta^2, \quad P_2(\xi, \eta, \zeta) := -\xi\eta - \xi\zeta - \eta\zeta.$

Basis of the space $\Pi_{\leq 3} \cap \ker P_1(-iD) \cap \ker P_2(-iD)$ **is of the form**

$$\Big\{1, x, y, z, y^2 - x^2, xz - xy, yz - xy, z^2 - x^2, \\ 3x^2y - 3x^2z - y^3 + z^3, -x^3 + 3x^2y + 3xy^2 - 6xyz - 2y^3 + 3yz^2, \\ -2x^3 + 3x^2y + 3xy^2 - 6xyz + 3xz^2 - y^3, \\ x^3 + 3x^2y - 3x^2z - 3xy^2 - y^3 + 3y^2z\Big\}.$$

1. **de Boor C. The polynomials in the linear span of integer translates of a compactly supported function. Constr. Approx. 3, 199–208 (1987)**

$$\mathcal{V}_{\text{de Boor}} := \operatorname{span}\left\{1, y, x + y^2\right\},$$

$$L_3 := 2\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}, \qquad x, y \in \mathbb{R}.$$

$$P_3(\xi, \eta) := 2i\xi + \xi^2 + \eta^2, \qquad \xi, \eta \in \mathbb{C}.$$

15×15 matrix $\widetilde{\mathrm{D}}_4 P_3(0,0)$ is of the form

$$\begin{bmatrix}
0 & 2 & 0 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & -6 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & -2 & 0 & -6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & -12 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & -6 & 0 & -6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 & -12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$L_3 := 2\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}, \qquad x, y \in \mathbb{R}.$$

$$P_3(\xi,\eta) := 2i\xi + \xi^2 + \eta^2, \qquad \xi,\eta \in \mathbb{C}.$$

$$\left[\widetilde{\mathcal{P}}_4\right]\left[\ker \widetilde{\mathrm{D}}_4 P_3(0,0)\right]$$

$$= \left[\begin{array}{c|cc|ccc|cccc|ccccc} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^4 & x^3y & x^2y^2 & xy^3 & y^4 \end{array}\right]$$

$$\times \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] = \boxed{\left[\begin{array}{ccccc} 1 & y & x + y^2 & 3xy + y^3 & 3x^2 + 6xy^2 + 3x + y^4 \end{array}\right]}$$

A Poisson equation is:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) p(x,y) = e^{ix-y}\left(-2xy + 3x + y^2 + 2\right) \qquad (*)$$

$$P(\xi,\eta) := -\xi^2 - \eta^2, \qquad w := \begin{bmatrix}2 & 3 & 0 & 0 & -2 & 1\end{bmatrix}^T,$$

$$F(x,y) := -2xy + 3x + y^2 + 2 = \left[\widetilde{\mathcal{P}}_2(1,i)\right] w$$

$$= \begin{bmatrix}1 & x & y & x^2 & xy & y^2\end{bmatrix}\begin{bmatrix}2 & 3 & 0 & 0 & -2 & 1\end{bmatrix}^T.$$

$l := 3 \geq \deg F + |\alpha| = 2 + 1 = 3.$

The corresponding linear algebraic equation $\left[\widetilde{\mathrm{D}}_3 P(1,i)\right] v = \left[w^T, 0, \cdots, 0\right]^T$ is:

$$\left[\begin{array}{c|cc|ccc|cccc}
0 & 2i & -2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 4i & -2 & 0 & 6 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2i & -4 & 0 & 2 & 0 & 6 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 6i & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4i & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2i & -6 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] v = \begin{bmatrix}2 \\ 3 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0\end{bmatrix} \qquad (**)$$

A Poisson equation is:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) p(x,y) = e^{ix-y}\left(-2xy + 3x + y^2 + 2\right) \qquad (*)$$

$$\left[\begin{array}{c|cc|ccc|cccc} 0 & 2i & -2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 4i & -2 & 0 & 6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2i & -4 & 0 & 2 & 0 & 6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 6i & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4i & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2i & -6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] v = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (**)$$

Solution to algebraic equation $(**)$,
dim ker $\left[\widetilde{D}_3 P(1,i)\right] = \widetilde{d}(3) - \widetilde{d}(3-1) = d(3) = 4$, is:

$$v = \left(v_0, v_1, iv_1 + \left(\frac{5}{4} + \frac{i}{4}\right), -v_5 + \left(\frac{1}{4} + \frac{i}{4}\right), \left(-\frac{1}{2} - \frac{i}{2}\right)(-1 + (2+2i)v_5), v_5, \right.$$
$$\left.\left(\frac{1}{6} + \frac{i}{6}\right)(-1 + (3+3i)v_9), -3v_9 + \left(\frac{1}{2} - \frac{i}{2}\right), -\frac{1}{2} i(6v_9 - 1), v_9\right),$$
$$v_0, v_1, v_5, v_9 \in \mathbb{C}.$$

A Poisson equation is:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) p(x,y) = e^{ix-y}\left(-2xy + 3x + y^2 + 2\right) \qquad (*)$$

Solution to algebraic equation $(**)$ is:

$$v = \left(v_0, v_1, iv_1 + \left(\frac{5}{4} + \frac{i}{4}\right), -v_5 + \left(\frac{1}{4} + \frac{i}{4}\right), \left(-\frac{1}{2} - \frac{i}{2}\right)(-1 + (2+2i)v_5), v_5, \left(\frac{1}{6} + \frac{i}{6}\right)(-1 + (3+3i)v_9), -3v_9 + \left(\frac{1}{2} - \frac{i}{2}\right), -\frac{1}{2}i(6v_9 - 1), v_9\right), \quad v_0, v_1, v_5, v_9 \in \mathbb{C}.$$

Solution to PDE $(*)$ is:

$$p(x,y) = e^{ix-y}\left[\widetilde{\mathcal{P}}_3(x,y)\right] v = e^{ix-y}\Bigg(\left(\left(\frac{1}{6} + \frac{i}{6}\right)(-1 + (3+3i)v_9)x^3 + v_5\left(-x^2 - 2ixy + y^2\right) + \left(-3v_9 + \left(\frac{1}{2} - \frac{i}{2}\right)\right)\right) x^2 y - \frac{1}{2} i(6v_9 - 1)xy^2 + v_1(x + iy) + v_9 y^3 + v_0 + \left(\frac{1}{4} + \frac{i}{4}\right)x^2 + \left(\frac{1}{2} + \frac{i}{2}\right)xy + \left(\frac{5}{4} + \frac{i}{4}\right)y\Bigg), \quad v_0, v_1, v_5, v_9 \in \mathbb{C}.$$

Another Poisson equation is:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) p(x,y) = e^{y-x}\left(-2xy + 3x + y^2 + 2\right) \qquad (*)$$

$$P(\xi,\eta) := -\xi^2 - \eta^2, \qquad w := \begin{bmatrix} 2 & 3 & 0 & 0 & -2 & 1 \end{bmatrix}^T,$$

$$F(x,y) := -2xy + 3x + y^2 + 2 = \left[\widetilde{\mathcal{P}}_2(1,i)\right] w$$

$$= \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 & 0 & -2 & 1 \end{bmatrix}^T.$$

$l := 2 \geq \deg F = 2.$

The corresponding linear algebraic equation $\left[\widetilde{\mathcal{D}}_2 P(i,-i)\right] v = w$ **is:**

$$\left[\begin{array}{c|cc|ccc} -2 & 2 & -2 & -2 & 0 & -2 \\ \hline 0 & -2 & 0 & 4 & -2 & 0 \\ 0 & 0 & -2 & 0 & 2 & -4 \\ \cline{2-6} 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{array}\right] v = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \qquad (**)$$

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$P(i, -i) \neq 0$

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Unique solutions to algebraic equation $(**)$ and PDE $(*)$ are:

$$v := \left(-5, -\frac{5}{2}, 2, 0, 1, -\frac{1}{2}\right),$$

$$\boxed{p(x,y) := e^{y-x}\left(xy - \frac{5x}{2} - \frac{y^2}{2} + 2y - 5\right)}$$

- **Now I have got a draft program (8,9 lines of code) on Wolfram Mathematica to obtain a polynomial basis of any dimension to solve any constant coefficient PDE (system of PDE's).**

Future trends

I shall try to obtain:

- ...matrix methods of polynomial solutions to PDE('s) with *polynomial coefficients*;
- ...matrix methods of investigation of polynomial solutions to *systems* of constant coefficient PDE's;
- ...matrix methods of investigation of *affine invariance/no invariance* of a polynomial solution space to constant coefficient PDE('s).
- Also I would like to supply my computer code in Mathematica by some *interface and maintenance*.

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To be continued. . .